

# Subjective Beliefs and Unintended Consequences in a Class of Differential Games with Economic Applications<sup>☆</sup>

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## Abstract

We discuss the implications of decisions based on subjective beliefs for a class of non-cooperative differential games with applications in the field of economics. In the framework proposed, players lack the relevant information to pursue their optimal strategies and have to base their decisions on subjective beliefs. Either by ignorance or choice, players in this class of games face the relevance paradox. This phenomena is a result of decision in incomplete information environments, driven by nonlinear mechanisms. We argue that solutions to this class of games are optimal if decisions based on subjective beliefs are consistent with the existence of a stable self-confirming equilibrium (SCE). To demonstrate these conjectures, we put forward two growing economy models, set up as differential games, where a discrete set of asymmetric players seeks to maximize consumption utility, given the evolution of their asset portfolio. To pursue this objective, players choose open loop strategies that are coupled by a risk premium mechanism describing the state of the game. Since they lack information about the state of the game, decisions have to be based on beliefs, and optimal outcomes require the existence of a stable SCE. We propose to answer the following question. Are players able to concur on SCE, where their expectations are self-fulfilling? In the first game proposed, we show that under some strict conditions, such as naive smooth beliefs, it is possible to describe completely the existence of a SCE solution and perform a qualitative analysis of equilibrium. Numerical results suggest that SCE are not locally stable solutions under this set of assumptions. In the second game proposed, we show that the game solution can only be described as a multi-objective optimization problem under uncertainty. We evaluate this solution numerically as a multi-criteria Hidden Markov Model(HMM), and show that under a linear learning regime, we have stable SCE solutions. We also show the existence of unintended consequences and strong emergence phenomena, as a result of persistent uncertainty.

*Keywords:* Nonlinear Differential Games, Subjective Beliefs, Evolutionary Multi-Objective Optimization, Self-Confirming Equilibrium, Hidden Markov Model, Unintended Consequences

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## 1. Introduction

The initial motivation for this proposal is related to the recent focus in literature on the analysis of nonlinear economies as competitive multi-agent frameworks. In modern economics literature this topic is framed by the seminal proposal of Grandmont [26]. In this paper, the author puts forward an extensive discussion on the vast implications and mathematical challenges of undertaking stability analysis in large socio-economic systems, with decentralized decision dynamics under incomplete information. Our aim in this paper is to discuss the implications of nonlinearity and incomplete information in growing economies, set up as non-cooperative differential games with asymmetric players. The papers by Clemhout and Wan [13], Vencatachellum [49], Bethamann [6] and Hori and Shibata [32] are some examples of the literature dealing with growing economies defined as multi-player non-cooperative dynamic games. Our specific proposal departs from a conjecture regarding the existence of optimal solutions to a class of exponentially discounted games, that lack the state-separability property, following the definitions of Dockner et al. [15]<sup>2</sup>. In this framework, players lack the relevant information to pursue their optimal strategies and have to base their decisions on subjective beliefs. Either by ignorance or choice, players in this class

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<sup>2</sup>Caputo [10] extends the notion of state-separability in a class of exponentially discounted games.

of games face the relevance paradox<sup>3</sup>. We argue that solutions to this class of games are optimal if decisions based on subjective beliefs are consistent with the existence of a stable SCE. The notion of SCE was first introduced by Fudenberg and Levine [21], to describe self-fulfilling solutions in repeated games with incomplete information. This topic has been gaining interest in modern macroeconomics theory. In a recent article, for example, Fudenberg and Levine [22] relates the concept of SCE with the famous *Lucas Critique*.

To demonstrate our conjectures, we describe in section 2 a general setup for a class of non-cooperative differential games, which have coupled open loop strategies, defined as an initial value problem<sup>4</sup>. Solutions in this class of games are optimal if player beliefs converge asymptotically to a SCE. To demonstrate the validity and implications of this hypothesis, we analyse two multi-player games, set up as endogenous growing economies, where agents seek to maximize their intertemporal consumption utility. To pursue this objective, players choose consumption and investment strategies and face a budget constraint describing the evolution of their net wealth. Player dynamics are coupled by a risk premium mechanism defined by the state of the game, while growth in this economy is driven by capital accumulation, following Romer [44] proposal. Given that this specific setup is closely related to theoretical finance frameworks, with roots in the Merton [37] investor problem, one can interpret this setup as a class of multi-player portfolio games<sup>5</sup>, if we consider productive capital as a risk free asset. In this context, players may opt between diversifying a financial portfolio composed by a risk free and a risky asset or leveraging their wealth to invest solely in the risk free asset. Another interesting interpretation is given by an economy populated by exchange rate speculators that can deposit their capital domestically and borrow capital from abroad, or invest in foreign currency and face exchange risk in foreign currency deposits.

To demonstrate our main hypotheses, we depart from a simple conjecture on the existence and qualitative analysis of solutions for dynamic games under incomplete information. We argue that if beliefs bound the state space of the game asymptotically, then it is possible to evaluate strategic equilibrium outcomes. If the belief function is known and fulfils *Lipchitz* continuity conditions, then it is possible to obtain a full qualitative description of the game dynamics and the stability of SCE can be evaluated at least locally. Equilibrium in this class of games can be fully described as a *Cauchy* boundary value problem, under this set of conditions, as long as we have knowledge of the value and gradient of the belief function, evaluated at the bounded equilibrium region. We portray this hypothesis in the game proposed in section 3. In this game, player beliefs impose a unique equilibrium. The existence of a SCE solution requires the existence of constraints on individual parameter distributions. Given a simplified hypothesis for the evolution of beliefs, assuming state independent control dynamics, we are able to give a full description of strategic dynamics and a qualitative analysis of state-space dynamics in the vicinity of the SCE solution.

When beliefs are no longer consistent with the existence of a unique solution, standard dynamical systems techniques cannot be applied. However, if beliefs guarantee that the state-space of the game is bounded asymptotically, then long run strategic dynamics can be described as a multi-objective optimization problem under uncertainty. We portray this hypothesis in section 4. In this section, we show that the game solution is defined by an individual equilibrium region in  $\mathfrak{R}^2$ , bounded by growth and transversality conditions. We propose that the asymptotic solution to this game can be posed as a static multi-objective expectation maximization problem. This is a complex geometric problem that may or may not have solutions consistent with a SCE. To check the existence of a SCE in this framework, we propose that this optimization problem can be evaluated numerically as a HMM<sup>6</sup>, where players learn a SCE in a co-evolving environment. This approach has several drawbacks. First, in incomplete information learning environments, it is unlikely that players will follow smooth strategies. The introduction of learning dynamics most probably does not allow for a rigorous mathematical description of the game dynamics and equilibrium solution. Başar and Olsder [3] describe this crucial dilemma in the following fashion.

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<sup>3</sup>In a dynamic game context the relevance paradox can be described in the following fashion. When deciding their strategies, players seek only the information that they perceive as relevant. However, this information might not be sufficient to allow players to pursue their optimal strategies. This problem can arise because players cannot access the relevant information, because they did not consider important information as crucial or cost effective. When this phenomenon dominates the decision making process, the player is trapped in a paradox. When decision makers are not aware of essential information to pursue their decisions in the best possible way, then strategic choices will lead to unintended consequences. Such consequences can be either positive or negative.

<sup>4</sup>This option of analysing a specific class of games and discuss specific applications as a long tradition in differential game theory. In a review of the first 25 years of economic applications in differential games, following the pioneering work of Isaacs [34], Clemhout and Wan [12] already suggests that the analysis of general sum multi-player differential games is limited to the study of specific applications. As the authors put it: "*A frontal assault can be promising only when a general theory of partial differential equations becomes available. So far, progress is made by solving special classes of differential games...*".

<sup>5</sup>We refer to Merton's thesis in the context of the authors earlier proposals on optimal control and economics dynamics that later influenced the modelling of open economies. Modern portfolio theory is based on the authors seminal articles, Merton [36], Merton [38] and Merton [40], which later led to the development of the famous Merton-Black-Scholes model, following Black and Scholes [8] and Merton [41].

<sup>6</sup>A HMM is a *Markov* decision process where agents lack information about the state evolution.

“Hence, a relaxation of the Lipschitz-continuity condition on the permissible strategies could make an optimal control problem quite ill-defined. In such a problem, the single player may be satisfied with smooth (but sub-optimal) strategies. In differential games, however, it is unlikely that players are willing to restrict themselves to smooth strategies voluntarily. If one player would restrict his strategy to be Lipschitz, the other player(s) may be able to exploit this. ... In conclusion, non-Lipschitz strategies cannot easily be put into a rigorous mathematical framework. On the other hand, in many games, we do not want the strategy spaces to comprise only smooth mappings.”<sup>7</sup>

Second, a consistent evaluation of game dynamics as a HMM requires the sampling and statistical analysis of numerical solutions, assuming different hypotheses. It is a well known fact that the simulation of a HMM and sampling of its results, following the *Markov Chain Monte Carlo* method, is not a computationally efficient task<sup>8</sup>. The analysis of solutions using this method poses several limitations. On the other hand, given that the existence of optimal solutions requires players to learn a SCE, all learning mechanisms considered have to fulfil this specific *Bayesian Incentive*, and solutions can be ordered according to the *Bayesian Efficiency* SCE criterion for optimality. This approach merges concepts from the fields of evolutionary multi-objective optimization, machine learning and game theory. We discuss this topic in detail in subsection 4.1, when describing the specific HMM proposed. In subsection 4.2 and subsection 4.3, we evaluate this game assuming two different learning mechanisms and portray the existence of strong emergence phenomena. In the first example, we show that agents are able to self-organize in an asymptotically robust SCE. In the second example, we show that belief-based decisions result in a stationary co-evolutionary dynamic process driven by permanent strategic interactions. The SCE attractor is now described by an invariant set that can only be evaluated statistically. We consider this outcome as a natural consequence of decision under subjective beliefs in nonlinear co-evolving environments. Individual belief-based decisions lead to unintended consequences that effectively change the environment faced by others, and as a consequence, their beliefs. This feedback loop drives the complex dynamics observed. Although this solution does not represent an optimal solution to the game proposed, we believe that this result provides a crucial link between the social sciences and decision theory paradigm of unintended consequences<sup>9</sup> and the mathematics and statistics paradigm of subjective probability<sup>10</sup>. When considered in a nonlinear co-evolutionary context, these two forces reveal, in our opinion, the evolutionary nature of decentralized competitive economic dynamics. The paradigm of subjectivity, as the driver of social and economic decisions, is a crucial concept to understand how unintended consequences arise and lead to complex evolutionary dynamics in a non-cooperative differential game framework.

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<sup>7</sup>Part II, chapter 5, section 5.3.

<sup>8</sup>The main difficulty faced when simulating and sampling a Markov process under uncertainty is related to the computational cost of performing inference in a large scale.

<sup>9</sup>The law of unintended consequences was a concept introduced by the sociologist Robert K. Merton, in his seminal essay Merton [42]. In very broad terms, Robert K. Merton defined the possible causes of the unintended consequences of purposive social action as ignorance, error, conventions and self or institutional interest. The author defines the process of social evolution as a consequence of social decision making based on subjective assumptions. Subjective decisions play a crucial role on the development of the complex inter-relations which organize society, and inevitably lead to individual, collective or institutional decisions with unintended consequences. The author describes this process in the following fashion. “*The empirical observation is incontestable: activities oriented toward certain values release processes which so react as to change the very scale of values which precipitated them. This process may in part be due to the fact that when a system of basic values enjoins certain specific actions, adherents are not concerned with the objective consequences of these actions but only with the subjective satisfaction of duty well performed. Or, action in accordance with a dominant set of values tends to be focused upon that particular value-area. But with the complex interaction which constitutes society, action ramifies, its consequences are not restricted to the specific area in which they were initially intended to center, they occur in interrelated fields explicitly ignored at the time of action. Yet it is because these fields are in fact interrelated that the further consequences in adjacent areas tend to react upon the fundamental value-system.*”.

<sup>10</sup>The paradigm of subjectivity, as a key concept in modern probability theory, is framed by the proposals of *Henri Poincaré*, *Frank Ramsey* and *Bruno De Finetti*. Although this discussion has older roots in philosophy and scientific thought, these three famous authors are fundamental for the development of key modern concepts in probability theory. For example, the formal definition of the general *Bayes* theorem, was first introduced by *Pierre-Simon Laplace*. This is usually referred to as the inductive reasoning theorem. However, it is *Henry Poincaré* that later forwarded a crucial justification for the introduction of the inverse probability reasoning concept in the field of calculus of probability, when he introduced the doctrine of conventionalism, and subjective probability theory. The crucial modern hypothesis of subjectivity can be attributed to *Bruno De Finetti* main paradigm. Probability as a measure of an event departing from an objective perspective simply does not exist. Therefore, all probabilities are subjective. *De Finetti* argues that there is always an inherent degree of uncertainty driving one’s beliefs about a given phenomenon. *Ramsey* is one of the first proponents of subjective probability, but describes this paradigm in terms of the degree of confidence or specific beliefs, one might have regarding the probability of an event. This concept is closer to *Poincaré’s* doctrine of conventionalism, since it allows for conventions to be determined and improved, assuming methods that increase the degree of confidence regarding the probability of a specific phenomenon. According to *Poincaré*, this could be achieved by proving the validity of conventions empirically. However, as *Ramsey* put it, the degree of confidence that one may have regarding the probability of a given phenomenon is also impaired by subjectivity. Therefore, there is always a degree of belief regarding any given probability measure. Gower [25] gives a detailed review, survey and discussion on this topic. The author focus the discussion on *Poincaré’s* vision and provides a detailed review of *Poincaré’s* proposals on conventions and subjectivity, which include date back to his work on non euclidean geometry and dynamical systems. Regarding *Ramsey* and *De Finetti*, their main contributions are found in Ramsey [43] and de Finetti [14], respectively.

## 2. General setup

In this section, we define the general setup for the class of differential games we wish to consider, and put forward the main conjectures for the existence of solutions driven by subjective beliefs. Consider the following general  $N$ -Player non-cooperative differential game<sup>11</sup> faced by player  $i \in N$ :

$$\begin{aligned} & \underset{u_i(t)}{\text{MAX}} \int_0^T \beta_i(t) \pi_i(u_i(t)) dt \\ & \text{subject to the solution of:} \\ & \dot{x}_i(t) = g_i(u_i(t), X(t)); \\ & x_i(0) = x_{i,0}. \end{aligned} \tag{1}$$

where:

- $N = \{1, \dots, n\}$ - Discrete set of players;
- $\beta_i(t)$ - Discount function for player  $i$ ;
- $u_i(t) = \{u_i^1, \dots, u_i^k\} \in \mathfrak{X}^k$ - Finite dimensional vector of player  $i$  controls;
- $\pi_i(u_i(t)) \in \mathfrak{X}^{k'}$ - Instantaneous pay-off for player  $i$ , where  $k' \leq k$ ;
- $x_i(t) = \{x_i^1, \dots, x_i^w\} \in \mathfrak{X}^w$ - Finite dimensional vector of player  $i$  states;
- $x_i(0) = \{x_i^1, \dots, x_i^w\} \in \mathfrak{X}^w$ - Finite dimensional vector of initial conditions on player  $i$  states
- $X(t) = [x_1(t), \dots, x_n(t)] \in \mathfrak{X}^{w'}$ - Finite dimensional vector of state variables, where  $w' \leq w$ .

