Driven diffusive systems and growing stationary configurations

Tom Rafferty[1,*], Paul Chleboun[2], Stefan Grosskinsky[1,2]
1 Centre for Complexity Science, University of Warwick, Coventry, CV4 7AL, United Kingdom
2 Mathematics Institute, University of Warwick
*** E-mail: t.rafferty@warwick.ac.uk**

Abstract

We study attractive particle systems with stationary product measures. We utilize the property of attractivity and its link to coupling to build a growth process that samples from the stationary measure of the zero-range process, on fixed and finite lattices, with computation times scaling linearly with the number of particles N. The zero-range process with constant jump-rates and a single defect site is known to exhibit a condensation transition, and we use L independent continuous time birth-processes to sample from the stationary measure that exhibits condensation. The birth-rate of the defect site is a time inhomogeneous process, where the intensity function (or time integrated birth-rate) exhibits the property of finite-time-blow-up. For this process, finite-time-blow-up implies that infinitely many events can occur in a finite window of time.

1 Introduction

Driven diffusive systems are models of non-equilibrium statistical mechanics, where particles move on a lattice or network, [1] [2]. We consider systems, where the rate these particles jump depends only on the number of particles at the exit site and the entry site, and the site a particle jumps to is given by some probability distribution that describes a random walker on the lattice. The jump-rates are a product of functions of the number of particles at entry and exit sites. Such systems exhibit factorisable stationary distributions known as product measures, which are independent of the dynamics of the random walker. The zero-range process is one particular example, where the jump-rate depends only on the number of particles at the exit site, hence the name zero-range.

We study attractive driven diffusive systems, which are known to be attractive if the jump-rate is decreasing (increasing) in the number of particles at the entry (exit) site, [3] [4]. We consider closed systems where the total number of particles N is conserved and attractivity implies that stationary distributions $\pi_{L,N}$ on a fixed lattice of size L are stochastically ordered in the number of particles, i.e. $\pi_{L,N} \leq \pi_{L,N+1}$. Coupling techniques are used to prove the property of attractivity and can often be interpreted as a method of simulating two, or more, systems simultaneously by forcing them to depend on each other via some non-trivial rules, [5] [6]. These rules are restricted such that when you observe one of the individual processes without observing the others, it behaves in the way it was originally constructed.

In this project, we use the property of attractivity and the coupling technique to grow configurations, which are used to sample from the stationary measure of the zero-range process. This growth rule allows us to sample from the stationary measure such that computation times growing linearly with N. This is vast improvement on the usual Markov Chain Monte Carlo (MCMC) techniques where the relaxation times, the time needed to generate an independent sample, are typically of order N^2 or N^3, [7]. The mixing or equilibration times, defined as the number of steps required to reach the stationary distribution [8], are typically larger than the relaxation times.

For the zero-range process with constant jump rates, i.e. where the jump-rates are independent of the number of particles at a site, a tight upper bound on the relaxation time is of order $(1/\rho+1)^2N^2$, where the density $\rho = N/L$, [9]. The zero-range process of this form is known to exhibit a transition to condensation, where a non-zero fraction of particles accumulate on one site, if there exists a defect-site where the jump-rate is slower than the surrounding sites [10] [11]. This transition occurs at a critical density (ρ_c), where below ρ_c there exists no condensate and the system is in a fluid phase and above ρ_c the system separates into a condensate and a fluid phase. At this transition point, equilibration times are typically much larger than for the spatial homogeneous system. We construct a time in-homogeneous birth-processes which allows us to sample from the stationary measure such that the computation time scales with N.

This report is organised as follows. In Section 2 we introduce the main mathematical properties of interacting particle systems that are involved in this project, while keeping the discussion as general as possible. We discuss

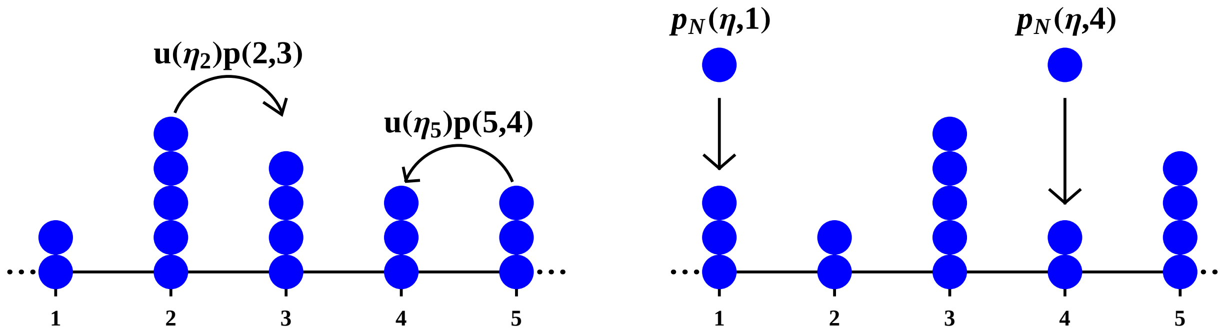


Figure 1. (Left) example dynamics of the zero-range process (4), with jump rates $u(\eta_x)$. For example, a particle at site 2 will jump at rate $u(\eta_2)$ and land on site 3 with probability $p(2,3)$. (Right) example growth dynamics, where a particle is added to site x with probability $p_N(\eta, x)$ and N denotes the number of particles currently in the system.

the concept of a generator, describing the time evolution of observables, and show the connection to the master equation, which governs the time evolution of probability distributions for the stochastic process. We also define what it means for a process to be attractive and connect this concept to the technique of coupling stochastic processes. Section 3 is where we define driven diffusive systems, the zero-range process and birth-death processes. We discuss some of the main properties of these processes such as their stationary measures, attractivity and condensation. In Sections 4 and 5 we discuss the main results of this project, focusing on the zero-range process. In the former, we use coupling techniques to define a growth rule to build stationary configurations related to the canonical measure, while in the latter we discuss the connection between pure-birth processes and the grand canonical measures. Finally, in Section 6 we discuss the results of this project and give a short summary of possible future work.

2 Background

2.1 The master equations and generator

Throughout this project we restrict our discussion to continuous-time Markov processes defined on a finite state-space given by $S = X^\Lambda$, where Λ is a finite lattice of the form $\{1, \ldots, L\}$. Configurations $\eta \in X^\Lambda$ are of the form $\eta_x \in X$ for all $x \in \Lambda$. Processes of this form may be characterised by the generator-matrix $G \in \mathbb{R}^{|S| \times |S|}$, where the matrix elements $c(\eta, \zeta)$ are the jump rates from state $\eta \in S$ to $\zeta \in S$ and the diagonal elements $-c(\eta, \eta)$ are the total exit rate of state $\eta \in S$. Therefore, the time evolution of a probability distribution vector $\mathbf{p}(t) = (p_\eta(t))_{\eta \in S}$ called the master equation, is given by

$$\frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(0)G \quad \Longleftrightarrow \quad \frac{d}{dt}p_\eta(t) = \sum_{\zeta \neq \eta} \Big(p_\zeta(t)c(\zeta, \eta) - p_\eta(t)c(\eta, \zeta) \Big).$$

Another characterisation of processes of this form is given by the generator $\mathcal{L}$ which governs the time-evolution of expected values of observables $f : S \to \mathbb{R}$. More explicitly, for a process X_t, we have

$$\frac{d}{dt}\mathbb{E}(f(X_t)) = \mathbb{E}\left(\mathcal{L}f(X_t)\right).$$

For general continuous-time Markov process on countable state spaces, the functional form of the generator $\mathcal{L}$, is given by

$$\mathcal{L}f(\eta) = \sum_{\{\zeta\in S|\zeta\neq\eta\}} c(\eta,\zeta)\Big(f(\zeta) - f(\eta)\Big),$$

and can be interpreted as the discrete derivative of f under a single transition in the process. We are now in a position to see the connection between the master equation and the generator $\mathcal{L}$ by setting our observable f to be the indicator function, $f(.) = \mathbb{I}_\eta(.)$, and using the notation $\int_S \mathcal{L}f(\eta)d\mu \equiv \mu(\mathcal{L}f(\eta)$,

$$\begin{aligned}\int_S \mathcal{L}\mathbb{I}_\eta d\mu \equiv \mu(\mathcal{L}\mathbb{I}_\eta) &= \sum_{\zeta\in S}\mu(\zeta)\sum_{\zeta'\in S} c(\zeta,\zeta')\Big(\mathbb{I}_\eta(\zeta') - \mathbb{I}_\eta(\zeta)\Big)\\ &= \sum_{\zeta\in S}\mu(\zeta)c(\zeta,\eta) - \mu(\eta)\sum_{\zeta'} c(\eta,\zeta'),\end{aligned}$$

which is exactly the form of the master equation for $\mu(\eta) = p_\eta(t)$. Since the two characterisations are equivalent, we use the most convenient formulation at the time.