We consider solutions to (1) consistent with players choosing open loop no feedback strategies,  $\eta(t) = \{X(0)\}, \forall t \in [0, T]$ , where  $\eta(t)$  is the information set available to players at period  $t$ . Players seek to maximize a concave pay-off function,  $\pi_i(u_i(t))$ . Pay-offs in this class of games are discounted at an individual constant exponential rate,  $\beta_i(t) = e^{-\rho_i t}$ , where  $\rho_i$  is player  $i$  discount rate. These conditions lead to solutions that can be defined as initial value problems, given by *Pontryagin* first order conditions. In this framework, *Pontryagin* maximum conditions are sufficient for the existence of an optimal solution to (1), provided that transversality conditions, following Arrow and Kurtz [2], are fulfilled<sup>12</sup>, thus guaranteeing that  $x_i(t)$  does not grow too fast. We assume the following conjecture regarding the existence of state-dependent optimal open loop strategies, for the game given in (1). The conjecture is that player  $i$  optimal control solutions to (1) can be defined generally in the following fashion,

$$\dot{u}_i(t) = f_i(u_i(t), X(t)). \tag{2}$$

When optimality conditions impose strategic solutions that are consistent with (2), we consider that players face the relevance paradox. To pursue their optimal strategies in this environment, players require full knowledge of the state of the game,  $\eta_i = \{X(t)\}$ , but when choosing the relevant information to pursue their strategies, either by ignorance or choice, players' knowledge about the state of the game is incomplete. As players lack crucial information to pursue their goals, they have to rely on subjective beliefs. In an open loop setup, we shall consider that belief dynamics are defined by a function,  $X_b^{(i)}$ , that depends on the information available to the player. We characterize beliefs as naive if there is no incentive to learn the true state of the game. Given that in the setup defined by (1), players only have information about the evolution of their individual state and their strategies, we define subjective beliefs in the following fashion: (i) player beliefs are independent of other player decisions,  $X_b^{(i)}(t) = v_i(u_i(t))$ ; (ii) player decisions are coupled but beliefs depend solely on individual outcomes observed,  $X_b^{(i)}(t) = v_i(u_i(t), x_i(t))$ ; and finally (iii) players can extrapolate past moments of the state of the game from individual outcomes,  $X_b^{(i)}(t) = v_i(u_i(t), x_i(t), X(t - \Delta t))$ <sup>13</sup>. Following this set of assumptions, the problem faced by player  $i$  under subjective beliefs, is generally defined by the following dynamical system:

<sup>11</sup>This specific class of games is contained in the broad general framework for exponentially discounted differential games with concave utility discussed in Dragone et al. [16].

<sup>12</sup>For this class of control problems, Arrow and Kurtz [2] have shown that first order conditions are sufficient, provided that transversality conditions, defined generally by  $\lim_{t \rightarrow \infty} e^{-\rho_i t} \Gamma_i(t) x_i(t) = 0$ , are fulfilled.  $\Gamma_i(t)$  are the co-state variables of the optimal control problem defined in (1), describing the marginal adjustment of the players control to the individual state evolution, when no feedback strategies are considered.

<sup>13</sup>In subsection 4.1 we show that this is a valid hypothesis. In this environment players observe forecasting errors and use their available information to extrapolate past outcomes of the state of the game.

$$\dot{u}_i(t) = f_i(u_i(t), X_b^{(i)}(t)); \quad (3)$$

$$\dot{x}_i(t) = g_i(u_i(t), X(t)). \quad (4)$$

To evaluate the game defined in general by (3) and (4), we require the existence of solutions consistent with asymptotic convergence to a equilibrium solution. A necessary condition for the existence of subjective belief solutions to (??), is that the following set of conditions is fulfilled,

$$\lim_{t \rightarrow \infty} X_b^{(i)}(t) = \bar{X}_b^{(i)} \wedge \lim_{t \rightarrow \infty} X = \bar{X} \wedge f_i(\bar{u}_i, \bar{X}_b^{(i)}), g_i(\bar{u}_i, \bar{X}) = 0, \quad (5)$$

where  $\bar{X}_b^{(i)}$ ,  $\bar{X}$  and  $\bar{u}_i$ , define an unique equilibrium solution that bounds the state-space of the game asymptotically. Solutions to (3) and (4) are optimal solutions to (1), if the transversality conditions are fulfilled for a equilibrium satisfying (5), and beliefs match the true state outcomes in the long run,

$$|X_b^{(i)}(t) - X(t)| \rightarrow 0 \quad \text{and} \quad f_i(\bar{u}_i, \bar{X}), g_i(\bar{u}_i, \bar{X}) = 0 \quad \text{as} \quad t \rightarrow \infty. \quad (6)$$

Condition (6) implies that optimal open loop solutions to (1), given control solutions defined by (2), and strategic decisions driven by beliefs, following (3), require the existence of a SCE solution. Given that the game solution does not provide any information regarding the evolution of beliefs, a qualitative evaluation of the SCE using standard methods requires specific assumptions about belief dynamics. If the belief function guarantees that a SCE is achieved asymptotically and (3) is smooth, then it is possible to perform a full qualitative analysis of the game defined in (1). In the next section, we portray this hypothesis assuming a naive specification for the evolution of beliefs, in a game where conditions impose a unique equilibrium solution. In section 4, beliefs are no longer consistent with a unique solution, but equilibrium, growth and transversality conditions bound the state-space of the game asymptotically. In this framework, the existence of a SCE can only be evaluated, as a learning outcome in a co-evolutionary environment. Beliefs are no longer consistent with the existence of smooth strategies. We evaluate the existence of SCE solutions numerically, as a HMM, for two different learning mechanisms.

### 3. A consumption/investment game with investment bias and coupled institutional risk premium

To demonstrate these conjectures, we consider two nonlinear extensions of the general investor problem and set it up as a non-cooperative differential game under incomplete information. In this section, we consider an economy populated by a discrete set of players,  $N = \{1, \dots, n\}$ , that seeks to maximize their intertemporal pay-offs, given by a consumption utility function,  $U_i(C_i)$ , subject to the evolution of individual net financial assets,  $B_i(t) \in \mathfrak{R}$ , describing the budget constraint of each player, and productive capital accumulation,  $K_i(t) \in \mathfrak{R}^+$ , where  $i \in N$ . In order to pursue this objective, agents choose open loop,  $\eta(t) = \{X(0)\}$ , no feedback consumption,  $C_i(t) \in \mathfrak{R}^+$  and investment strategies,  $I_i(t) \in \mathfrak{R}^+$ , and discount future consumption exponentially at a constant rate  $\rho_i \in \mathfrak{R}^+$ , in a game of infinite duration,  $\forall t \in [0, T]$  and  $T = \infty$ . Player decisions are coupled by a risk premium mechanism that depends on the overall evolution of the state of the game, defined by  $X(t) = \{B(t), K(t)\}$ , where  $B(t) = \sum_{i \in N} B_i(t)$  and  $K(t) = \sum_{i \in N} K_i(t)$ , following the proposal by Bardhan [5] on convex risk premium dynamics. The objective of each players is to maximize the flow of discounted consumption pay-offs,

$$U_i(C_i) = \int_0^T u_i(C_i(t)) e^{-\rho_i t} dt, \quad \text{with} \quad u_i(C_i(t)) = C_i(t)^{\gamma_i}, \quad (7)$$

where  $\gamma_i$  is the intertemporal substitution elasticity between consumption in any two periods, measuring the willingness to substitute consumption between different periods. We impose the usual constraint on the intertemporal substitution parameter,  $0 < \gamma_i < 1$ , such that  $u_i'(C_i(t)) > 0$ . This specification for utility belongs to the family of constant relative risk aversion (CRRA) utility functions and is widely used in economic optimization setups, where savings behaviour is crucial, such as economic growth problems. This setup also guarantees the concavity of the utility function,  $u_i''(C_i(t)) < 0$ . This is a necessary condition to define optimal solutions to our open loop differential game as an initial value problem.

Each player faces a budget constraint describing the evolution of net financial assets,  $\dot{B}_i(t)$ . We consider that players are bond buyers when  $B_i(t) < 0$  and bond sellers when  $B_i(t) > 0$ . Each player uses their financial resources to finance consumption and investment activities, and to repay interest on their outstanding bonds or reinvest in financial assets. Players have revenues arising from productive capital investments,  $r_k K_i(t)$ , where  $r_k \in \mathfrak{R}^+$  is the marginal revenue of investment in productive capital, and receive interest payments on holdings of financial assets,

if they are bond investors. Interest payments follow a convex specification defined by,  $rB_i(t)[1 + d_i B(t)/K(t)]$ , where  $r \in \mathfrak{R}^+$  stands as usual for the real market interest rate and  $d_i \in \mathfrak{R}$  is an institutional measure of risk premium, resulting from capital markets sentiments on the quality of the bonds issued by a specific player.

This assumption on risk premium is justified by bias arising from historical and psychological perceptions. A higher value of  $d_i$  means that holding player  $i$  debt bonds yields a higher risk for other investors, but investment by player  $i$  in financial assets pays a greater premium. A smaller value of  $d_i$  means that holding player  $i$  debt bonds yields a smaller risk for investors but investment by player  $i$  in financial assets pays a smaller premium. Such outcomes are reinforced if agents' financial situations match the aggregate financial situation of the economy. However, bond contract holders are rewarded with smaller interest premiums when the aggregate economy is a net seller of bond contracts, and agents selling debt contracts benefit from smaller interest premiums when the aggregate economy is a net buyer of bonds. This setup can be interpreted in terms of the degree of financial development of a given economy and on distortions arising from the complex link between microeconomic and macroeconomic outcomes. This phenomenon can also be explained by historical, political and economic factors, which bias investors' sentiments towards successful players, disregarding real economic information. It can also result from information costs, which deter investors from acquiring relevant information on the state of a specific economy and rely on individual or collective market beliefs<sup>14</sup>.

We additionally assume that agents face convex investment costs in their budget constraint, following the seminal proposal by Hayashi [28] on convex investment installation costs per unit of productive capital. In our non-cooperative differential game framework, convex investment costs are defined as,  $I_i(t)(1 + h_i I_i(t)/K_i(t))$ , where parameter  $h_i \in \mathfrak{R}$  has the following interpretation: if (i)  $h_i < 0$ , institutional conditions impose bias on investment in productive assets, if (ii)  $h_i > 0$ , institutional conditions impose bias on investment in financial assets. A thorough discussion on this topic in economic growth models can be found in Turnovsky [47]. Eicher et al. [17] are a recent example of a similar application in growth theory literature, where convex investment adjustment costs and risk premium are considered in the budget constraint. Finally, players accumulate productive capital exponentially following their strategic investment decisions,  $\dot{K}_i(t) = I_i(t) - \delta K_i(t)$ , and productive capital accumulation is subject to depreciation, which is defined by the common capital depreciation rate,  $\delta \in \mathfrak{R}^+$ . We assume that players playing open loop strategies do not commit to a common investment strategy, but their decisions will be such that they always fulfil growth,  $\liminf_{t \rightarrow \infty} I_i(t) K_i(t)^{-1} > \delta$ , and optimality conditions.

Following the description of the decision problem faced by each member of this economy, the non-cooperative game faced by player  $i \in N$ , is defined by the following dynamic optimization problem:

$$\begin{aligned} & \text{MAX}_{C_i(t), I_i(t)} \int_0^{\infty} e^{-\rho_i t} C_i(t)^{\gamma_i} dt \\ & \text{subject to the solution of:} \\ & \dot{B}_i(t) = C_i(t) + I_i(t) \left(1 + h_i \frac{I_i(t)}{K_i(t)}\right) + rB_i(t) \left(1 + d_i \frac{B(t)}{K(t)}\right) - r_k K_i(t); \\ & \dot{K}_i(t) = I_i(t) - \delta K_i(t); \end{aligned} \tag{8}$$

satisfying the transversality conditions, (A.8) and (A.9), guaranteeing that solutions to (8) do not grow too fast.

Due to its simplicity, the framework proposed in (8) can have different interpretations. These interpretations depend on the economic context we choose to consider<sup>15</sup>. This game can be interpreted as an economy populated by investors that seek to finance their intertemporal consumption, given the returns of their portfolios. Investors may finance consumption by investing their initial capital in a portfolio composed of a risk-free asset and a risky asset that is linked to other players investment decisions and market institutional conditions. This type of investor chooses to diversify their portfolios to finance present and future consumption. Otherwise, investors may choose to hold only risk-free assets in their portfolio and finance their investment and consumption decisions through the accumulation of financial debt. In this particular case, investors use risk-free asset returns to roll on existing debt contracts. Present and future accumulation of risk-free assets guarantee the sustainability of the net borrower position. Another interesting interpretation, is given by an economy populated by exchange rate speculators that can deposit their capital domestically and borrow capital from abroad, or invest in foreign currency

<sup>14</sup>Stiglitz and Weiss [46] have shown that even in cases of individual borrowing, because of informational asymmetries or problems associated with moral hazard, risk premium or credit constraints, or both, are known to exist.

<sup>15</sup>The simplicity of these proposals have their advantages and its drawbacks. The main advantages of the simplified framework proposed in this section and the next is, in our opinion, its mathematical tractability and the flexibility it allows in terms of economic interpretation. On the other hand, the extreme simplicity of these proposals does not take into account the diversity and the consequent complexity of real economic phenomena. Moreover, these setups are loosely related to neoclassical economic theory, but they do not take into account all theoretical assumptions usually required in mainstream orthodox economics. We consider that this trade-off eventually arises when one considers nonlinear economies in a differential game framework. The economic modeller must take into account this natural trade-off, and seek a reasonable balance between the mathematical tractability of the proposed problem and its economic interpretation.

and face exchange risk in foreign currency deposits. Throughout the remainder of this paper, we shall follow a twofold approach regarding the interpretation of solutions to the games discussed in section 3 and section 4. When convenient, we shall consider that these setups define growing economies populated by agents that invest in productive/domestic assets and choose to be either net borrowers or net lenders. However, the alternative interpretation as a portfolio investor game is also reasonable and provides interesting insight on some of the results arising from these specific proposals.

We now put forward the solution to the open loop case, following the necessary and sufficient first order maximum conditions given in Appendix A.1. We start by deriving the optimal *Keynes-Ramsey* consumption strategies<sup>16</sup>. The first *Keynes-Ramsey* consumption strategy is obtained as usual by taking the time derivative of (A.2) and substituting both these expressions in the co-state condition (A.4). After some manipulations we obtain,

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1} \left( \rho_i - r - rd_i \frac{B(t)}{K(t)} \right). \quad (9)$$

To obtain the second *Keynes-Ramsey* consumption strategy, we first have to take the time derivative of the optimality condition on investment decisions (A.3). We obtain the following differential equation,

$$\dot{q}_i(t) = -\dot{\lambda}_i(t) \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right) - \lambda_i(t) 2h_i \frac{\dot{I}_i(t)}{K_i(t)} + \lambda_i(t) 2h_i \frac{I_i(t)}{K_i(t)} \frac{\dot{K}_i(t)}{K_i(t)}. \quad (10)$$

Substituting (10) and optimality conditions (A.2) and (A.3) in the co-state condition defined in (A.5), where  $\dot{\lambda}(t)$  is again given by the time derivative of (A.2), we obtain the second *Keynes-Ramsey* consumption strategy<sup>17</sup>:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1} \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right)^{-1} \left[ (\rho_i + \delta) \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right) - h_i \frac{I_i(t)^2}{K_i(t)^2} - r_k - 2h_i \frac{\dot{I}_i(t)}{K_i(t)} + 2h_i \frac{I_i(t)}{K_i(t)} \frac{\dot{K}_i(t)}{K_i(t)} \right]. \quad (11)$$

Now we need to impose the optimal accumulation rule that guarantees indifference between consumption strategies for player  $i$ . Setting (9) equal to (11), and substituting the capital accumulation equation, (A.7), this rule defines strategic investment decisions<sup>18</sup>, which are given by,

$$\dot{I}_i(t) = \frac{I_i(t)^2}{2K_i(t)} + \left( r + rd_i \frac{B(t)}{K(t)} \right) I_i(t) + \left( r + rd_i \frac{B(t)}{K(t)} + \delta - r_k \right) \frac{K_i(t)}{2h_i}. \quad (12)$$

The optimal open loop solution to the game defined in (8) is thus given by consumption, (9), investment, (12), net financial assets, (A.5), and productive capital dynamics, (A.6). This system defines a solution described by a set of non-stationary variables. It is necessary to define a scaling rule consistent with the existence of stationary dynamics. We define a stationary dynamical system by taking advantage of the scaled invariance of the dynamics, and redefine the variables,  $X_{m,i}(t)$ , in terms of domestic capital units:

$$Z_{m,i}(t) = \frac{X_{m,i}(t)}{K_i(t)} \Rightarrow \dot{Z}_{m,i}(t) = \frac{\dot{X}_{m,i}(t)}{K_i(t)} - \frac{X_{m,i}(t)}{K_i(t)} \frac{\dot{K}_i(t)}{K_i(t)}, \quad (13)$$

where  $m \in \{1, 2, 4\}$  and  $X_{m,i}(t)$  defines consumption,  $C_i(t)$ , net foreign assets,  $B_i(t)$ , investment,  $I_i(t)$ , and  $Z_{m,i}(t)$  each corresponding scaled variable for each player  $i \in N$ , respectively. Following the scaling rule given in (13), we obtain the stationary dynamical system defining the general dynamic solution for the non-cooperative game given in (8), in terms of player  $i$  dynamics,

<sup>16</sup>By *Keynes-Ramsey* consumption rules, we mean the intertemporal dynamic consumption decisions that are obtained for this control variable in an optimal control problem with a constant intertemporal discount rate. In macroeconomics literature these dynamic equations are known by *Keynes-Ramsey* consumption rules, following the work by the two famous Cambridge scholars, which related intertemporal consumption decisions with the discounted value of expected future incomes and optimal savings for capital accumulation. It is our opinion that in economy optimization problems with two capital accumulation state variables, this rule is not unique, since state defined income accumulation can vary in its source. Therefore it is reasonable to impose two possible consumption paths that satisfy the optimal investment condition. In this model, optimal investment decisions impose an indifference rule on the intertemporal marginal adjustment between different assets, in order to allow for distinct capital accumulation decisions. This mechanism has the following interpretation, investors will always choose to accumulate assets that adjust faster to optimum outcomes rather than invest in assets that yield longer adjustment rates. In economics jargon the co-state variables represent the shadow price (or marginal value) of a specific asset.

<sup>17</sup>Condition (11) defines optimal consumption paths assuming income arising from the accumulation of productive capital while the condition (9) defines consumption through net assets accumulation.

<sup>18</sup>We would like to stress that this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining strategic investment dynamics is obtained when substituting directly (9) while deriving (11).

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t) \left[ \frac{\rho_i - r - rd_i Z_2(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1} \right], \quad (14)$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t) [1 + h_i Z_{4,i}(t)] + Z_{2,i}(t) [r + rd_i Z_2(t) - Z_{4,i}(t) + \delta] - r_k, \quad (15)$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + [r + rd_i Z_2(t) + \delta] Z_{4,i}(t) + [r + rd_i Z_2(t) + \delta - r_k] \frac{1}{2h_i}, \quad (16)$$

where:

$$K_i(t) = K_i(0) e^{\int_0^t (Z_{4,i}(s) - \delta) ds} \Rightarrow K(t) = \sum_{i \in N} K_i(0) e^{\int_0^t (Z_{4,i}(s) - \delta) ds}, \quad (17)$$

$$Z_2(t) = \frac{\sum_{i \in N} Z_{2,i}(t) K_i(t)}{\sum_{i \in N} K_i(t)}. \quad (18)$$

Since players follow open loop strategies, we have to assume that individual solutions to this game can only be correctly described by strategic decisions based on subjective beliefs about the state of the game. Following the general discussion in section 2, and the assumption in (3), we define individual beliefs about the evolution of  $Z_2(t)$  as  $Z_2^{b,i}(t)$ . The intuition is straightforward. When choosing their strategies, players discard the use of relevant information about the state of the economy. To pursue these strategies, players have to base their decisions on beliefs. Beliefs can be of a static, dynamic and/or stochastic nature, as long as they are consistent with the existence of a strategic equilibrium that bounds the state-space of this game asymptotically. We now consider that individual strategic dynamics are given by (19) and (20), below, instead of (14) and (16), respectively. The individual state dynamics continue to be defined by (15), while strategic dynamics for player  $i$  are now given by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t) \left[ \frac{\rho_i - r - rd_i Z_2^{b,i}(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1} \right]; \quad (19)$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + [r + rd_i Z_2^{b,i}(t) + \delta] Z_{4,i}(t) + [r + rd_i Z_2^{b,i}(t) + \delta - r_k] \frac{1}{2h_i}. \quad (20)$$

We now focus on the description of asymptotic solutions to this game. Following the conjecture on the existence of solutions to this class of games, defined by condition (5), beliefs have to be consistent with the existence of an asymptotic strategic equilibrium solution. In this specific setup, the evolution of beliefs has to be consistent with  $\lim_{t \rightarrow \infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} \wedge \dot{Z}_{1,i}(\bar{Z}_2^{b,i}) = 0$ . Setting  $\dot{Z}_{1,i}(t) = 0$ , the existence of individual equilibrium beliefs consistent with feasible asymptotic consumption outcomes,  $\bar{Z}_{1,i} \in \mathfrak{X}^+$ , requires that

$$\lim_{t \rightarrow \infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} = \frac{\rho_i - r - (\gamma_i - 1)(\bar{Z}_{4,i} - \delta)}{rd_i}, \quad (21)$$

is fulfilled. Condition (21) defines a unique belief solution about the long run evolution of the state of the game that depends on individual investment decisions and parameter asymmetries. This result has both advantages and disadvantages. If we consider that belief dynamics are given by a *Lipschitz* continuous function, consistent with  $\lim_{t \rightarrow \infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i}$ , then, given some simplifying assumptions, it is possible to perform qualitative analysis using standard dynamical systems techniques. On the other hand, the existence of an asymptotic SCE solution, as defined by (6), consistent with an optimal solution to (8), requires that we impose further constraints on this game solution, such that the existence of a unique solution fulfilling  $\lim_{t \rightarrow \infty} Z_2(t) = \bar{Z}_2^{b,1}, \dots, \bar{Z}_2^{b,n}$  is guaranteed. This set of conditions requires that individual parameter asymmetries are distributed in a unique fashion. To demonstrate this result, it is convenient to define first the remaining necessary conditions for the existence of equilibrium solution consistent with a bounded state-space for this game dynamic solution. We start by defining scaled investment equilibrium,  $\bar{Z}_{4,i}$ . Setting  $\dot{Z}_{4,i}(t) = 0$  and substituting  $\bar{Z}_2^{b,i}$  by the result in (21), we obtain:

$$\lim_{t \rightarrow \infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \frac{-\left[\rho_i + \delta\gamma_i - (\gamma_i - 1)(2h_i)^{-1}\right] \pm \sqrt{\left[\rho_i + \delta\gamma_i - (\gamma_i - 1)(2h_i)^{-1}\right]^2 - (2 - 4\gamma_i)(\rho_i + \delta\gamma_i - r_k)(2h_i)^{-1}}}{1 - 2\gamma_i}. \quad (22)$$

Now, it is convenient to redefine domestic/productive capital accumulation in the long run, as a function of long run investment decisions. Taking the asymptotic limit of expression (17), long run productive capital dynamics can be expressed as a function of (22),

$$\lim_{t \rightarrow \infty} t^{-1} \log K_i(t) = \bar{Z}_{4,i} - \delta. \quad (23)$$

Recall that strategic consumption dynamics are defined endogenously by  $Z_{4,i}(t)$  and  $Z_2^{b,i}(t)$  dynamic transitions to equilibrium. Specific assumptions regarding belief dynamics have to be taken into account in this context. We discuss this topic with further detail later in this section and focus now on the definition of state equilibrium. State dynamics are defined asymptotically in the following fashion. First, we assume that there is a unique equilibrium solution for individual state dynamics,  $\bar{Z}_{2,i}$ , obtained from solving  $\dot{Z}_{2,i} = 0$ . We then assume that aggregate state dynamics,  $\bar{Z}_2$ , is given by the asymptotic limit of (18), given the result in (23) for productive capital dynamics in the long run. In the long run the state of the game is defined by the following expression,

$$\lim_{t \rightarrow \infty} Z_2(t) = \bar{Z}_2 = \frac{\sum_{j \in L} \bar{Z}_{2,j} K_j(0)}{\sum_{j \in L} K_j(0)}, \quad (24)$$

where player  $j \in L$  corresponds to the subset of players that have scaled investment strategies consistent with  $\bar{Z}_{4,j} = \max(\bar{Z}_{4,i})$ <sup>19</sup>. This result has a straightforward interpretation. Aggregate risk premium dynamics are driven by the game investment leaders in the long run. The long run risk premium condition given in (24) defines a relative measure that takes into account the financial situation of the ensemble of leaders in this economy and weights it against their initial productive capital endowments. In the long run market forces price aggregate risk premium following the financial outcomes of the players choosing more aggressive, and therefore riskier investment strategies. Leveraged based aggressive investment strategies raise the game bond premium. While aggressive investment strategies from players with a diversified portfolio lower the game bond premium. This long run market risk premium measure can be justified by the existence of information costs that deter investors from acquiring relevant information. Under these circumstances, it is a reasonable decision to price aggregate risk based on a sample of aggressive investors and their portfolio decisions.

Following this last definition, we define individual state equilibrium,  $\bar{Z}_{2,i}$ , as the solution to  $\dot{Z}_{2,i}(t) = 0$ . Where we assume that  $\bar{Z}_1$  is a unique asymptotic outcome arising from belief and investment transitions to equilibrium. Individual state dynamics in the long run are given by:

$$\lim_{t \rightarrow \infty} Z_{2,i}(t) = \bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i} (1 + h_i \bar{Z}_{4,i})}{r + rd_i \bar{Z}_2 - \bar{Z}_{4,i} + \delta}. \quad (25)$$

The result in (25) confirms that the existence of a unique equilibrium solution for  $Z_{2,i}(t)$  requires the existence of a SCE, guaranteeing the existence of a unique solution to (24). To guarantee the existence of a SCE, we impose a specific distribution for the individual institutional risk premium parameter. This constraint can be considered, because investment equilibrium does not depend on  $d_i$ , following the result in (22). Substituting  $\bar{Z}_2^{b,i}$  by  $\bar{Z}_2$  in (21), and then solving in terms of  $d_i$ , we obtain the parameter condition guaranteeing beliefs are consistent with a unique SCE asymptotically. Individual risk premium distributions consistent with a SCE are now defined by:

$$d_i = \frac{\rho_i - r - (\gamma_i - 1)(\bar{Z}_{4,i} - \delta)}{r \bar{Z}_2}. \quad (26)$$

From the result in (26), it is straightforward to confirm that (25) no longer depends on  $\bar{Z}_2$  outcomes. Since investment equilibrium in the long run is not a function of  $d_i$ , institutional risk premium can be determined as a function of (22) and (25). The result in (26) can be interpreted as the optimal price of risk in a market with perfect information about the optimal state of the game, which sets individual institutional risk premium in accordance with the economy aggregate outcomes, player specific preferences and long run investment decisions. We are now able to define the final set of conditions for the existence of a SCE to (8). Before putting forward this set of conditions, it is convenient to determine under what conditions are the transversality constraints fulfilled, in order to guarantee that an optimal SCE solution, arising from the *Pontryagin* maximum conditions, exists. For that purpose we rearrange expressions (A.8) and (A.9) in terms of scaled variables and substitute the co-state variables by the optimality conditions, (A.2) and (A.3). The transversality conditions are now given by:

<sup>19</sup>The asymptotic limit of a sum ratio of equal exponential terms with different coefficients is given by the sum ratio of the coefficients of the fastest growing exponential terms.

$$\lim_{t \rightarrow \infty} -\gamma_i (Z_{1,i}(t) K_i(t))^{\gamma_i-1} Z_{2,i}(t) K_i(t) e^{-\rho_i t} = 0; \quad (27)$$

$$\lim_{t \rightarrow \infty} \gamma_i (Z_{1,i}(t) K_i(t))^{\gamma_i-1} (1 + 2h_i Z_{4,i}(t)) K_i(t) e^{-\rho_i t} = 0. \quad (28)$$

Recall now that we defined long run capital dynamics in (23), as a function of scaled investment equilibrium,  $\bar{Z}_{4,i}$ . Assuming that we only accept equilibrium solutions for the dynamical system defined by (15), (19) and (20), following the results in (21), (22), (25) and (26), we can rearrange the transversality conditions given in (27) and (28) in a intuitive fashion, by taking the scaled limit of the logarithm and solving the transversality constraints as an asymptotic inequality<sup>20</sup>. Conditions (27) and (28) are now given by the following inequalities:

$$\lim_{t \rightarrow \infty} t^{-1} \log \left[ -\gamma_i \bar{Z}_{1,i}^{\gamma_i-1} K_i(0)^{\gamma_i} \bar{Z}_{2,i} e^{[(\gamma_i-1)(\bar{Z}_{4,i}-\delta)+\bar{Z}_{4,i}-\delta-\rho_i]t} \right] < 0; \quad (29)$$

$$\lim_{t \rightarrow \infty} t^{-1} \log \left[ \gamma_i \bar{Z}_{1,i}^{\gamma_i-1} K_i(0)^{\gamma_i} (1 + 2h_i \bar{Z}_{4,i}) e^{[(\gamma_i-1)(\bar{Z}_{4,i}-\delta)+\bar{Z}_{4,i}-\delta-\rho_i]t} \right] < 0. \quad (30)$$

From (29) or (30) it is straightforward to obtain the transversality constraint for the existence of an optimal solution as a function of the equilibrium condition describing individual scaled investment long run decisions. Given the long run growth restriction,  $\bar{Z}_{4,i} > \delta$ , the optimal growth constraint for the problem defined in (8) is given by:

$$\delta < \bar{Z}_4 < \delta + \frac{\rho_i}{\gamma_i}. \quad (31)$$

We now focus on the description of strategic dynamics, assuming that beliefs do not depend on the state evolution. To simplify our approach, we assume that beliefs are a function of investment strategies,  $Z_2^{b,i}(t) = S_i(Z_{4,i}(t))$ , and fulfil the equilibrium condition defined in (21), such that  $S_i(\bar{Z}_{4,i}) = \bar{Z}_2^{b,i}$ . This naive hypothesis regarding players' beliefs has both advantages and disadvantages. The main advantage is that it allows for a great deal of simplification and a full description of strategic dynamics. This approach paves the way for a full qualitative analysis of this game in the vicinity of a SCE. However, this analysis is based on a weak asymptotic argument. We can only guarantee the validity of qualitative outcomes when we assume that  $Z_2(t)$  non-autonomous transitions, arising from  $Z_{4,i}(t)$  dynamics, stay in the vicinity of the SCE solution. We start by redefining consumption dynamics. Given the general hypothesis on the evolution of beliefs, we can redefine  $Z_{1,i}(t)$  as a function of  $Z_{4,i}(t)$ :

$$Z_{1,i}(t) = Z_{1,i}(0) e^{\int_0^t (S_i(\bar{Z}_{4,i}) - S_i(Z_{4,i}(s))) ds}. \quad (32)$$

Now recall that the existence of a SCE solution, requires the existence of a stable equilibrium for  $Z_{4,i}(t)$ . Substituting  $Z_2^{b,i}(t)$  by  $S_i(Z_{4,i}(t))$  in (20), the general condition for stability of  $Z_{4,i}(t)$  solutions can be defined as,

$$\bar{Z}_{4,i} > \frac{\rho_i + \delta \gamma_i}{\gamma_i - rd_i S_i'(\bar{Z}_{4,i})} + \frac{rd_i S_i'(\bar{Z}_{4,i})}{2h_i [\gamma_i - rd_i S_i'(\bar{Z}_{4,i})]}. \quad (33)$$

The result in (33) has several implications. When we consider  $S_i'(\bar{Z}_{4,i}(t)) = 0$ , the stability condition reduces to  $\bar{Z}_{4,i} > \delta + \rho_i/\gamma_i$ . This result is not consistent with the constraint required for the existence of an optimal solution. Beliefs cannot be static in this setup. We consider then a simple hypothesis and define belief dynamics as,

$$Z_2^{b,i}(t) = \frac{\rho_i - r - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{rd_i}. \quad (34)$$

Following (34), player consumption dynamics reduce to  $Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$ . This result has the following economic interpretation. Players choose beliefs such that their consumption outcome relative to their wealth, measured by productive assets accumulation, is stable throughout the duration of the game. This result is in accordance with the *Life Cycle* hypothesis for intertemporal consumption<sup>21</sup>. Given the uncertainty regarding

<sup>20</sup>Recall that a dynamic process that scales exponentially,  $w(t) \sim e^{\psi t}$ , can be defined asymptotically in the following fashion,  $\lim_{t \rightarrow \infty} t^{-1} \log w(t) = \psi$ . If  $\psi > 0 \Rightarrow w(t) \rightarrow \infty$ . If  $\psi < 0 \Rightarrow w(t) \rightarrow 0$ .