2.2 Stationary measures

Stationary measures are probability distributions that are conserved in time under the dynamics of the process. The processes we are interested converge as time tends to infinity to the stationary distributions associated to the process, which are unique under certain conditions. Such a property is called ergodicity. For a continuous-time Markov process X_t, a probability distribution vector $\mathbf{p}^\star(t)$ is stationary if

$$\mathbf{p}^\star(t)G = 0 \text{ which is equivalent to } \frac{d}{dt}p^\star_\eta(t) = \sum_{\zeta\neq\eta}\Big(p^\star_\zeta(t)c(\zeta,\eta) - p^\star_\eta(t)c(\eta,\zeta)\Big) = 0 \quad \text{for all } \eta\in S.$$

Or equivalently, we may discuss stationary measures using the generator characterisation of a process. A probability measure $\mu\in\mathcal{P}(S)$ is stationary if

$$\int_S \mathcal{L}f(\eta)d\mu \equiv \mu(\mathcal{L}f) = 0 \quad \text{for all observables } f. \tag{1}$$

2.3 Coupling, stochastic monotonicity

Coupling is an extremely powerful technique in probability theory and is of particular use in interacting particle systems. In particular, it is intimately connected with the concept of attractive processes. The definition of a coupling is as follows: a coupling of two probability distributions μ and ν is a pair of random variables (X,Y) defined on a single probability space such that the marginal distribution of X is μ and the marginal distribution of Y is ν. That is, a coupling (X,Y) satisfies $\mathbb{P}(X=x) = \nu(x)$ and $\mathbb{P}(Y=y) = \nu(y)$, [8]. In this project, we consider a coupling of two stochastic processes which a method of forcing the processes to depend on each other via some non-trivial rules. By the definition of coupling, if we observe one of the processes without observing the others, the process behaves as it is originally constructed.

The coupling technique is linked to the concept of stochastic monotonicity. To discuss stochastic monotonicity, we first consider the partial ordering of two configurations η and ζ defined on a state-space of the form $S = X^\Lambda$ where Λ is a lattice or network and for interacting particle systems $X\subseteq\mathbb{N}$. $\eta\leq\zeta$ if we have $\eta_x\leq\zeta_x$ for all $x\in\Lambda$. If a continuous function, $f: S\to\mathbb{R}$, preserves this partial ordering, the function is said to be increasing, i.e. $f(\eta)\leq f(\zeta)$ for all $\eta\leq\zeta$. The concept of stochastic monotonicity states that; for probability measures μ_1,μ_2 on S: $\mu_1\leq\mu_2$ provided that $\mu_1(f)\leq\mu_2(f)$ for all increasing f, [5].

The link between stochastic monotonicity and coupling is given by the following theorem (Strassen); for probability measures μ_1,μ_2 on S then $\mu_1\leq\mu_2$ if and only if there exists a coupling μ, on the product space, $S\times S$, such that $\mu\Big(\{\eta = (\eta^1,\eta^2) : \eta^1\leq\eta^2\}\Big) = 1$, i.e. the probability of observing partial order is one [6].

A process $(\eta(t) : t \geq 0)$ on S is attractive if the property of stochastic monotonicity, or the partial ordering of configurations, is preserved through time. Therefore, if the driven diffusive system is attractive then canonical stationary measures are stochastically ordered in the number of particles on a fixed lattice of length L, that is $\pi_{L,N} \leq \pi_{L,N+1}$. This implies, there exists a coupling between a particle system with N particles and the one with $N+1$ particles where the stationary measure of the coupled process defines a growth process. This growth rule allows us to sample from the stationary measure $\pi_{L,N}$ with computation time scaling linearly with the number of particles.

3 Models

3.1 Driven diffusive systems

Driven diffusive systems, or lattice gas models, are continuous time Markov processes with state space $X = \mathbb{N}^\Lambda$, where Λ is any countable set, e.g. $\{1, \dots, L\}$. Let $p(x,y)$ be the irreducible, finite range transition probabilities of a single random walker on Λ with $p(x,x) = 0$. For each $x \in \Lambda$, we define $u_x, v_x : \mathbb{N} \to [0,\infty)$ to be two non-negative functions of the number of particles, η_x, at site x, and the product $u_x(n)v_y(m)$ is called the jump-rate, where

$$\begin{aligned} &u_x(n) = 0 \quad \Leftrightarrow \quad n = 0, \\ &v_x(n) > 0 \quad \text{for all} \quad n \geq 0 \end{aligned} \tag{2}$$

for all $x \in \Lambda$. A particle at site x will jump to site y with a rate dependent only on the number of particles at the exit and entry sites, given by $u_x(\eta_x)v_y(\eta_y)p(x,y)$. The process $(\eta(t) : t \geq 0)$ on X is defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x,z\in\Lambda} u_x(\eta_x)v_z(\eta_z)p(x,z)(f(\eta^{x\to z}) - f(\eta)), \tag{3}$$

with $\eta^{x\to z}$ denoting the configuration after a particle has jumped from site x to site z.

3.2 Zero-range process

The zero-range process, [1], is a driven diffusive model of the from (3), where the jump rates depend only on the number of particles at the exit site. Therefore, $v_x(n) \equiv 1$ for all $x \in \Lambda$. The zero-range process $(\eta(t) : t \geq 0)$ on X is then defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x,z\in\Lambda} u_x(\eta_x)p(x,z)(f(\eta^{x\to z}) - f(\eta)). \tag{4}$$

The zero-range process is called homogeneous, if for all $x \in \Lambda$ and all $k \in \mathbb{N}$ we have $u_x(k) \equiv u(k)$. See Figure 1, for example dynamics of the zero-range process.

3.3 Stationary product measure

In this section, we discuss the explicit form of the stationary measures for driven diffusive systems. We consider the driven diffusive process $(\eta(t) : t \geq 0)$ on the one-dimensional lattice $\Lambda = \mathbb{Z}/L\mathbb{Z} = \{1, \dots, L\}$ with periodic boundary conditions, with state space $S = \mathbb{N}^\Lambda$, functions $u_x(n)$ & $v_x(n)$ and jump probabilities $p(x,y)$ with N particles. The stationary measures are known to be product measures, which means the measure is factorisable and therefore, the single-site distributions are independent. In the grand-canonical ensemble, the stationary measures are parametrised by a number ϕ called the fugacity. The fugacity parameter controls the average number of particles at a site whereas the actual number of particles in the system is a random variable. In the canonical ensemble, i.e. the number of particles is fixed, the process is irreducible and hence, has a unique stationary measure.

3.3.1 The grand-canonical stationary measure

Under certain conditions, the process defined by the generator (3) exhibits stationary product measures, [12] and for the zero-range process [4], $\nu_\phi^\Lambda[\eta] = \prod_{x\in\Lambda} \nu_\phi^x[\eta_x]$ for each $\phi \geq 0$. Where the single-sites are distributed according to the measures ν_ϕ^x, which are of the form

$$\nu_\phi^x[\eta_x = n] = \frac{w_x(n)(\lambda_x\phi)^n}{z_x(\phi)} \quad \text{and } w_x(n) = \prod_{k=1}^{n} \frac{v_x(k-1)}{u_x(k)}, \tag{5}$$

provided that the partition function (normalisation)

$$z_x(\phi) = \sum_{n=0}^{\infty} \frac{w_x(n)(\lambda_x\phi)^n}{z_x(\phi)} < \infty \quad \text{for all } x \in \Lambda. \tag{6}$$

The fugacity parameter, ϕ, controls the average number of particles per site and $(\lambda_x : x \in \Lambda)$ is a harmonic function solving $\sum_{x\in\Lambda} (\lambda_x p(x,y) - \lambda_y p(y,x)) = 0$ for all $y \in \Lambda$. We restrict our discussion to processes with $\lambda_x \equiv 1$ for all $x \in \Lambda$. For the existence of a stationary product measure (5), we require that $z_x(\phi) < \infty$ for each $x \in \Lambda$. We denote the domain of definition of the stationary product measure D_ϕ^Λ, where

$$D_\phi^\Lambda = \{\phi \geq 0 : z_x(\phi) < \infty \text{ for all } x \in \Lambda\}.$$

Since $z_x(\phi)$ is a power series in ϕ, the domain D_x is of the form $[0, \phi_c^x)$ or $[0, \phi_c^x]$, where $\phi_c^x = \left(\lambda_x \limsup_{n\to\infty} w_x(n)^{1/n}\right)^{-1}$ is the radius of convergence of the power series $z_x(\phi)$. Therefore, the domain of the product measure (5) is given by

$$D_\phi^\Lambda = [0, \phi_c^\Lambda) \quad \text{or} \quad [0, \phi_c^\Lambda] \quad \text{where} \quad \phi_c^\Lambda = \inf_{x\in\Lambda} \phi_c^x. \tag{7}$$

The family of measures

$$\{\nu_\phi^L : \phi \in [0, \phi_c]\}$$

is called the grand-canonical ensemble.