<sup>21</sup>The *Life Cycle* consumption hypothesis was forwarded by Brumberg and Modigliani [9] and Friedman [20]. This theory suggests that individuals make saving and consumption decisions, in order to maintain a stable consumption pattern throughout their lives. Evidence suggests that the *Life Cycle* hypothesis is not consistent with saving and consumption patterns observed in older members of the population. Older generations show patterns of precautionary saving that can be explained by: (i) intergenerational altruism; (ii) increased caution in spending; and (iii) poor retirement planning based on optimistic assumptions about life expectancy.

state outcomes, players rather choose consumption profiles that are not distorted by investment transitions to equilibrium. Substituting (34) in (20) we can redefine investment decisions as,

$$\dot{Z}_{4,i}(t) = Z_{4,i}(t)^2 + \frac{b}{a}Z_{4,i}(t) + \frac{c}{a}, \text{ where } a = \frac{1}{2} - \gamma_i, \quad b = \rho_i + \delta\gamma_i - \frac{\gamma_i - 1}{2h_i} \text{ and } c = \frac{\rho_i + \delta\gamma_i - r_k}{2h_i}. \quad (35)$$

Equation (35) defines a *Ricatti* equation that has an explicit solution. Before describing the solution to the above differential equation, it is convenient to investigate the properties of its coefficients, given the optimality and stability constraints discussed in (31) and (33). First, recall that stable optimal solutions to (35) require that  $\delta < \bar{Z}_{4,i} < -b(2a)^{-1} < \delta + \rho_i\gamma_i^{-1}$ . This condition implies that  $\text{sign}(b) \neq \text{sign}(a)$ , for solutions consistent with  $\bar{Z}_{4,i} > 0$ . Second, feasible solutions to (35),  $\bar{Z}_{4,i} \in \mathfrak{R}^+$ , require that  $b^2 > 4ac$ . Given these definitions, the solution to (35), is given by (36), below, following the solution to the general *Ricatti* equation described in Appendix B. Investment strategic dynamics are defined explicitly as,

$$Z_{4,i}(t) = -\frac{\sqrt{b^2 - 4ac} \tanh\left\{\frac{\sqrt{b^2 - 4ac}}{2a}\left[t + \frac{2a}{\sqrt{b^2 - 4ac}} \operatorname{arctanh}\left(-\frac{2aZ_{4,i}(0)+b}{\sqrt{b^2 - 4ac}}\right)\right]\right\} + b}{2a}. \quad (36)$$

We now conclude the description of the conditions for the existence of a SCE solution to (8), with the definition of equilibrium for  $Z_{2,i}(t)$ . Substituting (26) and  $\bar{Z}_{1,i} = Z_{1,i}(0)$  in (25),  $\bar{Z}_{2,i}$  is given by the following expression,

$$\bar{Z}_{2,i} = \frac{r_k - Z_{1,i}(0) - \bar{Z}_{4,i}(1 + h_i\bar{Z}_{4,i})}{\rho_i - \gamma_i(\bar{Z}_{4,i} - \delta)}. \quad (37)$$

Having described the conditions for the existence of SCE solutions consistent with a stable strategic equilibrium, for the non-cooperative game given in (8). We now focus on the qualitative description of this solution. We base our approach on a weak argument for asymptotic stability. This argument is based on the results described in (21) to (37), which guarantee that a SCE is always achieved asymptotically and independent of other players decisions, when institutional risk premium is defined by a unique distribution that depends on the asymptotic outcome of the state of the game. Since in the long run there are no longer transitions driven by  $Z_{1,i}(t)$  and  $Z_2(t)$  dynamics, when we assume  $Z_{4,i}(t)$  dynamics always converges to the equilibrium defined in (22), we can evaluate qualitatively the local stability of the SCE strategies by testing the stability of the system describing scaled net assets dynamics,  $\{\dot{Z}_{2,1}(t), \dots, \dot{Z}_{2,n}(t)\}$ , following the individual state dynamic condition defined in (15). We start by defining the  $n$  by  $n$  *Jacobian* matrix describing individual state dynamics in the vicinity of SCE,

$$J = \begin{bmatrix} \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,2}(t)} & \dots & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,n}(t)} \\ \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,2}(t)} & \dots & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,n}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,2}(t)} & \dots & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,n}(t)} \end{bmatrix}_{Z_{2,i}(t)=\bar{Z}_{2,i}}, \quad (38)$$

where the partial derivatives of this *Jacobian* are given generally by the following expressions:

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} = r - Z_{4,i}(t) + \delta + rd_i \left[ Z_2(t) + Z_{2,i}(t) \frac{K_i(t)}{\sum_{i \in N} K_i(t)} \right]; \quad (39)$$

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)} = rd_i Z_{2,i}(t) \frac{K_w(t)}{\sum_{i \in N} K_i(t)}, \quad w \neq i \wedge w, i \in N. \quad (40)$$

To evaluate these derivatives in equilibrium, we have to distinguish between investment and non investment leaders. If players  $i, w \in N$  are investment leaders,  $i, w \in L$ , then (39) and (40) evaluated in equilibrium come,

$$\left. \frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} \right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + rd_i \left[ \bar{Z}_2 + \bar{Z}_{2,i} \frac{K_i(0)}{\sum_{j \in L} K_j(0)} \right], \quad i \in L, \quad (41)$$

$$\left. \frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)} \right|_{Z_i(t)=\bar{Z}_i} = rd_i \bar{Z}_{2,i} \frac{K_w(0)}{\sum_{j \in L} K_j(0)}, \quad w \neq i \wedge w \in L. \quad (42)$$

If players  $i, w \in N$  are not investment leaders,  $i, w \notin L$ , then (40) vanishes and (39) reduces to,

$$\left. \frac{\partial \bar{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} \right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + rd_i \bar{Z}_2. \quad (43)$$

The local stability of SCE solutions for the game defined in (8) can be easily evaluated numerically. If all the eigenvalue solutions of (38) have negative real part, then we can state that SCE solutions are at least locally weakly asymptotically stable. A robust argument for local asymptotic stability would have to take into account transitions to equilibrium arising from  $Z_{4,i}(t)$  decisions and  $Z_2(t)$  non-autonomous dynamics. The result in (36) can be used to test the robustness of numerical results. We discard this analysis in this paper and focus on the evaluation of (38). To evaluate the dynamics in the vicinity of equilibrium for this game, we assume a simple hypothesis for initial consumption and productive/riskless asset endowments. We shall consider that  $Z_{1,i}(0)$  is given by random outcomes distributed according to  $Z_{1,i}(0) \sim U(0, 1)$ , while  $K_i(0)$  is given by random outcomes distributed according to  $K_i(0) \sim \exp(1)$ . To test the qualitative dynamics in the vicinity of a SCE, we have to consider the existence of a robust population. By robust population, we mean a discrete set of  $n = 1000$  agents with uniform randomly drawn individual characteristics,  $\rho_i, \gamma_i \sim U(0, 1)$  and  $h_i \sim U(-10, 10)$ , such that  $\bar{Z}_{4,i}$  outcomes, defined by (22), fulfil the optimal growth constraint, (31), for the range institutional scenarios,  $r_k \in [0.01, 0.25]$  and  $r \in [0.03, 0.5]$ , where  $\delta = 0.03$ <sup>22</sup>. For simplification reasons, we consider that the state of the game is driven by a fixed pool of investors, which is defined by a fixed share of the population. We set this share at 30%, and consider that the aggregate risk faced by investors, (24), is obtained from the share of aggressive players with higher rates of investment per unit of capital. This assumption is consistent with risk setting in real markets. The LIBOR spread for example, is defined by a similar institutional mechanism, where only the average interest rate on credit transactions faced by a pool of the large financial institutions is considered.

In Figure 1, below, we portray the parameter distributions describing the computed robust population. In this setup, robust populations are characterized by a large set of impatient investors with high intertemporal elasticity and bias towards domestic/productive assets. This distribution of characteristics is consistent with the existence of an unique stable equilibrium for  $\bar{Z}_{4,i}$ , given by the positive root of (22). Further numerical analyses, assuming different hypothesis, suggests that the distributions portrayed, provide a good picture of a robust population, given the set of conditions required for the existence of stable strategic solutions and a SCE.

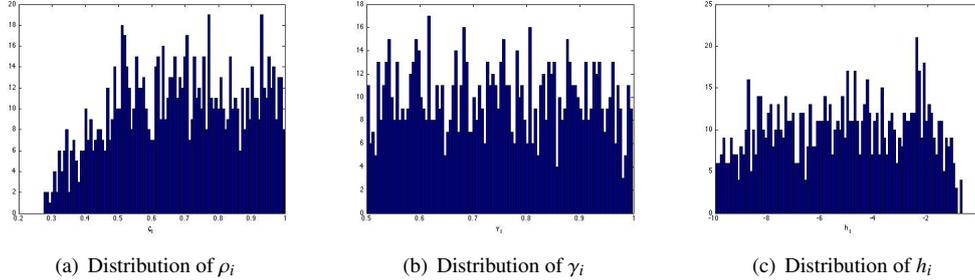


Figure 1: Robust population distributions.

Figure 2, below, portrays the qualitative results for the computed SCE solution, the sample density of  $\bar{Z}_2$  and the mean and standard deviation for  $d_i$  distribution. The stability diagram shows that SCE solutions are repelling for all the range of institutional scenarios considered<sup>23</sup>. Our numerical analysis suggests the existence of saddle solutions, when different scenarios are considered. However, these solutions have very few stable dimensions. This outcome is robust to different distributions of  $Z_{1,i}(0)$  and  $K_i(0)$ , and also to different hypothesis regarding the share size of the population of leaders. We conclude that the optimal SCE solution proposed, based on the naive evolution of beliefs following (34), is inherently an unstable game solution for (8).

We attribute this dramatic result to the existence of SCE solutions that impose a negative institutional risk premium on the majority of players. This outcome implies that investors in financial assets get a negative real yield on their investment, since the game premium is always negative<sup>24</sup>. Moreover, leveraged players do not benefit from

<sup>22</sup>Given that there are two possible solutions to  $\bar{Z}_{4,i}$ , following (22), the routine tests the robustness of each solution. When both equilibrium are robust, eigenvalue solutions to (38) are computed by choosing randomly one of the solutions. Our extensive numerical analysis suggests that for a given set of parameters only one of the roots of (22) is consistent with (31).

<sup>23</sup>In these simulations we only evaluate the real part of the two leading eigenvalues,  $\min\{\text{Re}(\Lambda)\}$  and  $\max\{\text{Re}(\Lambda)\}$ , where  $\Lambda$  stands for the set of eigenvalues solving the characteristic polynomial of (38) in the vicinity of SCE. By unstable solutions we mean that the SCE is a completely unstable solution, which implies that the SCE solution is time-reverse stable.

<sup>24</sup>It is possible that some bond investors get a negative nominal yield, which means that they pay interest on their investments.

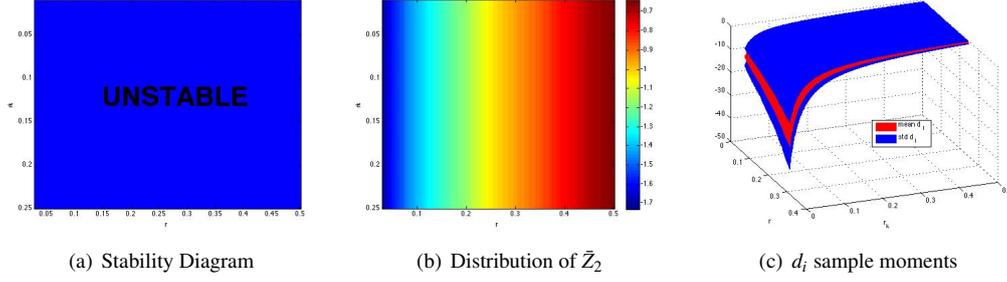


Figure 2: Stability diagram and risk premium distributions.

this arrangement either, since this setup imposes a positive premium on their debt. This SCE arrangement does not benefit any type of investor. In our opinion, this result is a consequence of players' belief rigidity and rationality. An optimal unique SCE solution to (8) can only be imposed when we consider the existence of a credit market with complete information that is able to price individual risk optimally by taking into account player's asymmetries. This is a result of players' belief rigidity. However, players do not concur on this solution, because they believe they will be better off pursuing other objectives, even though only the SCE solution described is consistent with an optimal solution. For agents facing uncertainty, this decision can be considered as a rational one. Finally, other hypotheses regarding the evolution of beliefs could be considered, such as naive beliefs that at least allow for consumption transitions. Given that our numerical results suggest that SCE solutions are unstable for a wide range of consumption distributions, a qualitative analysis based on the two step approach proposed would not be able to take into account the impact of this transitions on the stability of a SCE. Therefore, we conclude that the existence of stable SCE solutions for (8) requires that we take into account the role of the controls and beliefs gradients evaluated in the vicinity of a SCE. An interesting exercise would involve the description of the class of smooth belief functions that is consistent with the existence of stable SCE institutional scenarios. Alternatively, we could drop all naive assumptions regarding the evolution of beliefs and evaluate the existence of learning mechanisms consistent a SCE. We discuss this approach in the next section for a game where players' risk premium dynamics are given by the evolution of his state conditions.