3.3.2 The canonical stationary measure

Models of the form (3) conserve the number of particles and are irreducible on the state space $X_{L,N} = \{\eta \in \Lambda_L | \sum_L (\eta) = N\}$. Thus, it has a unique stationary measure $\pi_{L,N}$ on $X_{L,N}$ which, can be written as a conditional product measure

$$\pi_{L,N}[\eta] = \nu_\phi^\Lambda\left[\eta \Big| \sum_L (\eta) = N\right] = \frac{\mathbb{I}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda_L} w_x(\eta_x)\lambda_x^{\eta_x}, \tag{8}$$

where $Z_{L,N} = \sum_{\eta\in X_{L,N}} \prod_{x\in\mathbb{N}^{\Lambda_L}} w_x(\eta_x)\lambda_x^{\eta_x}$ defines the canonical partition function and the family of measures

$$\{\pi_{L,N} : N \in \mathbb{N}\}$$

is called the canonical ensemble.

3.4 The attractive zero-range process and condensation

Driven diffusive systems are known to be attractive if the jump-rates are increasing at the exit site and decreasing at the entry site, [3] and for the zero-range process see [4]. Therefore, canonical stationary measures $\pi_{L,N}$ are stochastically ordered in the number of particles, $\pi_{L,N} \leq \pi_{L,N+1}$. It is also known that the homogeneous attractive zero-range process does not exhibit condensation [12]. Hence, we restrict our discussion on condensation to the non-homogeneous zero-range process which is known to exhibit condensation [10]. To discuss condensation in particle systems given by the generator (3), we must first introduce the average density of particles, $R_x(\phi)$, for a site $x \in \Lambda$, as follows

$$\rho_x(\phi) \equiv R_x(\phi) = \nu_\phi^x(\eta_x) = \sum_{k=1}^{\infty} k w_x(k)(\lambda_x \phi)^k.$$

Let ϕ_c be the radius of convergence of the grand canonical partition function defined in equation (7). Since the density function and partition function have the same radius of convergence, the critical density at site $x \in \Lambda$ is defined as

$$\rho_x^c \equiv R_x^c = \lim_{\phi \nearrow \phi_c} R_x(\phi) \in [0, \infty].$$

Therefore, condensation can occur for non-homogeneous systems if there exists a single site $d \in \Lambda$ such that the critical density $R_d^c = \infty$, and for the non-defect sites, $x \in \Lambda \setminus \{d\}$, the critical densities are finite. For finite lattices, Λ, condensation of this form implies that in the limit of infinite particle number, $N \to \infty$, we have $\frac{1}{N} \lim_{N\to\infty} \eta_d = 1$, [10]. i.e. almost all particles condense on the defect site.

3.5 Numerical methods

To compare our methods of growing configurations to the stationary product measure of the zero-range process, we have to numerically calculate the canonical partition function, $Z_{L,N}$, which can be written as $Z_{L,N} = \nu_1^\Lambda(\sum_L (\eta) = N)$. The single-site marginals under the grand canonical measure are of the form

$$\pi_{L,N}(\eta_x = k) = \frac{w(k) Z_{L-1,N-k}}{Z_{L,N}}.$$

Therefore, utilising the product form of the stationary measures, two iterative formulas for calculating the partition are given by

$$Z_{L,N} = \sum_{k=0}^{N} w(k) Z_{L-1,N-k} \quad \text{where} \quad w(k) = \prod_{i=0}^{k} u(i)^{-1} = Z_{1,k},$$

and

$$Z_{2L,N} = \sum_{k=0}^{N} Z_{L,k} Z_{L,N-k},$$

therefore, one can compute the partition function easily for system sizes of the form $L = l^2+1$, where $l \in \{0, 1, 2, \dots\}$.

3.6 Birth-death process

In this project, we consider birth-death processes to construct a continuous time growth process, which are used to simulate the condensation phenomena, discussed above, in the zero-range process with a single defect site. A birth-death process is a continuous-time Markov process with state space $S = \mathbb{N}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S, \quad i \xrightarrow{\beta_i} i-1 \text{ for all } i \geq 1.$$

The rate α_i is called the birth-rate and β_i the death-rate. The generator-matrix for the birth-death process is therefore given by

$$G = \begin{pmatrix} -\alpha_0 & \alpha_0 & 0 & \dots & \dots \\ \beta_1 & -\alpha_1 - \beta_1 & \alpha_1 & 0 & \dots \\ 0 & \beta_2 & -\alpha_2 - \beta_2 & \alpha_2 & 0 \\ \vdots & 0 & \beta_3 & -\alpha_3 - \beta_3 & \alpha_3 \end{pmatrix}.$$

The master equation, describing the evolution of the distribution of the process X_t, is as follows

$$\frac{d}{dt}\mathbb{P}(X_t = n) = \alpha_{n-1}\mathbb{P}(X_t = n-1) + \beta_{n+1}\mathbb{P}(X_t = n+1) - (\alpha_n + \beta_n)\mathbb{P}(X_t = n). \tag{9}$$

4 Results: Growth for homogeneous zero-range processes with no condensation

In this section, we focus on the homogeneous zero-range process, where the jump rate is an increasing function of particle number on a finite lattice with L sites, denoted Λ_L. We construct a coupling of two zero-range process and show sufficient conditions for a coupling measure to be stationary by using relations between the stationary product measure, (8). We also connect this coupling measure to a discrete time growth rule, $p_N(\eta, x)$ in the following way:

Consider the zero-range process (4) with configuration $\eta \in X_{L,N}$. Assuming the configuration η is generated from the stationary measure $\pi_{L,N}$, equation (8), we may generate a sample from the stationary measure $\pi_{L,N+1}$ by adding a particle to site $x \in \Lambda_L$ with probability $p_N(\eta, x)$. Therefore, for all $\xi \in X_{L,N+1}$ we need

$$\pi_{L,N+1}(\xi) = \sum_x \pi_{L,N}(\xi - \delta_x) p_N(\xi - \delta_x, x). \tag{10}$$

See Figure 1 for example dynamics of the growth process $p_N(\eta, x)$.

4.1 Coupling the zero-range process

Let $(\eta(t) : t \geq 0)$ and $(\xi(t) : t \geq 0)$ be two zero-range processes defined via the same jump-rates such that $\eta \in X_{L,N}$ and $\xi \in X_{L,N+1}$. Since zero-range processes conserve total mass, we have that $\eta(t)$ and $\xi(t)$ contain N and $N+1$ particles, respectively for all time. We focus on the homogeneous attractive zero-range process therefore, we consider site-independent jump rates which are increasing in the number of particles, i.e.

$$\text{if } n \geq m \quad \text{then} \quad u(n) \geq u(m).$$

We may construct a coupling on the joint state space $(X_{L,N}, X_{L,N+1})$ between process η and ξ such that, $\xi = \eta + \delta_y$ for some $y \in \Lambda_L$. The extra particle in the $\xi(t)$ process is called a second class particle. The coupling is constructed such that

1. The marginals of the coupled process are two zero-range processes with N and $N+1$ particles respectively, defined by the generator (4). As a consequence the stationary coupled process is a coupling of measures $\pi_{L,N}$ and $\pi_{L,N+1}$, [4].
2. Particles move together as much as possible.

This coupling is often called a basic coupling. The coupled process behaves via the following rules; for the site with the second class particle

$$\begin{aligned} \left.\begin{array}{l} \xi_y = n+1 \\ \eta_y = n \end{array}\right\} &\xrightarrow{u(\xi_y) - u(\eta_y)} \left\{\begin{array}{l} \xi_y = n \\ \eta_y = n \end{array}\right. \\ \left.\begin{array}{l} \xi_y = n+1 \\ \eta_y = n \end{array}\right\} &\xrightarrow{\quad u(\eta_y) \quad} \left\{\begin{array}{l} \xi_y = n \\ \eta_y = n-1 \end{array}\right. \end{aligned} \tag{11}$$

For the remaining sites, both processes jump at rate $u(\eta_x) = u(\xi_x)$. Since we construct the coupling by fixing the ξ process to be of the form $\eta + \delta_y$, we may map the state space of the coupled process to $(X_{L,N}, \Lambda_L)$. Therefore, configurations in the coupling are of the form (η, y), where $y \in \Lambda_L$ is the site of the second class particle. The generator for the coupled process is given as

$$\begin{aligned} \mathcal{L}f(\eta, y) = &\sum_{x,z \in \Lambda_L} [u(\eta_x) p(x,z) (f(\eta^{x \to z}, y) - f(\eta, y))] \\ &+ \sum_{z \in \Lambda_L} (u(\eta_y + 1) - u(\eta_y)) p(y,z) (f(\eta, z) - f(\eta, y)). \end{aligned} \tag{12}$$

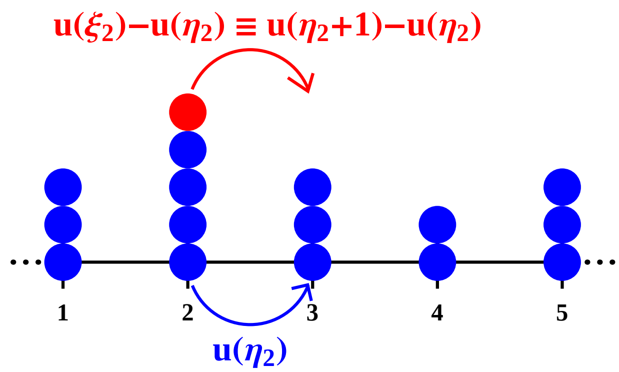


Figure 2. Example configuration of the coupled dynamics. The η process is shown in blue and the second class particle is shown in red. The jump-rates are defined according to equation (11). The coupling can only be constructed for increasing jump-rates u as we need $u(\xi_x) - u(\eta_x) \geq 0$ for all $x \in \Lambda_L$ for the dynamics to be well defined.

See Figure 2 for exampled dynamics of the coupled process.

Consider a probability measure $\mu \in \mathcal{P}(X_{L,N}, \Lambda_L)$, acting on the state space of the coupled process, which is the unique stationary measure of the coupled process. The stationary measure is unique since the process in an irreducible Markov process on a finite state space. This implies the marginals of μ are given by $\pi_{L,N}$ and $\pi_{L,N+1}$. Therefore, μ is a coupling of two stationary zero-range processes. We use the stationary measure μ to define a growth rule as follows:

Statement 1 *Let $\mu(\eta, y) = \mu(y|\eta)\pi_{L,N}(\eta)$ be the stationary measure of the coupled process,* (12)*, and $\mu(y|\eta) = \alpha_\eta(y)$ be the location of the second class particle given the configuration $\eta \in X_{L,N}$. Then*

(a) $\alpha_\eta(y)$ is a valid growth rule according to (10).

(b) For totally asymmetric dynamics $\alpha_\eta(y)$ must satisfy the following condition

$$\alpha_\eta(y)\left[u(\eta_y+1) - u(\eta_y)\right] - \alpha_\eta(y-1)\left[u(\eta_{y-1}+1) - u(\eta_{y-1})\right] = \sum_x u(\eta_x)\left[\alpha_{\eta^{x\to x-1}}(y) - \alpha_\eta(y)\right]. \quad (13)$$

Proof (a) For each $x \in \Lambda_L$ we have

$$\mu(\xi - \delta_x, x) = \pi_{L,N}(\xi - \delta_x)\alpha_{\xi-\delta_x}(x),$$

and we also know

$$\begin{aligned}
\pi_{L,N+1}(\xi) &= \sum_\eta \mu(\eta, y) \\
&= \sum_{\xi\,:\,\eta=\xi-\delta_y} \mu(\xi - \delta_y, y) \\
&= \sum_{y\,:\,\eta=\xi-\delta_y} \mu(\xi - \delta_y, y) \\
&= \sum_{y\,:\,\eta=\xi-\delta_y} \pi_{L,N}(\xi - \delta_y)\alpha_{\xi-\delta_y}(y).
\end{aligned}$$

Therefore, $\alpha_\eta(y)$ is a valid growth rule according to equation (10).

(b) A measure $\mu \in \mathcal{P}(X_{L,N}, X_{L,N+1})$ is stationary if the following equality holds (1)

$$\mu(\mathcal{L}f) = 0 \quad \text{for all observables } f. \tag{14}$$

This implies

$$\begin{aligned}\mu(\mathcal{L}f) = &\sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y) \sum_{x,z \in \Lambda_L} [u(\eta_x) p(x,z)(f(\eta^{x \to z}, y) - f(\eta, y))] \\ &+ \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y) \sum_{z \in \Lambda_L} (u(\eta_y + 1) - u(\eta_y)) p(y,z)(f(\eta, z) - f(\eta, y)) = 0. \end{aligned}\tag{15}$$

For simplicity of computation, only consider totally asymmetric dynamics in the zero-range process. That is, dynamics of the form $p(x,y) = \begin{cases} 1 & \text{if } y = x+1 \\ 0 & \text{otherwise} \end{cases}$. Therefore, equation (15) maybe simplified to the following form

$$\begin{aligned}&\sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y) \sum_{x \in \Lambda_L} u(\eta_x)(f(\eta^{x \to x+1}, y) - f(\eta, y)) \\ &+ \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y)(u(\eta_y + 1) - u(\eta_y))(f(\eta, y+1) - f(\eta, y)) = 0. \end{aligned}\tag{16}$$

To show the stationarity condition (13) we make two changes of variables.

1. For all $x, y \in \Lambda_L$, we change the variable in the sum over η. Such that, we highlight the jumps from configurations $\eta' = \eta^{x+1 \to x}$ into η;

$$\begin{aligned}\sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y) \sum_{x \in \Lambda_L} u(\eta_x) f(\eta^{x \to x+1}, y) &= \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \sum_{x \in \Lambda_L} \mu(\eta^{x+1 \to x}, y) u(\eta_x^{x+1 \to x}) f(\eta, y) \\ &= \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \sum_{x \in \Lambda_L} \pi_{L,N}(\eta^{x+1 \to x}) \alpha_{\eta^{x+1 \to x}}(y) u(\eta_x + 1) f(\eta, y). \end{aligned}\tag{17}$$

The following identity holds for the canonical stationary product measure $\pi_{L,N}$ on $X_{L,N}$

$$\pi_{L,N}(\eta^{x+1 \to x}) = \frac{u(\eta_{x+1})}{u(\eta_x + 1)} \pi_{L,N}(\eta) \tag{18}$$

Hence, equation (17) can be transformed to the following form

$$\begin{aligned}&\sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y) \sum_{x \in \Lambda_L} u(\eta_x) f(\eta^{x \to x+1}, y) \\ &= \sum_{\eta \in X_{L,N}} \sum_{x,y \in \Lambda_L} \pi_{L,N}(\eta) \alpha_{\eta^{x+1 \to x}}(y) u(\eta_{x+1}) f(\eta, y). \end{aligned}\tag{19}$$

2. Replace y with $y-1$, to highlight the extra particle jumping into position y;

$$\begin{aligned}&\sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y)(u(\eta_y + 1) - u(\eta_y)) f(\eta, y+1) \\ &= \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \mu(\eta, y-1)(u(\eta_{y-1} + 1) - u(\eta_{y-1})) f(\eta, y) \\ &= \sum_{\eta \in X_{L,N}} \sum_{y \in \Lambda_L} \pi_{L,N}(\eta) \alpha_\eta(y-1)(u(\eta_{y-1} + 1) - u(\eta_{y-1})) f(\eta, y) \end{aligned}\tag{20}$$

Thus, substituting equations (19) and (20) into equation (16) we find the stationarity condition (13).

Remark

(a) If we assume that $\alpha_\eta(y)$ depends only on the configuration at site y and not the total configuration, we can reduce (13) to the following form

$$\alpha_\eta(y)\left[u(\eta_{y+1})+u(\eta_y+1)\right]-\alpha_\eta(y-1)\left[u(\eta_{y-1}+1)-u(\eta_{y-1})\right]=u(\eta_{y+1})\alpha_{\eta^{y+1\to y}}(y)+u(\eta_y)\alpha_{\eta^{y\to y-1}}(y). \quad (21)$$

(b) We may extend this result to more general dynamics, other than totally asymmetric and homogeneous, to find the following condition on α_η

$$\begin{aligned}&\sum_{x,z\in\Lambda_L} u_z(\eta_z)p(x,z)\alpha_{\eta^{z\to x}}(y)+\sum_{z\in\Lambda_L}\left[u_z(\eta_z+1)-u_z(\eta_z)\right]p(z,y)\alpha_\eta(z)\\ &=\sum_{x,z\in\Lambda_L} u_x(\eta_x)p(x,z)\alpha_\eta(y)+\sum_{z\in\Lambda_L}\left[u_y(\eta_y+1)-u_y(\eta_y)\right]p(y,z)\alpha_\eta(y). \end{aligned} \quad (22)$$

Therefore, if we find an $\alpha_\eta(y)$, which satisfies equations (13) or (22) we have a valid growth rule for the zero-range process.