#### 4. A consumption/investment game with investment bias and coupled endogenous risk premium

To introduce strategic interactions in our framework, we propose an extension to the game discussed in the previous section. We now drop the institutional risk premium hypothesis and consider that individual risk premium depends on the ratio of net financial assets to productive capital. Interest payments/revenues are now given by  $rB_i(t) [1 + (B_i(t)/K_i(t))(B(t)/K(t))]$ . In this setup, institutional conditions driving risk premium no longer depend on market driven beliefs, but on information regarding the player financial balances. The inclusion of an additional nonlinearity in the risk premium mechanism allows for the introduction of several novel features in this economy. Bond buyers are now rewarded when the aggregate economy is a net buyer of bond contracts, and penalized when the economy is a net issuer of bonds. Bond issuers benefit from smaller interest premiums, when the aggregate economy is a net buyer of bonds and are penalized if the aggregate economy is a net issuer of bonds. Given this very brief introduction, the non-cooperative differential game faced by player  $i$  is given by the following dynamic optimization problem:

$$\begin{aligned}
 & \text{MAX}_{C_i(t), I_i(t)} \int_0^{\infty} e^{-\rho_i t} C_i(t)^{\gamma_i} dt \\
 & \text{subject to the solution of:} \\
 & \dot{B}_i(t) = C_i(t) + I_i(t) \left(1 + h_i \frac{I_i(t)}{K_i(t)}\right) + rB_i(t) \left(1 + \frac{B_i(t)}{K_i(t)} \frac{B(t)}{K(t)}\right) - r_k K_i(t); \\
 & \dot{K}_i(t) = I_i(t) - \delta K_i(t)
 \end{aligned} \tag{44}$$

satisfying the transversality conditions, (A.18) and (A.19), guaranteeing that solutions to (44) do not grow too fast. The optimal *Keynes-Ramsey* consumption strategies for (44), following the procedure described in section 3, and given the maximum conditions in Appendix A.2, are now defined by:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1} \left( \rho_i - r - 2r \frac{B_i(t)}{K_i(t)} \frac{B(t)}{K(t)} \right); \tag{45}$$

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1} \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right)^{-1} \left[ (\rho_i + \delta) \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right) - h_i \frac{I_i(t)^2}{K_i(t)^2} - r_k - 2h_i \frac{\dot{I}_i(t)}{K_i(t)} + 2h_i \frac{I_i(t)}{K_i(t)} \frac{\dot{K}_i(t)}{K_i(t)} - r \frac{B_i(t)^2}{K_i(t)^2} \frac{B(t)}{K(t)} \right]. \quad (46)$$

We obtain the optimal rule that guarantees indifference between consumption strategies for player  $i$ , by imposing equality between consumption strategies, (45) and (46), and substituting by the capital accumulation equation (A.17). Again, this rule is given by the dynamics of strategic investment decisions,

$$\dot{I}_i(t) = \frac{I_i(t)^2}{2K_i(t)} + \left( r + 2r \frac{B_i(t)}{K_i(t)} \frac{B(t)}{K(t)} \right) I_i(t) - \frac{r}{2h_i} \frac{B_i(t)^2}{K_i(t)} \frac{B(t)}{K(t)} + \left( r + 2r \frac{B_i(t)}{K_i(t)} \frac{B(t)}{K(t)} + \delta - r_k \right) \frac{K_i(t)}{2h_i}. \quad (47)$$

Following the scaling rule given in (13), we obtain the stationary dynamical system defining the general solution for the non cooperative game given in (44), in terms of player  $i$  dynamics:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t) \left[ \frac{\rho_i - r - 2rZ_{2,i}(t)Z_2(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1} \right]; \quad (48)$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t) [1 + h_i Z_{4,i}(t)] + Z_{2,i}(t) [r + rZ_{2,i}(t)Z_2(t) - Z_{4,i}(t) + \delta] - r_k; \quad (49)$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + [r + 2rZ_{2,i}(t)Z_2(t) + \delta] Z_{4,i}(t) - \frac{r}{2h_i} Z_{2,i}(t)^2 Z_2(t) + [r + 2rZ_{2,i}(t)Z_2(t) + \delta - r_k] \frac{1}{2h_i}; \quad (50)$$

where productive capital dynamics,  $K_i(t)$ , and aggregate state dynamics,  $Z_2(t)$ , are given by (17) and (18), respectively. A quick glance at the strategic controls of player  $i$ , defined by (48) and (50), shows that players are again trapped in the relevance paradox. The pursuit of optimal strategies requires information that is not available to the players. We follow the same set of assumptions regarding the existence of feasible strategic solutions based on beliefs, given in section 3. This involves assuming the existence of individual beliefs,  $Z_2^{b,i}(t)$ , as previously. Substituting  $Z_2^{b,i}(t)$  in (48) and (50), we obtain the belief-based controls for player  $i$ . The dynamical solution to (44) is now given by the evolution of individual state dynamics, (49), while strategic dynamics are defined by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t) \left[ \frac{\rho_i - r - 2rZ_{2,i}(t)Z_2^{b,i}(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1} \right]; \quad (51)$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + [r + 2rZ_{2,i}(t)Z_2^{b,i}(t) + \delta] Z_{4,i}(t) - \frac{r}{2h_i} Z_{2,i}(t)^2 Z_2^{b,i}(t) + [r + 2rZ_{2,i}(t)Z_2^{b,i}(t) + \delta - r_k] \frac{1}{2h_i}. \quad (52)$$

Following the discussion in section 2, we have to guarantee that the dynamic solution to (44) is consistent with the existence of belief solutions that bound the state-space of the game asymptotically. First, we check under what conditions beliefs are consistent with the existence of asymptotic strategic equilibrium. This assumption requires that the set of conditions,  $\lim_{t \rightarrow \infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} \wedge \dot{Z}_{1,i}(t), \dot{Z}_{4,i}(t) = 0$ , is fulfilled. From (51) and (52) we obtain:

$$\bar{Z}_2^{b,i} = \frac{\rho_i - r - (\gamma_i - 1)(\bar{Z}_{4,i} - \delta)}{2r\bar{Z}_{2,i}}; \quad (53)$$

$$\bar{Z}_2^{b,i} = \frac{\frac{\bar{Z}_{4,i}^2}{2} - (r + \delta)\bar{Z}_{4,i} - (r + \delta - r_k)(2h_i)^{-1}}{2r\bar{Z}_{2,i}\bar{Z}_{4,i} - r(2h_i)^{-1}\bar{Z}_{2,i}^2 + r(h_i)^{-1}\bar{Z}_{2,i}}. \quad (54)$$

From the results in (53) and (54) we will show that belief solutions consistent with strategic equilibrium are no longer unique, but given by a continuous curve of equilibrium solutions. This result has dramatic implications. However, as we demonstrate later on, it has a straightforward interpretation in the context of a portfolio game. To justify our claim we start by eliminating  $\bar{Z}_2^{b,i}$  from (53) and (54). After rearranging we obtain,

$$\left( \frac{1}{2} - \gamma_i \right) \bar{Z}_{4,i}^2 + \left( \frac{\gamma_i - 1}{4h_i} \right) \bar{Z}_{2,i} \bar{Z}_{4,i} + \left[ (\gamma_i - 1) \left( \delta - \frac{1}{2h_i} \right) + \rho_i + \delta \right] \bar{Z}_{4,i} + \left( \frac{r - \rho_i - (\gamma_i - 1)\delta}{4h_i} \right) \bar{Z}_{2,i} + \frac{\gamma_i \delta + \rho_i - r_k}{2h_i} = 0. \quad (55)$$

Equilibrium condition (55) describes a quadratic curve relating individual investment decisions and financial outcomes. It defines an asymptotic frontier solution, where player beliefs are always fulfilled. This frontier is always defined by a hyperbola<sup>25</sup>, since  $[(\gamma_i - 1) / (4h_i)]^2 > 0$ . The frontier defined in (55) is defined by the right hand side of a hyperbola for feasible values of investment strategies,  $\bar{Z}_{4,i} \in \mathfrak{R}^+$ . We show this result numerically later on. First, it is convenient to rearrange (55) as an equilibrium solution for  $\bar{Z}_{2,i}$  in terms of  $\bar{Z}_{4,i}$ . Rearranging (55) we obtain,

$$\bar{Z}_{2,i} = -4h_i \frac{\left(\frac{1}{2} - \gamma_i\right) \left(\bar{Z}_{4,i}\right)^2 + \left[(\gamma_i - 1) \left(\delta - \frac{1}{2h_i}\right) + \rho_i + \delta\right] \bar{Z}_{4,i} + (\gamma_i \delta + \rho_i - r_k) (2h_i)^{-1}}{(\gamma_i - 1) \bar{Z}_{4,i} + r - \rho_i - (\gamma_i - 1) \delta}. \quad (56)$$

In this setup, beliefs provide only information about the game asymptotic frontier. We do not have any information about the value of beliefs in equilibrium. Under these circumstances, the definition of naive assumptions regarding the evolution of beliefs is no longer an option. The analysis of equilibrium and the evaluation of SCE solutions in this framework is a complex geometric problem. We propose to analyse this solution as a stochastic decision process under uncertainty. To illustrate our approach, we start by defining equilibrium for consumption strategies,  $\bar{Z}_{1,i}$ . Setting  $\dot{Z}_{2,i} = 0$  and solving for  $\bar{Z}_{1,i}$ , we obtain,

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i} \left(1 + h_i \bar{Z}_{4,i}\right) - \bar{Z}_{2,i} \left(r + r \bar{Z}_{2,i} \bar{Z}_{2,i} - \bar{Z}_{4,i} + \delta\right), \quad (57)$$

where  $\bar{Z}_2$  is again defined by the investment leaders' actions, following the result in (24). From (57), it is clear that  $\bar{Z}_{1,i}$  outcomes depend solely on investment decisions, given the result in (56). Since the state of the game,  $\bar{Z}_2$  is also a function of  $\bar{Z}_{4,i}$  outcomes, we have to check under what conditions  $\bar{Z}_{4,i}$  is bounded, to bound the state-space of the game asymptotically. In section 3 we showed that investment decisions are bounded by growth and transversality conditions. The interval defined in (31) is again a necessary condition for the existence of optimal solutions to the game defined in (44). However, as solutions might not be unique, it is convenient to redefine the notion of acceptable solutions to the system given by (49), (51) and (52), in the following fashion. We assume now, that solutions can be described as an ergodic invariant set<sup>26</sup> with a well defined invariant probability measure, whose expectation operator we denote by  $\langle \cdot \rangle$ , such that  $\lim_{t \rightarrow \infty} t^{-1} \int_0^t Z_{m,i}(t) \rightarrow \langle Z_{m,i} \rangle$ . If this invariant set is unique,  $\langle Z_{m,i} \rangle = \bar{Z}_{m,i}$ , then solutions are consistent with a SCE when players concur in a unique outcome for  $\bar{Z}_2$ . Following this definition, we can redefine  $K_i(t)$  in the long run, by taking the asymptotic limit of (17). Long run productive/domestic capital dynamics are defined by

$$\lim_{t \rightarrow \infty} t^{-1} \log K_i(t) \rightarrow \langle Z_{4,i} \rangle - \delta. \quad (58)$$

Following the result in (58), the constraint on scaled investment activities that imposes the existence of long run dynamics consistent with exponential growth,  $\lim_{t \rightarrow \infty} K_i(t) \rightarrow \infty$ , is given by  $\langle Z_{4,i} \rangle > \delta$ . To obtain the transversality constraint, we follow the same procedure described in section 3, given in (27) to (30), but taking into account solutions described by the invariant probability measure of an ergodic invariant set,  $\langle Z_{m,i} \rangle$ , with well defined lower and upper bounds given by  $\liminf_{t \rightarrow \infty} t^{-1} \int_0^t Z_{m,i}(t) \geq \min(\bar{Z}_{m,i})$  and  $\limsup_{t \rightarrow \infty} t^{-1} \int_0^t Z_{m,i}(t) \leq \max(\bar{Z}_{m,i})$ , respectively. The growth and transversality constraint is now given in terms of the invariant probability measure for investment decisions,

$$\delta < \langle Z_{4,i} \rangle < \delta + \frac{\rho}{\gamma_i}, \text{ where } \min(\bar{Z}_{4,i}) > \delta \text{ and } \max(\bar{Z}_{4,i}) < \delta + \frac{\rho}{\gamma_i}. \quad (59)$$

The solution to (44) assuming the existence of subjective beliefs, is given by an asymptotic frontier bounded by growth and transversality conditions. It is no longer possible to evaluate this game using qualitative dynamical methods. It is possible to provide a geometric description of the possible individual game outcomes and put forward some conjectures regarding feasible solution scenarios. To portray this feature, we compute the equilibrium outcomes for a feasible set of investment strategies, as defined in (59), given a set of parameters values,  $\{r, r_k, \delta, \rho_i, \gamma_i, h_i\} = \{0.02, 0.1, 0.03, 0.05, 0.3, -0.01\}$ , and assuming a random sample for state outcomes,

<sup>25</sup>This outcome can be related to Merton [39] proposal on the efficient portfolio frontier, which has been one of the main workhorses of modern financial and portfolio decision theory. The efficient portfolio theory suggests that the relationship between the expected value and the standard deviation of a given portfolio is correctly described by the right hand side of an hyperbola.

<sup>26</sup>By invariant set we refer to solutions of the scaled dynamical system that can be defined as distributions obtained from bounded trajectories of  $\lim_{t \rightarrow \infty} Z_{m,i}(t)$  in the phase space. We follow the definition of an invariant set composed by asymptotic limit sets of points given in Guckenheimer and Holmes [27]. Let  $\phi_t$  be a flow such that the  $\alpha$  limit set of  $x$  for  $\phi_t$  is the set of accumulation points of  $\phi_t(x)$ ,  $t \rightarrow -\infty$ . The  $\omega$  limit set of  $x$  for  $\phi_t$  is the set of accumulation points of  $\phi_t(x)$ ,  $t \rightarrow \infty$ . The  $\alpha$  and  $\omega$  limits of  $x$  are its asymptotic limit sets.

$\bar{Z}_2 \sim N(\alpha, 1)$ , where  $\alpha \sim N(0, 0.1)$  is also a random number. Results are given in Figure 3 below, for a sample with one thousand random draws. The picture on the left portrays individual state outcomes for the range of investment strategies, which are given by the right hand side of the hyperbola defined in (55). The picture on the right portrays the expected values and standard deviations for consumption outcomes.

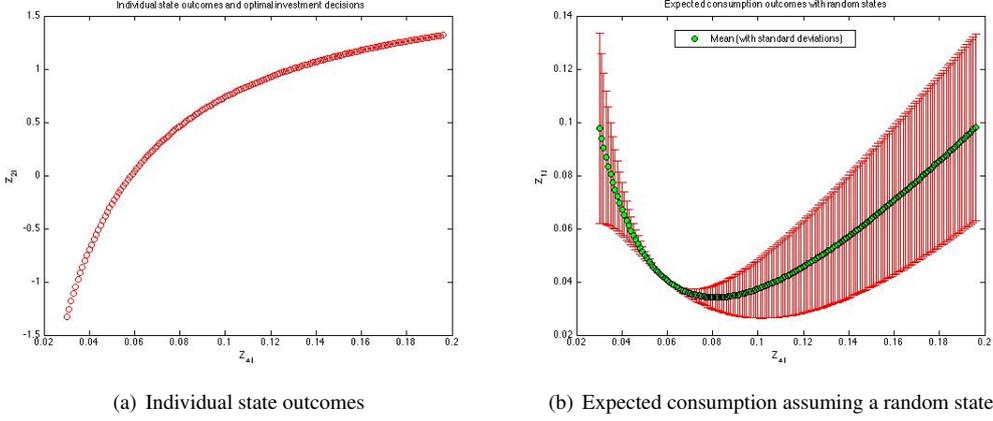


Figure 3: Individual state and consumption outcomes assuming a random state

Figure 3 portrays the indecision faced by players in an environment with incomplete information. The distribution of consumption outcomes suggests that this player may be undecided between following conservative or aggressive investment strategies. The explanation for this result is straightforward. Condition (57) defines a 4<sup>th</sup> order polynomial in terms of  $\bar{Z}_{4,i}$ . Given different parameter and  $\bar{Z}_2$  sample outcomes, numerical simulations suggest that we may have one global maximum, interior or on the edges, or two local maxima, most likely on the edges. The geometric definition of a maximum for player  $i$  depends on his beliefs about the state of the game, which in turn depends on other players' beliefs. Further, if we consider that players beliefs evolve, as they try to learn state outcomes, their actions lead to changes in the environment faced by other players. Under these circumstances a geometric approach is not an option. Game asymptotic solutions are, in our opinion, best described as a multi-objective optimization problem, where the existence of multiple decision criteria in a co-evolutionary framework has to be taken into account. We are interested in evaluating two possible game outcomes. If game dynamics, given a set of rules, are consistent with strategic coordination, leading to an unique asymptotic equilibrium solution, then we can consider that this solution is consistent with a SCE. We argue that this portfolio game between asymmetric investors has solutions consistent with the *return to the dollar* property<sup>27</sup>. Given the existence of strategic interactions and uncertainty driving the evolution of consumption outcomes, it is also likely that under certain conditions this property is not fulfilled. To test these hypotheses, we propose to evaluate this game asymptotic solution as a HMM.

#### 4.1. Asymptotic dynamics under uncertainty

Following the discussion in the previous section, we propose that the non-cooperative game defined in (44) can be evaluated as a multi-objective maximization expectation problem. Given the general asymptotic solution, defined generally in (56) to (59), player  $i$ 's objective is now correctly described by the following optimization problem:

$$\text{MAX}_{\bar{Z}_{4,i}} E \left[ \bar{Z}_{1,i} \right], \quad (60)$$

subject to (56) and (59). In this framework, players seek to maximize the expected consumption outcomes under uncertainty,  $E \left[ \bar{Z}_{1,i} \right] = \bar{Z}_{1,i}(\bar{Z}_{2,i}, \bar{Z}_{2,i}, E \left[ \bar{Z}_2^i \right])$ , defined by equilibrium condition (57), assuming individual expectations about state outcomes,  $\bar{Z}_2 = E \left[ \bar{Z}_2^i \right]$ , for the optimal range of investment strategies, defined in (59).

To evaluate the decision problem defined in (60), we propose to model it as a HMM. For that purpose we shall consider a multi-criteria decision process, where players co-evolve in a stochastic learning environment.

<sup>27</sup>This definition was introduced in the famous paper by Georgescu-Roegen [24], to describe the robustness of equilibrium in many agent systems with decentralized strategic decisions. The author coined this expression to describe the asymptotic dynamics on the production-possibility frontier, which he characterized in some cases as lacking the *return to the dollar* property or of being of the saddle type.

The application of stochastic processes to evaluate multi-objective optimization problems, is an inter-disciplinary approach that merges concepts of evolutionary optimization, machine learning and non-cooperative game theory. This approach focuses on the development of evolutionary algorithms capable of evaluating outcomes in complex interacting environments<sup>28</sup>. In a review on modern evolutionary multi-objective optimization methods, Zitzler et al. [51] summarizes this approach in the following fashion. "*The term evolutionary algorithm (EA) stands for a class of stochastic optimization methods that simulate the process of natural evolution.*"

We now focus on the description of the specific HMM setup. First, we define the strategic decision space for player  $i$ , as a one dimensional lattice with reflective boundary conditions. Player  $i$ 's strategic space is defined by a discrete bounded set,  $\Lambda_i \in \mathfrak{R}^{\sigma_i}$ , where  $\min(\Lambda_i) > \delta$ ,  $\max(\Lambda_i) < \delta + \rho_i/\gamma_i$  and  $\sigma_i = \#\Lambda_i$ . Given a small number  $v$ <sup>29</sup>, the set,  $\Lambda_i$ , describing player  $i$  strategic decision space can be defined as,

$$\Lambda_i = \left\{ \delta + v, \delta + 2v, \dots, \delta + \frac{\rho_i}{\gamma_i} - v \right\}. \quad (61)$$

Given an investment strategy value,  $\bar{Z}_{4,i} \in \Lambda_i$ , player  $i$ 's feasible set of strategic actions,  $\Delta\bar{Z}_{4,i}$ , is defined by: (i)  $\{\Delta\bar{Z}_{4,i}^-, \Delta\bar{Z}_{4,i}^0, \Delta\bar{Z}_{4,i}^+\} = \{-v, 0, v\}$ , if  $\delta + v < \bar{Z}_{4,i} < \delta + \rho_i\gamma_i^{-1} - v$ ; (ii)  $\{\Delta\bar{Z}_{4,i}^0, \Delta\bar{Z}_{4,i}^+\} = \{0, v\}$ , if  $\bar{Z}_{4,i} = \delta + v$ ; (iii)  $\{\Delta\bar{Z}_{4,i}^-, \Delta\bar{Z}_{4,i}^0\} = \{-v, 0\}$ , if  $\bar{Z}_{4,i} = \delta + \rho_i\gamma_i^{-1} - v$ . Figure 4, below, portrays the evolution of player  $i$  investment strategies, for a given  $\bar{Z}_{4,i} \in \Lambda_i$ ,

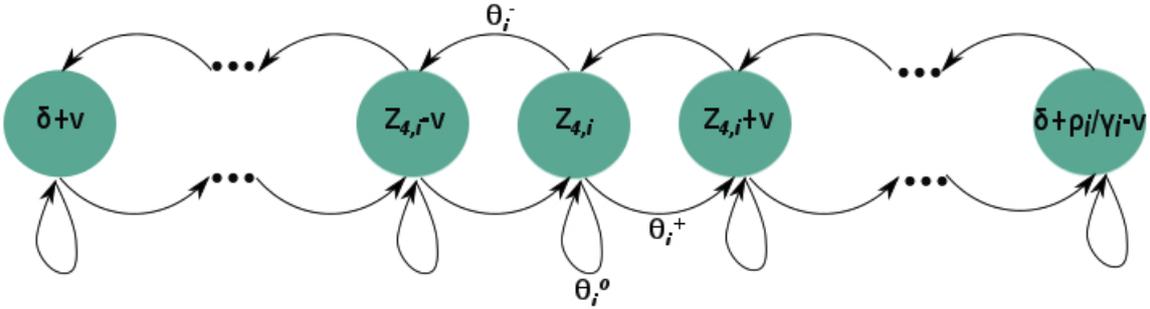


Figure 4: Player investment transitions in a lattice

where  $\theta_i \in \mathfrak{R}^+$  are player  $i$  transition rates between reachable  $\bar{Z}_{4,i}$  states. Transition rates fulfil the usual probability transition rules for a stochastic matrix describing a *Markov* chain over the finite state-space of strategies: (i)  $0 \leq \theta_i \leq 1$ ; and (ii)  $\theta_i^- + \theta_i^0 + \theta_i^+ = 1$ . The total probability of decreasing, maintaining or increasing investment is given by,  $P(\Delta\bar{Z}_4^-) = n^{-1} \sum_{i \in N} \theta_i^-$ ,  $P(\Delta\bar{Z}_4^0) = n^{-1} \sum_{i \in N} \theta_i^0$  and  $P(\Delta\bar{Z}_4^+) = n^{-1} \sum_{i \in N} \theta_i^+$ , respectively. By setting strategic actions in this fashion, we seek that the HMM proposed is able to capture the co-evolutionary nature of decision under uncertainty suggested by our framework. Players follow investment paths and adjust these as the environment changes, instead of radically changing their investment strategies with every state innovation. This option is consistent with investment strategies in financial markets, where radical trading decisions might trigger market movements that lead to important portfolio losses. It also guarantees that an absorbing state,  $\sum_{i \in N} \theta_i^0 = n \Rightarrow P(\Delta\bar{Z}_4^0) = 1$ , consistent with a SCE solution, as defined in (6), exists.

As mentioned previously, we shall assume that individual transition rates,  $\theta_i$ , are computed following a simple multi-criteria decision process that takes into account the uncertainty faced by players, given the available information about the state of the game. In the absence of information, modern machine learning theory suggests the use of variational methods for inference and forecasting purposes<sup>30</sup>. A popular approach to inference in

<sup>28</sup>This approach is gaining ground in economics literature. In a recent review on these topics, Castillo and Coello [11] discuss the potential applications of these inter-disciplinary methods in the fields of economic and finance. This approach are already widely used in modern portfolio optimization theory. Steuer et al. [45] gives an overview of the inter-disciplinary aspects of multi-criteria portfolio optimization. The author suggests that the inclusion of further decision criteria, can improve existing models of expected portfolio maximization under uncertainty. Hens and Schenk-Hoppe [29] shows that mean-variance portfolio strategies in incomplete markets are not evolutionary stable, whereas diversified portfolio strategies consistent with the CAPM rule are evolutionary stable. Elliott et al. [18] proposes a regime switching HMM for mean-variance portfolio selection and provides a recent survey on the application of stochastic processes in finance. Finally, Ahmed and Hegazi [1] provides a discussion on three inter-disciplinary aspects of portfolio optimization: (i) multi-objective optimization; (ii) dynamical re-balancing; and (iii) evolutionary game theory.

<sup>29</sup>For simulation purposes, we assume  $v = 10^{-3}$ .

<sup>30</sup>For an introduction to this topic see Bishop [7], chapter 10, or Barber [4], chapter 21.

HMM environments, is the *Expectation Maximization*(EM) algorithm, which requires some knowledge about the evolution of the state of the system. We discard the use of sophisticated machine learning methods, because of efficiency problems arising when performing inference in a large scale. To overcome this issue, we propose the use of simple *Bayesian* learning mechanisms. In subsection 4.1, we demonstrate the effectiveness of this approach to describe game outcomes consistent with the existence of SCE solutions. *Bayesian* inference methods require information about the evolution of the state of the game. In this setup, players can extrapolate relevant information by monitoring their forecast errors. The forecasting error,  $\epsilon_i$ , observed by player  $i$ , is generally given by,

$$\bar{Z}_{1,i} - E[\bar{Z}_{1,i}] = \bar{Z}_{1,i} [\bar{Z}_{4,i}, \bar{Z}_{2,i}, \bar{Z}_2] - \bar{Z}_{1,i} [\bar{Z}_{4,i}, \bar{Z}_{2,i}, E(\bar{Z}_2^i)] = \epsilon_i. \quad (62)$$

Substituting  $\bar{Z}_{1,i}$  and  $E[\bar{Z}_{1,i}]$ , by the equivalent steady-state expression, following (57), we obtain,

$$\bar{Z}_2(\tau) = E[\bar{Z}_2^i(\tau)] + \frac{\epsilon_i(\tau)}{r\bar{Z}_{2,i}^2(\tau)}, \quad (63)$$

where  $\tau$  describes the time period of the continuous time HMM<sup>31</sup>. Finally, following (63), we define player  $i$  forecast rule, as a simple error correcting mechanism based on the last outcome of the state of the game:

$$E[\bar{Z}_2^i(\tau + \Delta\tau)] = E[\bar{Z}_2^i(\tau)] + \frac{\epsilon_i(\tau)}{r\bar{Z}_{2,i}^2(\tau)} \Rightarrow E[\bar{Z}_2^i(\tau + \Delta\tau)] = \bar{Z}_2(\tau). \quad (64)$$

The result given in (64) defines the simplest inference rule available to players<sup>32</sup>, in this incomplete information setup. This result has several advantages. It allows for a simplification of the expected value computation process, thus greatly reducing the computational time required for the simulation of the HMM, and consequently, the cost of introducing other decision criteria. The existence of information about past moments of the game paves the way for the introduction and evaluation of different reinforcement learning mechanisms. The SCE optimality condition can then be used to rank these mechanisms according to their performance<sup>33</sup>. We propose that players' beliefs about the evolution of the state of the game have to be consistent with *Bayesian* learning in a competitive environment, in order to preserve the co-evolutionary nature of this decision process. This assumption involves introducing as decision criteria, an individual measure of uncertainty, which we define as  $\Phi_i$ . Such that  $\Phi_i$  is a function of the evolution of  $\bar{Z}_2$ . Since  $\epsilon_i(\tau)$  fits this profile already, we assume for simplicity that  $\Phi_i$  evolves according to some linear function of the observed forecasting errors,  $\Phi_i(\epsilon_i(\tau))$ . Following this set of assumptions and the result in (64), we assume individual beliefs about future values of  $\bar{Z}_2$  are normally distributed,

$$\begin{cases} \bar{Z}_2^i(\tau + \Delta\tau) \sim N(\bar{Z}_2(\tau), \Phi_i^2) & \text{if } \tau > 0, \\ \bar{Z}_2^i(\tau + \Delta\tau) \sim N(0, 1) & \text{if } \tau = 0. \end{cases} \quad (65)$$

We are now able to describe the computation of individual transition rates. At a given period,  $\tau$ , the players objective is to take investment decisions that are consistent with the long run maximization of their expected outcomes under uncertainty. At a given  $\bar{Z}_{4,i} \in \Lambda_i$ , players decide what is the best action to undertake given the full range of achievable investment decisions to the left and right of  $\bar{Z}_{4,i}$ , which are defined by the following discrete sets,  $\Lambda_i^L = \{\delta + v, \dots, \bar{Z}_{4,i} - v\}$  and  $\Lambda_i^R = \{\bar{Z}_{4,i} + v, \dots, \delta + \rho_i/\gamma_i - v\}$ , respectively. To evaluate the best path to follow, players compute the maximum expected consumption values, given (64), for all the range of the strategic space,  $E_{\tau+\Delta\tau}[\bar{Z}_{1,i}(\Lambda_i, \bar{Z}_2(\tau))]$ , and determine the following quantities,  $Z_{1,i}^{L,\max} = \max\{E_{\tau+\Delta\tau}[\bar{Z}_{1,i}(\Lambda_i^L, \bar{Z}_2(\tau))]\}$ ,  $Z_{1,i}^{R,\max} = \max\{E_{\tau+\Delta\tau}[\bar{Z}_{1,i}(\Lambda_i^R, \bar{Z}_2(\tau))]\}$  and  $Z_{1,i}^C = E_{\tau+\Delta\tau}[\bar{Z}_{1,i}(\bar{Z}_{4,i}, \bar{Z}_2(\tau))]$ . The best expected achievable outcome is defined generally by  $Z_{1,i}^{best} = \max\{Z_{1,i}^{L,\max}, Z_{1,i}^C, Z_{1,i}^{R,\max}\}$ , while the remaining outcomes are defined as  $Z_{1,i}^{worst,1}$  and

<sup>31</sup>As we are describing the asymptotic dynamics of the game defined in (44), we shall use this notation when referring to asymptotic time. In a continuous time *Markov* process, time evolution is defined as a random distributed exponential variable, following  $\Delta\tau \sim \exp(n)$ , where  $E[\Delta\tau] = n^{-1}$ , defines the average waiting time for player  $i$ .

<sup>32</sup>The result in (63) shows that players can learn the past moments of  $\bar{Z}_2$ . If expectations were driven solely by beliefs, then the learning rule would be a belief-based decision. See Feltovich [19] for a discussion on this topic.

<sup>33</sup>In economic theory the reinforcement learning assumption has the following interpretation. It justifies how equilibrium may or may not arise when agents have bounded rationality. The consistent expectations equilibrium (CEE) hypothesis by Hommes and Sorger [31], for example, suggests that linear forecasting decisions, based on past available information, represent simple rational adaptive learning rules for nonlinear incomplete information environments, that are consistent with the existence of self-fulfilling belief solutions. As the authors put it, "...agents form expectations about future variables in such a way that their beliefs are consistent with the observed realizations in a linear statistical sense. In other words, it is supposed that agents act like econometricians using linear statistical techniques and, in doing so, they do not make systematic forecasting errors...". An application of this hypothesis in a nonlinear dynamic optimization setup can be found in Hommes and Rosser [30]. For a thorough review on the topic of multi-agent reinforcement learning see Tuyls and Nowé [48].