4.2 Constant jump rates

In this section, we consider the zero-range process (4) with jump rates of the form $u(k)=1$ for all $k\geq 1$. According to the coupled process defined in Section 4.1, the second class particle placed at $y\in\Lambda_L$ must wait for the site to become empty in the η process before being allowed to jump. i.e. $u(\eta_y+1)-u(\eta_y)=1$ if $\eta_y=0$ and $u(\eta_y+1)-u(\eta_y)=0$ if $\eta_y\geq 1$. In this case, the growth probability

$$p_N(\eta,x)=\frac{(\eta_y+1)}{L+N}, \quad (23)$$

satisfies equation (10). To compare the growth distribution with the theoretical canonical measure, we plot the tail of the distributions given by the equation

$$\begin{aligned}Tail_\eta(n)&=\frac{\#\{i\in\Lambda_L|\eta_i\geq n\}}{L}\\ t_x(n)&=\mathbb{I}_{\eta_x\geq n}\end{aligned} \quad (24)$$

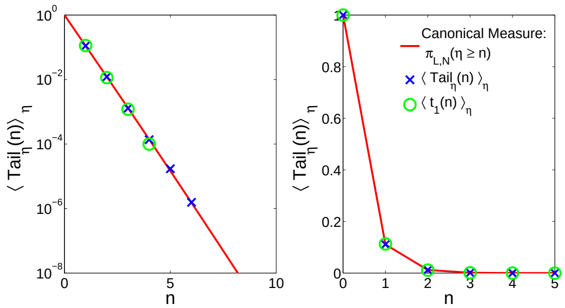
$\langle Tail_\eta(n)\rangle_\eta$ is the average tail over realisations of η and therefore, both a spatial average and an average over realisations. $\langle t_x(n)\rangle_\eta=\langle \mathbb{I}_{\eta_x\geq n}\rangle_\eta$ is the average over realisations of η only considering the single-site $x\in\Lambda_L$. Figure 3 shows simulation of the growth process compared with the numerical values of the single site stationary measure $\pi_{L,N}[\eta_x\geq n]$.

To show analytically, the growth rule $p_N(\eta,y)=\frac{\eta_y+1}{L+N}$ is a valid growth process for a configuration $\eta\in X_{L,N}$ and rates rates $u(k)=1$ for $k\geq 1$, we must first calculate the stationary measure, $\pi_{L,N}$, which is given by

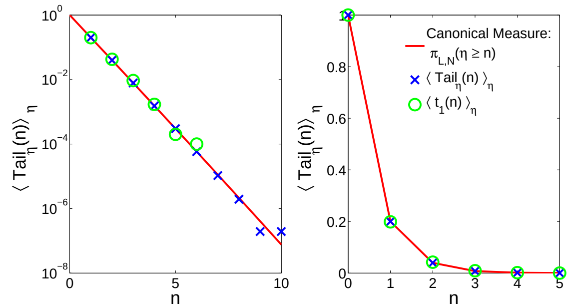
$$\pi_{L,N}(\eta)=\frac{1}{Z_{L,N}},$$

since the stationary weights $w(n)\equiv 1$ for all $n\in\mathbb{N}$. Each configuration η has equal weight which implies $Z_{L,N}$ is equal to the number of configurations in $X_{L,N}$. Therefore,

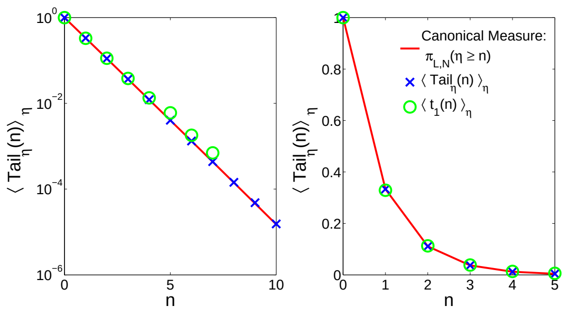
$$Z_{L,N}=\binom{N+L-1}{N}. \quad (25)$$



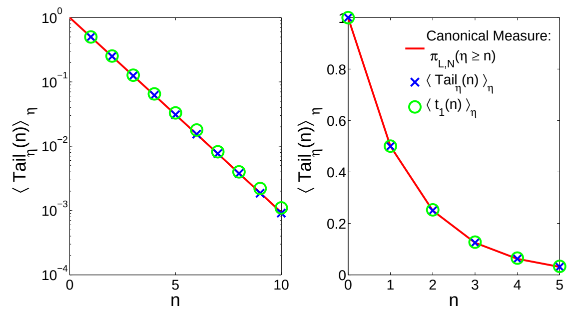
(a) $N = 64$



(b) $N = 128$



(c) $N = 256$



(d) $N = 512$

Figure 3. Comparing the growth process with the single-site marginal of the canonical stationary measure (red line) of the zero-range process for system size $L = 513$, 3a - $N = 64$ to 3d - $N = 512$. We compare both the tail of the distribution (left) using a semi-log plot and the distribution (right). The growth process (blue crosses) is given by $\langle Tail_\eta(n)\rangle_\eta$, and the single-site growth process (green circles) is given by $\langle t_x(n)\rangle_\eta = \langle \mathbb{I}_{\eta_x \geq n}\rangle_\eta$, i.e. the fraction of realisations where a single-site $x \in \Lambda_L$ has more than n particles, both averaged over 10000 realisations. See equation (24). Due to the constraint $\sum_{x\in\Lambda_L} \eta_x = N$ the configurations are slightly correlated, which leads to $\langle Tail_\eta(n)\rangle_\eta$ having a slightly higher tail than the canonical measure. However, the statistics for $\langle Tail_\eta(n)\rangle_\eta$ are obviously better than the single-site distribution, $\langle \mathbb{I}_{\eta_x \geq n}\rangle_\eta$, since there are $L \times 10000$ averages compared to 10000.

For a configuration $\xi \in X_{L,N+1}$, we have from equation (10)

$$\begin{aligned}
\pi_{L,N+1}(\xi) = \frac{1}{Z_{L,N+1}} &= \sum_{x\in\Lambda\,:\,\xi_x>0} \pi_{L,N}(\xi - \delta_x)\frac{\xi_x}{L+N} \\
&= \frac{1}{Z_{L,N}(L+N)} \sum_{x\in\Lambda\,:\,\xi_x>0} \xi_x \\
&= \frac{N+1}{Z_{L,N}(L+N)} \\
&= \frac{N!(L-1)!}{(N+L-1)!}\frac{N+1}{L+N} \\
&= \frac{(N+1)!(L-1)!}{(N+L)!} \\
&= \frac{1}{Z_{L,N+1}}.
\end{aligned}$$

Thus, the growth rule $p_N(\eta, x) \propto \eta_x + 1$ holds. Figure 3 shows simulation results of the growth process against numerical values of the canonical measure $\pi_{L,N}$.

In Section (4.1), we showed that for a measure of the form $\mu(\eta, y) = \alpha_\eta(y)\pi_{L,N}(\eta)$ to be stationary, α must have the property given in equation (13), which was derived using totally asymmetric dynamics, that is

$$\alpha_\eta(y)\left[u(\eta_{y+1}) + u(\eta_y + 1)\right] - \alpha_\eta(y-1)\left[u(\eta_{y-1}+1) - u(\eta_{y-1})\right] = u(\eta_{y+1})\alpha_{\eta^{y+1\to y}}(y) + u(\eta_y)\alpha_{\eta^{y\to y-1}}(y).$$

The growth rule, $\alpha_\eta(y) = \frac{\eta_y+1}{L+N}$, shown above to be a correct growth rule, does not in fact satisfy equation (13). Consider a configuration η, such that, $\eta_{y-1} = 0$ for some $y \in \Lambda_L$ and $\eta_z > 0$ for all $z \neq y-1 \in \Lambda_L$. Therefore, assume equation (13) holds for $\alpha_\eta(y) \propto \eta_y + 1$ then

$$\begin{aligned} 2\alpha_\eta(y) - \alpha_{\eta^{y+1\to y}}(y) - \alpha_{\eta^{y\to y-1}}(y) &= \alpha_\eta(y-1) \\ \implies 2\frac{\eta_y+1}{L+N} - \frac{\eta_y+2}{L+N} - \frac{\eta_y}{L+N} &= \frac{\eta_{y-1}+1}{L+N} \\ \implies 0 = \frac{1}{L+N} &\quad \text{contradiction.} \end{aligned}$$

We see the contradiction with the totally asymmetric coupled dynamics and growth. Hence, we see that the growth process is non-unique since a valid growth rule does not satisfy the stationary coupling condition (13).

Although the growth rule $\eta_y + 1$ does not necessarily satisfy equation (13), due to the totally asymmetric dynamics, we can show it does satisfy equation (22), assuming mean-field dynamics. Mean-field dynamics implies the network Λ_L to be totally connected and the jump probabilities $p(x, y)$ show no biases. i.e. $p(x, y) = 1/(L-1)$ for all $x \neq y \in \Lambda_L$.