$Z_{1,i}^{worst,2}$ . Finally, each agent determines the following quantities, taking into account the uncertainty regarding the best achievable outcome<sup>34</sup>:

$$\left\{ \begin{array}{ll} \Theta_i^{worst,1} = Z_{1,i}^{worst,1} (\bar{Z}_2(\tau) + \Phi_i) - Z_{1,i}^{best} (\bar{Z}_2(\tau) - \Phi_i) & \text{if } Z_{1,i}^{worst,1} (\bar{Z}_2(\tau) + \Phi_i) > Z_{1,i}^{best} (\bar{Z}_2(\tau) - \Phi_i) \\ \Theta_i^{worst,1} = 0 & \text{otherwise} \end{array} \right. ; \quad (66)$$

$$\left\{ \begin{array}{ll} \Theta_i^{worst,2} = Z_{1,i}^{worst,2} (\bar{Z}_2(\tau) + \Phi_i) - Z_{1,i}^{best} (\bar{Z}_2(\tau) - \Phi_i) & \text{if } Z_{1,i}^{worst,2} (\bar{Z}_2(\tau) + \Phi_i) > Z_{1,i}^{best} (\bar{Z}_2(\tau) - \Phi_i) \\ \Theta_i^{worst,2} = 0 & \text{otherwise} \end{array} \right. ; \quad (67)$$

$$\Theta_i^{best} = Z_{1,i}^{best} (\bar{Z}_2(\tau) + \Phi_i) - Z_{1,i}^{best} (\bar{Z}_2(\tau) - \Phi_i). \quad (68)$$

Finally, transition rates are given by a weighted average of all achievable outcomes that are greater than the lower confidence bound of  $Z_{1,i}^{best}$  expected outcomes,

$$\left\{ \begin{array}{l} \theta_i^{best} = \Theta_i^{best} (\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2})^{-1} \\ \theta_i^{worst,1} = \Theta_i^{worst,1} (\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2})^{-1} \\ \theta_i^{worst,2} = \Theta_i^{worst,2} (\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2})^{-1} \end{array} \right. . \quad (69)$$

To describe the mechanics driving this decision process, we resort to the example portrayed in Figure 3. In this setup, the best expected outcome for agent  $i$ , is in one of the extremes of  $\Lambda_i$ . It is not clear which one of the two extremes has the best expected value. However, by evaluating the landscape in this fashion, we know that the transition rates for a player with an interior strategy,  $\delta + v < \bar{Z}_{4,i} < \delta + \rho_i \gamma_i^{-1} - v$ , are ranked according to,  $0 \leq \theta_i^0 < \theta_i^-, \theta_i^+$  and  $\theta_i^- \approx \theta_i^+$ . This ordering of rates is, in our opinion, a reasonable weighting of the uncertainty faced by player  $i$ . This process allows players to re-evaluate their investment priorities in a co-evolutionary environment, where one player's decision may lead to drastic changes in the landscape. To illustrate the HMM described in this section, we simulate it numerically as a *Markov* chain. Our main objective is to show the crucial role played by uncertainty. As in this framework, players' expected outcomes are homogeneous, uncertainty is the only distinctive criteria driving player decisions<sup>35</sup>. This feature of the HMM allows us to focus on the dramatic consequences of uncertainty in the quality of solutions that may arise from (60). Our approach is purely illustrative. The simple metric proposed serves mainly the purpose of avoiding the computational costs of performing large scale inference and paves the way for a future statistical analyses of this HMM outcomes based on the *Markov Chain Monte Carlo* method. A more thorough analysis will involve assuming a more realistic decision criteria based on a consistent probability measure, which takes into account the heterogeneity and evolutionary nature of the individual inference process. In the next two sections, we demonstrate the framework proposed for two different hypothesis describing the evolution of individual uncertainty. In these simulations, we consider an institutional framework defined by  $\{r, \delta, r_k\} = \{0.05, 0.03, 0.11\}$ . Each experiment draws a distinct robust population<sup>36</sup>, with initial investment positions distributed randomly following,  $\bar{Z}_{4,i}(\tau = 0) \sim U_d(\Lambda_i)$ .

#### 4.2. Self-confirming equilibrium

Before putting forward the results of this experiment, it is convenient to frame the definition of SCE, given in (6), with the literature on evolutionary multi-objective optimization. In this field, the analysis of solutions relies on the definition of a *Pareto Dominance* criterion<sup>37</sup> or *Pareto Efficiency* frontier, to develop efficient algorithms

<sup>34</sup>Where  $Z_{1,i}^{worst,1} (\bar{Z}_2(\tau) \pm \Phi_i)$ ,  $Z_{1,i}^{worst,2} (\bar{Z}_2(\tau) \pm \Phi_i)$  and  $Z_{1,i}^{best} (\bar{Z}_2(\tau) \pm \Phi_i)$ , describe  $Z_{1,i}^L$ ,  $Z_{1,i}^C$  and  $Z_{1,i}^R$ , evaluated at  $\bar{Z}_2(\tau) \pm \Phi_i$ .

<sup>35</sup>The inclusion of individual memory, is a natural extension in this HMM. It allows the limited introduction of heterogeneity about expected state outcomes, without adding further complications to this setup. The general *Bayesian* forecast rule in an environment with individual memory, following the set of conditions described, would be given by  $\bar{Z}_i^j(\tau + \Delta\tau) \sim N(\bar{Z}_i^j(\tau), \bar{\Phi}_i(\bar{\epsilon}_i(\tau))^2)$ ,  $\forall \tau > 0$ , where  $\bar{Z}_i^j(\tau)$ , is the sample mean of player  $i$  observed posterior distribution up to period  $\tau$ , and  $\bar{\Phi}_i(\bar{\epsilon}_i(\tau))^2$ , is a function of the estimated sample variance given a prior belief that depends on the evolution of player  $i$  past forecast errors,  $\bar{\epsilon}_i(\tau)$ .

<sup>36</sup>By robust population, we mean a discrete set of  $n = 1000$  agents with randomly drawn distributed individual characteristics,  $\rho_i, \gamma_i \sim U(0, 1)$  and  $h_i \sim U(-10, 10)$ , that are consistent with  $\delta + \rho_i/\gamma_i < 1$ ,  $\#\Lambda_i \geq 10$  and  $\bar{Z}_{1,i} > 0, \forall \bar{Z}_{4,i} \in \Lambda_i$ , given a reasonable range of state outcomes,  $-1.5 < \bar{Z}_2 < 1.5$ . Following the definitions in section 3,  $K_i(0)$  is again given by random draws of an exponential distribution,  $K_i(0) \sim \exp(1)$ , and the evolution of the state of the game is driven by a 30% population share of aggressive investors.

<sup>37</sup>See Gajda et al. [23] for a formal definition of *Pareto Dominance* and a mathematical discussion on multi-objective optimization problems.

and order feasible solutions. A strategy is *Pareto Dominant*, if it lies in the *Pareto Efficient* frontier. This approach cannot be extended to decision problems under uncertainty. In incomplete information environments, solutions are by definition conjectural equilibria outcomes<sup>38</sup>, since equilibrium is a result of co-evolution under uncertainty. Hu and Wellman [33], for example, show that solutions to competitive multi-agent models with learning dynamics are highly sensitive to initial conditions. In evolutionary multi-objective optimization environments with uncertainty, a common approach is to order solutions in terms of some *Bayesian Efficiency* criterion. This involves considering a consistent *Bayesian* learning incentive. A reasonable definition of *Bayesian Efficiency* for the problem defined in (60), is given by the SCE condition. To be able to order solutions to (60), consistent with this notion of *Bayesian Efficiency*, we have to introduce an incentive compatible rule in the *Bayesian* learning mechanism of our HMM. Recall that in subsection 4.1, we mentioned that a SCE requires the existence of an absorbing state,  $\sum_{i \in N} \theta_i^0 = n$ . Following our definition in (65), a SCE is achieved for this HMM if  $\lim_{\tau \rightarrow \infty} \Phi_i(\epsilon(\tau)) = 0 \Rightarrow E[\bar{Z}_2^i(\tau + \Delta\tau)] = \bar{Z}_2^i(\tau), \forall i \in N$ . To include a valid *Bayesian Incentive* in this setup, we shall consider that when  $e_i(\tau) = 0$ , agents no longer forecast future outcomes under uncertainty,  $\Phi_i(0) = 0$ . We demonstrate the existence of SCE solutions for the HMM proposed, assuming that uncertainty depends linearly on the last observed forecasting error. In this experiment,  $\Phi_i$  evolves in the following fashion<sup>39</sup>,

$$\Phi_i(\tau + \Delta\tau) = 10^8 |\epsilon_i(\tau)|. \quad (70)$$

We start this presentation with the parameter distributions describing a random robust population. These are portrayed below, by the histograms of  $\rho_i$ ,  $\gamma_i$  and  $h_i$ . In this institutional setup, robust populations are characterized by a group of patient players with a high rate intertemporal substitution of consumption, where the vast majority of players is biased towards investment in productive/riskless assets.

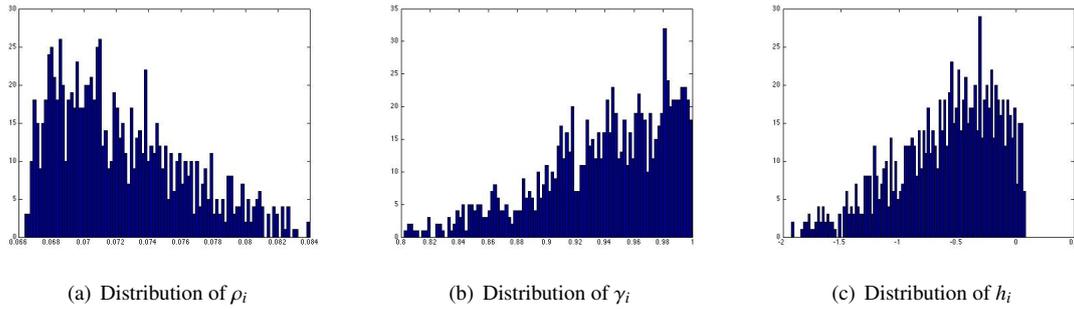


Figure 5: Robust population distributions.

The next figure portrays the HMM convergence to a SCE. The dynamics  $\bar{Z}_2(\tau)$ , on the right, describe an initial fast transition to the equilibrium region, followed by overshooting phenomenon arising before players concur on a SCE solution. In the figure on the left, we observe that uncertainty about investment decisions decreases during the initial convergence moment and suddenly increases before settling in the SCE solution. This behaviour during transitions is, in our opinion, linked to dynamics of learning under uncertainty. In a first phase, there is a clear trend driving strategies towards a region where agents outcomes are close to optimum. At this point, higher aggregate outcomes are achievable, but they come at the expense of increased uncertainty among players. This eventually leads to a review of the long run strategies by an increasing share of the population. The reason for this phenomenon is straightforward. There is a set of players that is now worst off. As the co-evolution process develops, players eventually concur on an equilibrium that best fits their individual interests. This self-organization process materializes in conjectural SCE solution when uncertainty vanishes completely from the system.

Figure 7 describes the asymptotic outcomes for the mean absolute forecast errors, aggregate state and investment dynamics. The figure on the left shows that the learning rule described in (70) is consistent with the existence of optimal solutions to (44), as the result  $\lim_{\tau \rightarrow \infty} \Phi_i(\epsilon(\tau)) = 0$  guarantees that (6) is fulfilled. Additional simulations suggest that this specific setup is always consistent with convergence to a SCE solution.

We conclude this discussion with an analysis of the effects of player asymmetries and portfolio decisions on consumption outcomes. These results are given below in Figure 8. The figure on the left shows that there is a

<sup>38</sup>We consider that SCE solutions are a particular stronger case of conjectural equilibrium solutions. See Wellman et al. [50] for a discussion on this topic in a continuous time pure exchange economy model. In economics literature a popular extension of this topic is the conjectural variations approach. Itaya and Shimomura [35] provides a thorough discussion on this method for multi-player public goods games.

<sup>39</sup>The constant  $10^8$  in (70), is used to portray the robustness of this linear learning rule in guaranteeing convergence to a SCE, for this HMM.

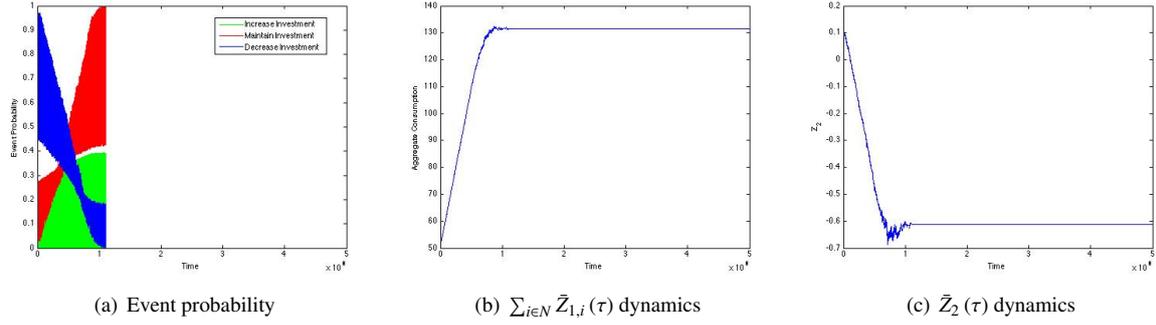


Figure 6: Total probability, consumption and game dynamics.

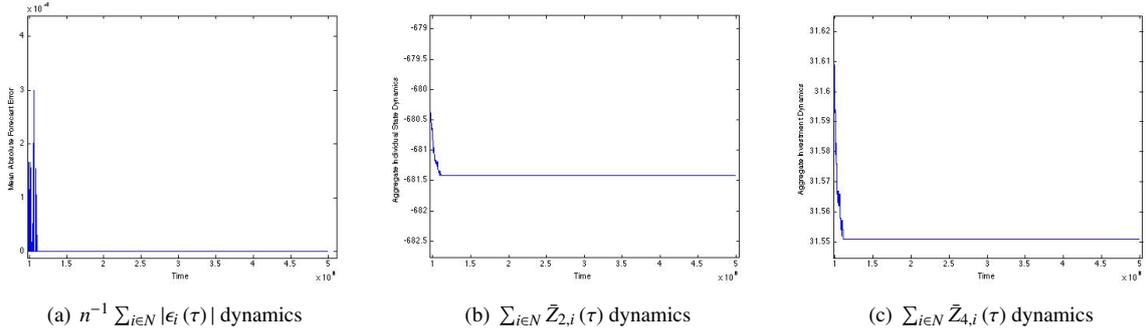


Figure 7: Asymptotic mean absolute forecast errors and aggregate dynamics.

linear/logarithmic relation between player patience and his consumption outcomes. This result is consistent with the fundamental law of growth theory, which states that agents with higher savings rate (smaller discount rates), are able to sustain higher consumption profiles in the future<sup>40</sup>. A similar relation is observed when players portfolios are compared against consumption outcomes. In the figure in the right, we observe that leveraged players have the smallest share of consumption to productive assets. These results suggest that players with diversified portfolios and higher propensity to save will perform better in this economy. The remaining figures portray the relation between intertemporal substitution of consumption and investment bias on the final consumption outcomes. The joint density plots show that the majority of players with higher  $\gamma_i$  and higher  $h_i$  have lower consumption profiles. However, there is no clear pattern arising in these two cases.

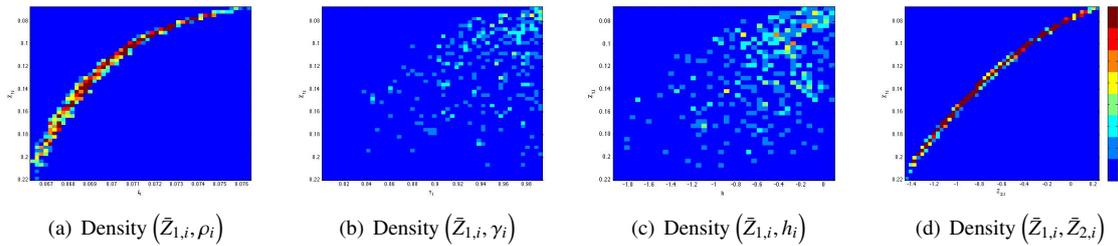


Figure 8: Joint densities of final outcomes versus player asymmetries.

#### 4.3. Unintended consequences

To portray game dynamics consistent with strategic interactions and unintended consequences driven by subjective beliefs as a possible outcome to (44), we now assume that agents have homogeneous beliefs about uncertainty, except when no forecast errors are observed. In this setup, uncertainty is no longer a linear function of  $\epsilon_i(\tau)$ , but given by a *Heaviside* step function. This specification allows us to show that above some threshold,

<sup>40</sup>This result holds for growing economies, where savings finance profitable investments in the present that increase capital in the future.

uncertainty plays a crucial role on the quality of solutions. It also also guarantees the existence of a *Bayesian Incentive* mechanism that is compatible with the existence of an absorbing state for this HMM. This simplification has the sole purpose of showing the co-evolutionary nature of solutions to (60). Solutions in this setup are not unique and asymptotic dynamics are better defined as invariant sets. Although these solutions no longer fulfil the *Bayesian Efficiency* criterion and, therefore, are non-optimal, numerical results suggest that consumption outcomes are quasi-optimal in the long run, when compared to the previous example. Uncertainty is now defined as

$$\Phi_i(\tau + \Delta\tau) = \begin{cases} 1 & \text{if } |e_i(\tau)| > 0 \\ 0 & \text{if } |e_i(\tau)| = 0 \end{cases} . \quad (71)$$

We start this presentation with the description of player asymmetries. A quick glance at the histograms given in Figure 5 and Figure 9 shows that there is no significant differences between the two samples.

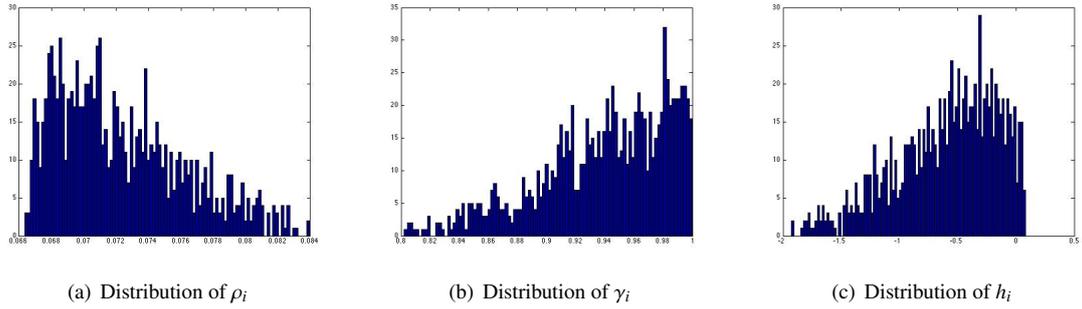


Figure 9: Robust population distributions.

Figure 10, below, portrays the main results for this experiment. Again, we observe a fast convergence to long run dynamics. However, asymptotic dynamics are no longer described by a unique SCE solution. The stochastic process does not converge to an absorbing state. This solution is best defined as an invariant set, and asymptotic dynamics are described by a stationary distribution. A quick inspection reveals that aggregate consumption outcomes converge to values close to the SCE solution portrayed in Figure 6. Although these two experiments cannot be directly compared, since initial values and population characteristics are not identical, this result suggests that the invariant set describing this game asymptotic dynamics can be characterized as a quasi-optimal solution to (60). The level of uncertainty faced by the ensemble of players has dramatic consequences on the quality and complexity of solutions to (44). The picture on the left, describing the dynamics of total probability, portrays the evolutionary mechanism driving the complex macro dynamics observed. In the long run, a minority of players is undecided between choosing conservative or aggressive investment strategies,  $0.1 < n^{-1} \sum_{i \in N} \theta_i^+ + n^{-1} \sum_{i \in N} \theta_i^- < 0.5$ . The indecision leads to unintended consequences, which in turn feeds strategic interactions and further changes in the environment. On the other hand, a majority of players maintains their long run investment strategies,  $0.5 < n^{-1} \sum_{i \in N} \theta_i^0 < 0.9$ , despite the changes imposed by the undecided minority. In a continuous co-evolution process, undecided investors drive the market and reshape the economic landscape. The dynamics of this economy are thus dominated by the decisions of a minority.

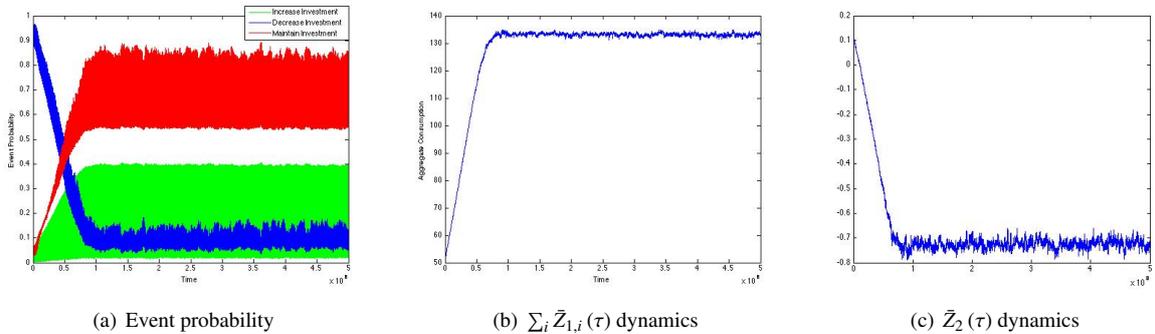


Figure 10: Probability, game and consumption dynamics.

We finish this presentation with a figure portraying the asymptotic dynamics of  $n^{-1} \sum_i |\epsilon_i(\tau)|$ ,  $\sum_i \bar{Z}_{2,i}(\tau)$  and  $\sum_i \bar{Z}_{4,i}(\tau)$ . The dynamics of  $n^{-1} \sum_i |\epsilon_i(\tau)|$  portray the consequences of persistently high uncertainty on the forecasting process. Errors in this setup alternate between stable and bursting periods with different amplitudes. The largest bursts identify the decisions that imposed greater changes on the state of the game. The consequences of these interactions are portrayed in the dynamics of  $\sum_i \bar{Z}_{2,i}(\tau)$  and  $\sum_i \bar{Z}_{4,i}(\tau)$ . Investment and portfolio decisions are asymptotically stationary but consistently drift away from the mean. Further insight on this and other matters requires a thorough statistical analysis of this stochastic process.

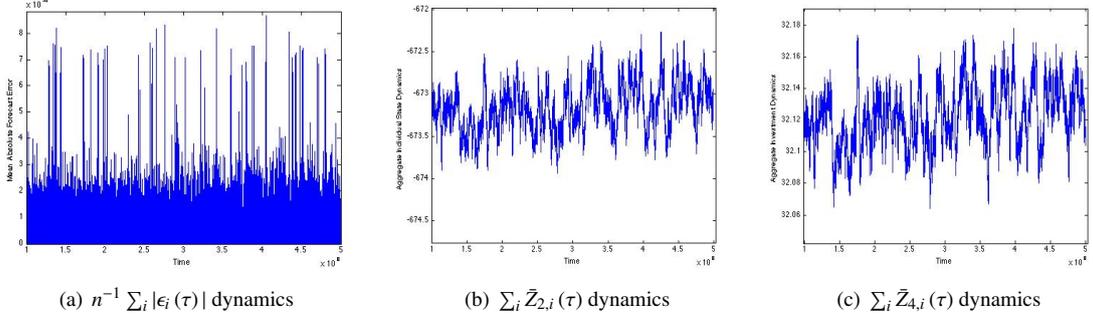


Figure 11: Asymptotic mean absolute forecast errors and aggregate dynamics.

## 5. Conclusions and further research

In this paper, we described a class of differential games, where players' optimal strategies require information that is not available to them. Decisions in this context, we argued, have to be based on subjective beliefs, and optimality requires the existence of a SCE solution. To demonstrate this conjecture, we proposed two extensions of the consumer-investor problem. In the first example discussed in section 3, beliefs impose a unique strategic solution and the existence of a SCE requires the existence of credit markets with complete information that are able to price risk optimally. Solutions can be evaluated qualitatively when beliefs are naive. Numerical results show that game solutions are not stable for a wide range of institutional scenarios. This result is not surprising, as these constraints impose a game environment where both creditors and debtors believe that they will be worse off. We concluded that stable SCE solutions require that the strict assumptions imposed on the evolution of beliefs are at least partially dropped. In section 4, we showed that the introduction of additional nonlinearities in the initial game is sufficient to introduce multiple equilibria. Solutions are consistent with a game with a state-space bounded asymptotically, but standard qualitative methods cannot be employed in this context. We propose that game solutions can be evaluated as a multi-objective optimization problem under uncertainty. Given the specific nature of this game, we proposed to evaluate game solutions as a multi-criteria HMM, consistent with learning in a co-evolutionary environment, and focused on the role played by uncertainty. First, we showed the existence of SCE solutions, when uncertainty depends linearly on forecast errors. When uncertainty is persistent, belief-based decisions lead to unintended consequences and strategic interactions that have dramatic effects on the macro evolution of this economy. Solutions are now of a quasi-optimal nature. This phenomenon is driven by a minority of players, which suggests the existence of strong emergence phenomena. Further insight on these two hypotheses requires the computation of samples and a thorough statistical analysis. It is our opinion that the statistical analysis of quasi-optimal solutions to nonlinear dynamic games is bound to play an important role in future economics literature. Finally, the introduction of learning heterogeneity represents a natural extension to the simple framework proposed. It would also be interesting to investigate the implications of introducing evolutionary adaptation in the decision process discussed.

## Appendix

### Appendix A. Optimal control conditions

#### Appendix A.1. Optimal control conditions for the non-cooperative game with investment bias and coupled institutional risk premium

The current value *Hamiltonian* for the non-cooperative game in (8) is:

$$H [B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)]^* = C_i(t)^{\gamma_i} + \lambda_i(t) \dot{B}_i(t) + q_i(t) \dot{K}_i(t), \quad (\text{A.1})$$

where  $\dot{B}_i(t)$  and  $\dot{K}_i(t)$  are given in (A.6) and (A.7). The general *Pontryagin* maximum conditions for the existence of optimal open loop solutions are given by:

*Optimality conditions*

$$\gamma_i C_i(t)^{\gamma_i-1} = -\lambda_i(t); \quad (\text{A.2})$$

$$q_i(t) = -\lambda_i(t) \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right); \quad (\text{A.3})$$

*Multiplier conditions*

$$\dot{\lambda}_i(t) = \lambda_i(t) \left( \rho_i - r - rd_i \frac{B(t)}{K(t)} \right); \quad (\text{A.4})$$

$$\dot{q}_i(t) = q_i(t) (\rho_i + \delta) + \lambda_i(t) \left( h_i \frac{I_i(t)^2}{K_i(t)^2} + r_k \right); \quad (\text{A.5})$$

*State conditions*

$$\dot{B}_i(t) = C_i(t) + I_i(t) \left( 1 + h_i \frac{I_i(t)}{K_i(t)} \right) + rB_i(t) \left( 1 + d_i \frac{B(t)}{K(t)} \right) - r_k K_i(t); \quad (\text{A.6})$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \quad (\text{A.7})$$

*Transversality conditions*

$$\lim_{t \rightarrow \infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \quad (\text{A.8})$$

$$\lim_{t \rightarrow \infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \quad (\text{A.9})$$

*Admissibility conditions*

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \quad (\text{A.10})$$

*Appendix A.2. Optimal control conditions for the non-cooperative game with investment bias and coupled endogenous risk premium*

The current value *Hamiltonian* for the non-cooperative game in (44) is:

$$H [B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)]^* = C_i(t)^{\gamma_i} + \lambda_i(t) \dot{B}_i(t) + q_i(t) \dot{K}_i(t), \quad (\text{A.11})$$

where  $\dot{B}_i(t)$  and  $\dot{K}_i(t)$  are given in (A.16) and (A.17). The general *Pontryagin* maximum conditions for the existence of optimal open loop solutions are given by:

*Optimality conditions*

$$\gamma_i C_i(t)^{\gamma_i-1} = -\lambda_i(t); \quad (\text{A.12})$$

$$q_i(t) = -\lambda_i(t) \left( 1 + 2h_i \frac{I_i(t)}{K_i(t)} \right); \quad (\text{A.13})$$

*Multiplier conditions*

$$\dot{\lambda}_i(t) = \lambda_i(t) \left( \rho_i - r - 2r \frac{B_i(t) B(t)}{K_i(t) K(t)} \right); \quad (\text{A.14})$$

$$\dot{q}_i(t) = q_i(t) (\rho_i + \delta) + \lambda_i(t) \left( h_i \frac{I_i(t)^2}{K_i(t)^2} + r \frac{B_i(t)^2 B(t)}{K_i(t)^2 K(t)} + r_k \right); \quad (\text{A.15})$$

*State conditions*

$$\dot{B}_i(t) = C_i(t) + I_i(t) \left( 1 + h_i \frac{I_i(t)}{K_i(t)} \right) + rB_i(t) \left( 1 + \frac{B_i(t)}{K_i(t)} \frac{B(t)}{K(t)} \right) - r_k K_i(t); \quad (\text{A.16})$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \quad (\text{A.17})$$

*Transversality conditions*

$$\lim_{t \rightarrow \infty} \lambda_i(t) B_i(t) e^{-\rho t} = 0; \quad (\text{A.18})$$

$$\lim_{t \rightarrow \infty} q_i(t) K_i(t) e^{-\rho t} = 0; \quad (\text{A.19})$$

*Admissibility conditions*

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \quad (\text{A.20})$$

## Appendix B. General Riccati equation solution

The general Riccati equation of interest is defined by the following first order ordinary differential equation,

$$\frac{\partial \chi(t)}{\partial t} = \chi(t)^2 + \frac{b}{a} \chi(t) + \frac{c}{a}, \quad (\text{B.1})$$

where  $a/b < 0$ ,  $b^2 - 4ac > 0$ . To solve the above Riccati equation it is convenient to start by dividing everything by the right hand side expression and then take integrals with respect to time. Equation (B.1) is now given by:

$$\int_0^t \frac{\frac{\partial \chi(s)}{\partial s}}{\chi(s)^2 + ba^{-1}\chi(s) + ca^{-1}} ds = \int_0^t 1 ds. \quad (\text{B.2})$$

The right hand side integral is given by  $t + o_1$ , where  $o_1$  is a constant of integration. We can now focus on the solution of the left hand side integral. We start by simplifying this integral by assuming the following substitution of the integration variable and integrand,  $u = \chi(s) \Rightarrow du = \frac{\partial \chi(s)}{\partial s} ds$ . This integral is now given by:

$$\int_0^t \frac{1}{u^2 + ba^{-1}u + ca^{-1}} du. \quad (\text{B.3})$$

The result in (B.3) suggests that we may be able to factor out constants by substitution, in order to obtain a solvable integral expression. We start by completing the square in (B.3),

$$\int_0^t \frac{1}{\frac{1}{4a^2} (4ac - b^2) + \left(u + \frac{b}{2a}\right)^2} du. \quad (\text{B.4})$$

The next substitution is straightforward. We consider,  $p = u + \frac{b}{2a} \Rightarrow dp = du$ , and (B.4) after rearranging comes,

$$\int_0^t \frac{1}{\frac{1}{4a^2} (4ac - b^2) + p^2} dp = \int_0^t \frac{1}{\frac{4ac-b^2}{4a^2} \left(\frac{4a^2}{4ac-b^2} p^2 + 1\right)} dp = \int_0^t \frac{-\frac{4a^2}{b^2-4ac}}{-\frac{4a^2}{b^2-4ac} p^2 + 1} dp. \quad (\text{B.5})$$

Factoring out constants and then setting  $y = -2a(b^2 - 4ac)^{-\frac{1}{2}} p$ , where  $dy = -2a(b^2 - 4ac)^{-\frac{1}{2}} dp$ , we obtain,

$$\frac{2a}{\sqrt{b^2 - 4ac}} \int_0^t \frac{1}{1 - y^2} dy. \quad (\text{B.6})$$

Since  $\int 1/(1 - y^2) dy = \text{arctanh}(y)$ , we substitute everything back and obtain the solution to (B.2),

$$\frac{2a}{\sqrt{b^2 - 4ac}} \text{arctanh}\left(-\frac{2a\chi(t) + b}{\sqrt{b^2 - 4ac}}\right) + o_2 = t + o_1, \quad (\text{B.7})$$

where  $o_2$  is a constant of integration. Taking arctanh from both sides the general solution to (B.1) is given by,

$$\chi(t) = -\frac{\sqrt{b^2 - 4ac} \tanh\left(\frac{1}{2a}(t + o)\sqrt{b^2 - 4ac}\right) + b}{2a}, \quad (\text{B.8})$$

where  $o = o_1 - o_2$ . Now we need to take into account the existence of an initial value,  $\chi(0)$ . Setting  $t = 0$  in (B.8), taking arctanh on both sides and then solving in terms of  $o$  we obtain:

$$o = \frac{2a}{\sqrt{b^2 - 4ac}} \operatorname{arctanh}\left[-\frac{(2a\chi(0) + b)}{\sqrt{b^2 - 4ac}}\right]. \quad (\text{B.9})$$

Substituting (B.9) in (B.8), we obtain the general explicit solution to (B.1),

$$\chi(t) = -\frac{\sqrt{b^2 - 4ac} \tanh\left\{\frac{\sqrt{b^2 - 4ac}}{2a}\left[t + \frac{2a}{\sqrt{b^2 - 4ac}} \operatorname{arctanh}\left(-\frac{2a\chi(0) + b}{\sqrt{b^2 - 4ac}}\right)\right]\right\} + b}{2a}. \quad (\text{B.10})$$

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