4.3 Independent random walkers

The zero-range process with jump-rates of the form $u(k) = k$ for all $k \in \mathbb{N}$ is equivalent to independent random walkers each jumping with rate 1 on Λ_L. The jump probabilities of the random walkers are translation invariant and therefore, have uniform stationary distribution. From this, we expect the growth probability $p_N(\eta, x)$ and stationary coupling measure $\alpha_\eta(y)$ to be independent of the configuration η. In fact, $\alpha_\eta(y) = \frac{1}{L}$ for all $y \in \Lambda_L$ solves equation (13). Since $\alpha_\eta(y) = \frac{1}{L}$ is independent of the configuration η, the following statement holds

$$\alpha_{\hat\eta}(y) = \frac{1}{L} = \alpha_\eta(y) \quad \text{for all} \quad \eta, \hat\eta \in X_{L,N}.$$

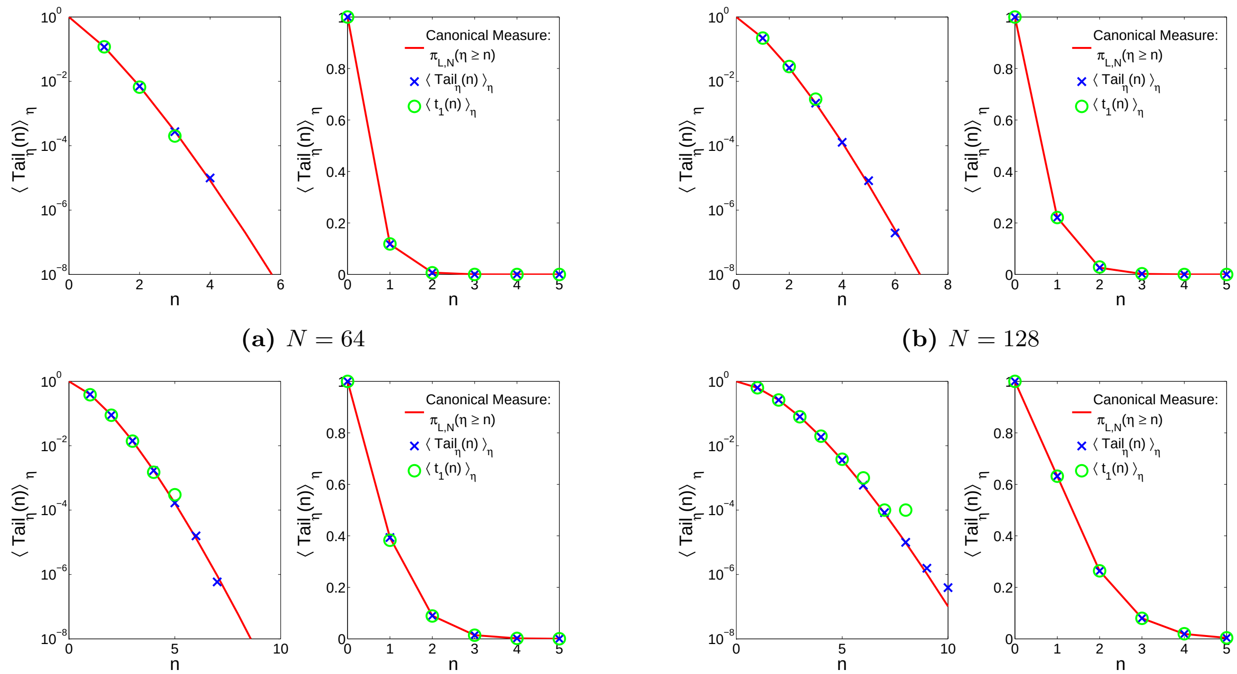
Therefore, substituting our ansatz into equation (21) with rates $u(k) = k$ we get

$$\begin{aligned} \alpha_\eta(y)\left[u(\eta_{y+1}) + u(\eta_y+1)\right] &= u(\eta_{y+1})\alpha_{\eta^{y+1\to y}}(y) + u(\eta_y)\alpha_{\eta^{y\to y-1}}(y) + \alpha_\eta(y-1)\left[u(\eta_{y-1}+1) - u(\eta_{y-1})\right] \\ \frac{1}{L}\left[\eta_{y+1} + (\eta_y+1)\right] &= \frac{1}{L}\left[\eta_{y+1} + \eta_y + (\eta_{y-1} + 1 - \eta_{y-1})\right] \\ &= \frac{1}{L}\left[\eta_{y+1} + \eta_y + 1\right]. \end{aligned}$$

Figure 4 shows simulation results of the growth process against numerical values of the canonical measure $\pi_{L,N}$.

5 Results: Growth via a pure-birth process

In this section, we consider growing configurations via a continuous time birth-death process defined in Section 3.6. A pure-birth process is a birth-death process where $\beta_i = 0$ for all $i \in S$. For given rates α_i of the pure-birth process X_t, we compare the distribution of X_t with the zero-range process and its associated grand canonical measure, (4) and (5). We calculate the distribution of X_t using generating functions, [6], and the master equation for the process, (9).



(a) $N = 64$ **(b)** $N = 128$

(c) $N = 256$ **(d)** $N = 512$

Figure 4. Comparing the growth process with the single-site marginal of the canonical stationary measure (red line) of the zero-range process for system size $L = 513$, 4a - $N = 64$ to 4d - $N = 512$. We compare both the tail of the distribution (left) using a semi-log plot and the distribution (right). The growth process (blue crosses) is given by $\langle Tail_\eta(n)\rangle_\eta$, and the single-site growth process (green circles) is given by $\langle t_x(n)\rangle_\eta = \langle \mathbb{I}_{\eta_x \geq n}\rangle_\eta$, i.e. the fraction of realisations where a single-site $x \in \Lambda_L$ has more than n particles, both averaged over 10000 realisations. See equation (24). Due to the constraint $\sum_{x\in\Lambda_L} \eta_x = N$ the configurations are slightly correlated, which leads to $\langle Tail_\eta(n)\rangle_\eta$ having a slightly higher tail than the canonical measure. However, the statistics for $\langle Tail_\eta(n)\rangle_\eta$ are obviously better than the single-site distribution, $\langle \mathbb{I}_{\eta_x \geq n}\rangle_\eta$, since there are $L \times 10000$ averages compared to 10000.

5.1 Pure-birth and the constant rate zero-range process

We consider birth-rates of the form

$$\alpha_i = i + 1 \quad \text{for all} \quad i \in S \tag{26}$$

and compute the distribution of the pure-birth process at time t by using the generating function

$$F(s,t) = \sum_{k=0}^{\infty} s^k p_k(t), \tag{27}$$

where $p_k(t) = \mathbb{P}(X_t = k)$. The generating function in this form has the following properties; the boundary conditions are given by

$$\begin{aligned} F(1,t) &= 1 \quad \text{for all } t \geq 0 \\ F(s,0) &= 1 \quad \text{for all } s, \end{aligned} \tag{28}$$

and for each $n \in S$ the distribution of X_t, $p_n(t)$, is given by

$$p_n(t) = \frac{1}{n!} \frac{\partial^n F}{\partial s^n}\bigg|_{s=0}. \tag{29}$$

Therefore, differentiating the generating function, see equation (27), and substituting the master equation (9) with rates defined in equation (26) we find that

$$\begin{aligned}
\frac{\partial}{\partial t}F(s,t) &= \sum_{k=0}^{\infty} s^k \frac{d}{dt}p_k(t) \\
\frac{\partial}{\partial t}F(s,t) &= \sum_{k=0}^{\infty} s^k \left[k p_{k-1}(t) - (k+1)p_k(t)\right] \\
\frac{\partial}{\partial t}F(s,t) &= s^2 \frac{\partial}{\partial s}F(s,t) + sF(s,t) - s\frac{\partial}{\partial s}F(s,t) - F(s,t). \qquad (30)
\end{aligned}$$

The explicit solution to the partial differential equation (30) with the boundary conditions given in (28) is of the form

$$F(s,t) = \frac{1}{e^t + s - se^t}. \qquad (31)$$

Hence, we may calculate the distribution of the pure-birth chain with birth rates $\alpha_k = k+1$ using equation (29).

$$\mathbb{P}(X_t = n) = e^{-t}(1-e^{-t})^n. \qquad (32)$$

Following equation (5), the single-site marginals for the spatial-homogeneous zero-range process with rates $u(k) = 1$ for all $k > 0$ are given by

$$\nu_\phi(n) = \frac{1}{z(\phi)}\phi^n$$

where the partition function (normalisation) is given by

$$z(\phi) = \sum_{n=0}^{\infty} \phi^n = \frac{1}{1-\phi} \quad \text{for all } \phi \in [0,1).$$

The domain of the marginal is $[0,1)$, since in this case the radius of convergence for the partition function is $\phi_c = 1$. Thus, the marginals of the stationary measure have the following form

$$\nu_\phi(n) = (1-\phi)\phi^n \quad \text{for all} \quad \phi \in [0,1). \qquad (33)$$

By setting the fugacity parameter, ϕ, in the grand-canonical measure of the zero-range process equal to $1-e^{-t}$, we see a direct comparison to the single-site marginal (33) with the distribution of the pure-birth chain at time t (32). More explicitly

$$\nu_{1-e^{-t}}(n) = e^{-t}(1-e^{-t})^n = \mathbb{P}(X_t = n). \qquad (34)$$

We may also connect the distribution of the pure-birth process to the canonical stationary measure $\pi_{L,N}$, which are independent of the fugacity parameter, ϕ. To do this we consider growing L pure-birth process independently and conditioning on there being N particles in total. Let $\zeta_x(t)$ be the number of particles in chain x at time t. Since we grow chains independently, it is easy to see the joint distribution of chains will be a product measure. Therefore, the joint distribution is given by

$$\begin{aligned}
\mathbb{P}(\zeta_1(t),\ldots,\zeta_L(t) | \sum_{k=0}^{L}\zeta_k(t) = N) &= \frac{1}{Z_{L,N}}\prod_{k=1}^{L} e^{-t}(1-e^{-t})^{\zeta_k(t)} \\
&= \frac{1}{Z_{L,N}} e^{-Lt}(1-e^{-t})^N
\end{aligned}$$

The normalisation, $Z_{L,N} = e^{-Lt}(1-e^{-t})^N \sum_{\xi:\sum\xi=N} 1$. We see the time dependence cancels out and the conditional measure is exactly the form of the canonical measure for the zero-range process with constant rates.

5.2 Pure-birth and independent random walkers

Similar to Section 5.1, we compare the pure-birth process with birth rates of the form

$$\alpha_i = \alpha > 0 \quad \text{for all} \quad i \in S \tag{35}$$

with the single site marginals for the grand-canonical measure of the zero-range process, with jump rates $u(k) = k$ for all $k \in \mathbb{N}$. Using the same method for analysing the generating function in Section 5.1, we find it satisfies the following condition

$$\frac{\partial}{\partial t} F(s,t) = \alpha F(s,t)(s-1)$$

which, for the given boundary conditions (28), has solution

$$F(s,t) = e^{-\alpha t(s-1)}.$$

The distribution of the pure-birth chain with birth rates given by (35) is of the form

$$\mathbb{P}(X_t = n) = \frac{(\alpha t)^n}{n!} e^{-\alpha t}. \tag{36}$$

To compare this distribution to the zero-range process we must first compute the single-site marginals (5).

$$\nu_\phi(n) = \frac{1}{z(\phi)} \frac{\phi^n}{n!}$$

where the partition function (normalisation) is given by

$$z(\phi) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} = e^{\phi} \quad \text{for all } \phi.$$

In this case the partition function converges for all ϕ and we see the domain of the marginal is $[0, \infty)$. The marginals of the stationary measure have the form of a Poisson distribution

$$\nu_\phi(n) = \frac{\phi^n}{n!} e^{-\phi}. \tag{37}$$

Again, we may directly compare the single-site marginals (37) with the distribution of the pure-birth chain at time t (36) by setting the fugacity parameter $\phi = \alpha t$. More explicitly

$$\nu_{\alpha t}(n) = \frac{(\alpha t)^n}{n!} e^{-\alpha t} = \mathbb{P}(X_t = n). \tag{38}$$

Similar to the previous section, we may find the canonical stationary measure of the zero-range process by conditioning on L independent birth-chains having N particles in total.

5.3 Growth and condensation

Consider a non-homogeneous zero-range process (4) with a single-site defect, labelled $d \in \Lambda_L$. It is known that for such a process, the presence of a defect site is sufficient conditions for condensation [10]. In this case, the defect site has a jump rate which is slower than the surrounding sites. For example, condensation occurs if the jump-rates are of the following form

$$\begin{aligned} u_x(k) &= 1 \quad \text{for all } x \in \Lambda_L \setminus \{d\} \quad \text{and } k \geq 1, \\ u_d(k) &= r < 1 \quad \text{for all } k \geq 0. \end{aligned} \tag{39}$$

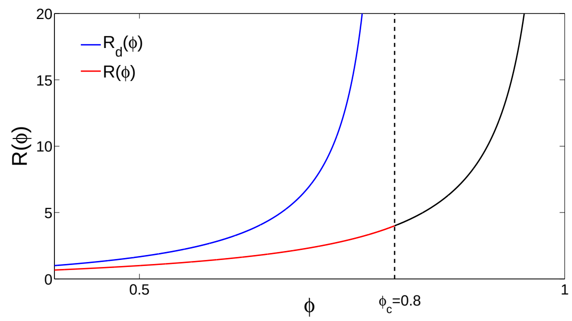


Figure 5. Plots of the density functions for both fluid sites (red) and condensed defect site (blue). This shows the density of the defect site diverges at the critical density, here $\phi_c = 0.8$, while the fluid sites have finite density.

Setting the jump rate of the non-defect site to one sets the time-scale for the dynamics.

To see the condensation transition, we must first calculate ϕ_c^Λ as defined in equation (7). For each $x \in \Lambda_L \setminus \{d\}$ we have

$$z_x(\phi) = \sum_{n=0}^{\infty} \phi^n = \frac{1}{1-\phi} < \infty \quad \text{for } \phi \in [0,1) \quad \text{so } \phi_c^x = 1.$$

For the defect site d, the stationary weight is of the form $w_d(n) = 1/r^n$, which implies,

$$z_d(\phi) = \sum_{n=0}^{\infty} \left(\frac{\phi}{r}\right)^n = \frac{1}{1-\phi/r} < \infty \quad \text{for } \phi \in [0,r) \quad \text{so } \phi_c^d = r.$$

Therefore, the domain of the product measure (5) is given by $D_\phi^\Lambda = [0,r)$. We see, for $r \in (0,1)$, the system exhibits condensation. This is because the critical densities for sites $x \in \Lambda \setminus \{d\}$ are of the form

$$R_x^c = \lim_{\phi \nearrow r} R_x(\phi) = R_x(r) = \frac{r}{1-r} < \infty$$

and for the defect site the critical density becomes

$$R_d^c = \lim_{\phi \nearrow r} R_d(\phi) = \lim_{\phi \nearrow r} \frac{\phi}{r-\phi} = \infty.$$

Thus, we find for the zero-range process with rates defined in equation (39), and $r \in (0,1)$, the system will split into two subsets; a set of sites with finite critical density, often called the fluid phase, and a set of sites with infinite critical density, called the condensed phase.

5.3.1 The canonical partition function

For the zero-range process with one defect site, labelled $d \in \Lambda_L$, the canonical measure becomes

$$\pi_{L,N}(\eta) = \frac{\prod_{x\in\Lambda_L} w_x(\eta_x)}{Z_{L,N}} = \frac{1}{r^{\eta_d}} \frac{1}{Z_{L,N}}.$$

To calculate the partition function $Z_{L,N}$, we first consider a system with k particles on the defect site. Since the stationary weights, w, of the non-defect sites are equal to one, the number of configurations η with $\eta_d = k$ is given by

$$\binom{N-k+L-2}{N-k}.$$

This is the number of ways of placing $N-k$ particles on $L-1$ sites. Therefore, the partition function is given by

$$Z_{L,N} = \sum_{i=0}^{N} r^{-i} \begin{pmatrix} N-i+L-2 \\ N-i \end{pmatrix}.$$

5.3.2 Growth

We may generalise the growth rate α_i to become time-dependent to sample from the stationary measure of the zero-range process with one-defect site (39), which exhibits condensation. In Section 5.1, we show that the distribution of a pure-birth process with rates $\alpha_i = i+1$ is equivalent to the single-site marginal of the grand-canonical measure of the zero-range process, with rates $u(k) = 1$. We consider L independent pure-birth chains growing in time, where the defect site d grows with rates as both a function of time and position, and the non-defect sites grow with rate (26), more explicitly

$$\begin{aligned} \alpha_i^d(t) &= (i+1)h(t) \quad \text{for all} \quad i \in S, \\ \alpha_i^x(t) &= (i+1) \quad \text{for all} \quad x \neq d \text{ and for all } i \in S \end{aligned} \tag{40}$$

where $h(t)$ is some unknown function of time t. We make this generalisation because we want to constrain our growth process to have the correct marginals for all time. Therefore, we must find an explicit form for $h(t)$ such that marginals of the growth process correspond to the marginal of the zero-range process. As in Section 5.1, we may find the distribution for the defect site using a generating function and the master equation. The master equation is given by

$$\frac{d}{dt} p_k^d(t) = h(t) \left[k p_{k-1}^d(t) - (k+1) p_k^d(t) \right],$$

where $p_k^d(t) = \mathbb{P}(X_t^d = k)$ is the distribution of the defect site. The time derivative of the generating function (27) is given by

$$\frac{\partial}{\partial t} F(s,t) = h(t) \left[s^2 \frac{\partial}{\partial s} F(s,t) + sF(s,t) - s\frac{\partial}{\partial s} F(s,t) - F(s,t) \right],$$

and its explicit solution

$$F(s,t) = \frac{1}{e^{H(t)} + s - se^{H(t)}} \quad \text{where} \quad H(t) = \int_0^t h(s) ds.$$

The function $H(t)$ is called the intensity function. Therefore,

$$\mathbb{P}(X_t^d = n) = e^{-H(t)} \left(1 - e^{-H(t)}\right)^n$$

Once again, we may compare the single-site marginal for the zero-range process with the distribution of the birth-process. The marginal for a defect site, $u(k) = r$, is given by $\nu_\phi^d(n) = \left(\frac{\phi}{r}\right)^n \left(1 - \frac{\phi}{r}\right)$. Thus, the single-site marginal and the pure-birth distribution are equivalent for

$$\mathbb{P}(X_t^d = n) = \nu_\phi^d(n) \implies \frac{\phi}{r} = 1 - e^{-H(t)}, \tag{41}$$

while for the non-defect sites, $\nu_\phi^x(n) = \phi^n(1-\phi)$,

$$\mathbb{P}(X_t^x = n) = \nu_\phi^x(n) \implies \phi = 1 - e^{-t}. \tag{42}$$

Since the fugacity parameter ϕ is a fixed quantity parametrising the stationary measure of the zero-range process, we may solve explicitly for $h(t)$ as follows

$$r\left(1 - e^{-H(t)}\right) = 1 - e^{-t}$$
$$\implies H(t) = \int_0^t h(s)ds = -\log\left(1 - \frac{1-e^{-t}}{r}\right)$$
$$\implies h(t) = \frac{1}{1+(r-1)e^t}.$$

The inverse of the intensity function $H(t)$ is given by

$$H^{-1}(t) = -\log\left(1 - r(1-e^{-t})\right). \tag{43}$$

Notice for $r \in (0,1)$, the functions $h(t), H(t) \to \infty$ as $t \to T^\star = -\log(1-r) < \infty$, associated with a finite time blow-up. Finite time blow-up implies as $t \to T^\star$ the birth-rate of the defect site, $\alpha_i^d(t) = h(t)(i+1)$, will tend to infinity and therefore, condensation will occur at the defect site. Using the limit $t \to T^\star$, the critical fugacity may be obtained using equation (42),

$$\phi_c = \lim_{t \to T^\star}\left(1 - e^{-t}\right) = r.$$

We also regain the correct densities at both the fluid and condensed sites, in the limit of $t \to T^*$.

5.3.3 Simulating the growth process

The process defined by birth-rates (40) is an example of a non-homogeneous Poisson process, where the growth-rates depend on both space and time. To simulate such a process, we use the inverse of the intensity function and points sampled from a homogeneous Poisson process. We generate a sequence of events $\{E_1^d, \dots, E_n^d\}$, where the waiting time between events E_i^d and E_{i-1}^d are exponentially distributed with rate α_i, and the label d is there to indicate the defect site. Then, the sequence of events $\{T_1^d = H^{-1}(E_1^d), \dots, T_n^d = H^{-1}(E_n^d)\}$ are the event times of the non-homogeneous process with jump-rates $\alpha_i h(t)$. Figure 6 shows the transform from a homogeneous Poisson process to an non-homogeneous Poisson process using the intensity function $H(t)$ and jump-rates $\alpha_i = i+1$. More explicitly, for events E_{i-1}^d and E_i^d, the waiting time is given by

$$E_i^d - E_{i-1}^d \sim exp(i+1), \tag{44}$$

then, the waiting time between events in our non-homogeneous process are

$$T_i^d - T_{i-1}^d \sim H^{-1}(exp(i+1)). \tag{45}$$

We can simulate the $L-1$ non-defect sites jointly using the following property of the exponential distribution, and hence the waiting time between events,

$$\text{if } \tau_i \sim exp(r_i) \quad \text{for any i, then} \qquad \min\{\tau_1, \dots, \tau_{L-1}\} \sim exp(\sum_i r_i).$$

Let $\{\hat{E}_1, \dots, \hat{E}_n\}$ be the sequence of events happening in the non-defect site then, by the property above, and since the birth-rate of a single-site x is $\alpha_\eta^x = \eta_x + 1$, the waiting time between events $\hat{E}_{i-1}$ and $\hat{E}_i$ becomes

$$\hat{E}_i - \hat{E}_{i-1} \sim exp(N_t + L - 1)$$

where, $N_t = -\eta_d + \sum_{i=0}^{L} \eta_i$ is the number of particles in the non-defect site at time t. The particle is then added to site $x \neq d$, which contains k particles with probability

$$\frac{\alpha_\eta^x}{N_t + L - 1}.$$

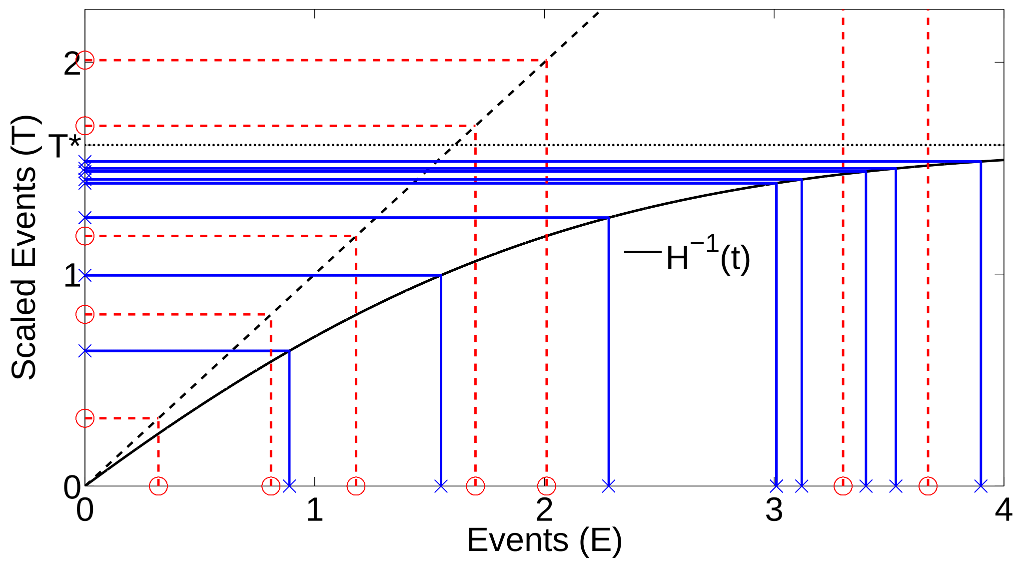


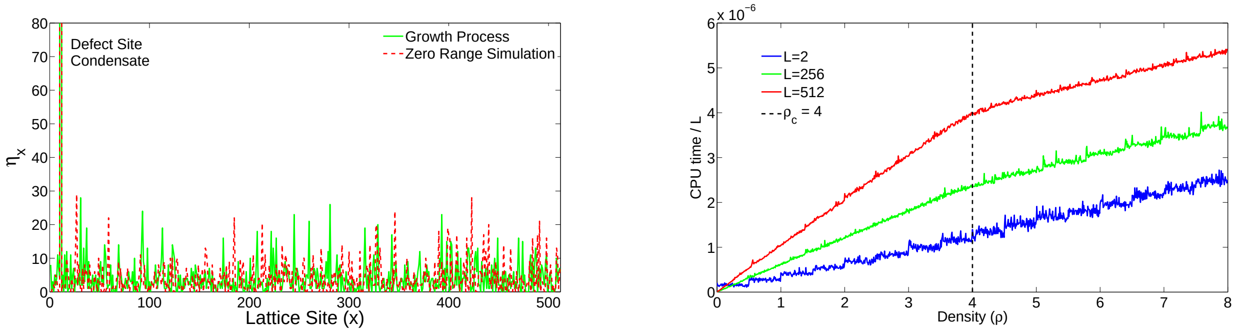
Figure 6. Illustration of the distribution of events generated by a non-homogeneous Poisson process (blue lines) and a homogeneous Poisson process (red dashed). Considering only the non-homogeneous process, the unscaled events, (E), are distributed along the x-axis, with exponentially distributed waiting times (44), and the scaled events, T, along the y-axis with waiting times given by equation (45). The scaled events are the actual events used by the process. The events for the defect site correspond to the non-homogeneous process (blue lines) and are found using the inverse of the intensity function, (43), which diverges in finite time and therefore, the defect site events will be constrained in the interval $[0, T^{*}]$. Now considering the homogeneous process (red dashed), since the birth-rates are not time-dependent no scaling is involved hence, unscaled events are exactly the scaled events. The black dashed line is the function $y = x$, used since there is no time scaling for the homogeneous process.

We use a binary search algorithm to place the particle at the correct site.

Figure 6 shows visually how to simulate the non-homogeneous process, where the actual events the system uses are along the y-axis. We include both a non-homogeneous (blue lines) and homogeneous process (red dashed) to show the difference in waiting times and how the non-homogeneous process will eventually dominate, as $t \to T*$. Figure 7a shows two example configurations comparing the simulated zero-range process with our growth process, and the condensate is placed at site 10. Figure 7b shows the cpu-times of the growth process for three system sizes. The cpu-time grows linearly with density above and below the critical density. However, the rate at which the cpu-time grows is different. Since the growth process utilizes the binary search algorithm less often the rate is slower above the critical density, which is confirmed with the system of two sites, $L = 2$, as the binary search algorithm is never used.

5.3.4 Non-homogeneous growth and independent random walkers

In Section 5.2, we saw that the single-site marginals for the zero-range process with rates $u(k) = k$, where equivalent to the distribution of a pure-birth process, with constant growth rate $\alpha > 0$. In fact, the distribution is a Poisson distribution with parameter $\phi = \alpha t$. We also find that the radius of convergence of the grand canonical partition function, $z(\phi) = e^{\phi}$, is $[0, \infty)$; the critical density, $R^c = \infty$ and therefore, the zero-range process does not exhibit condensation. Similar to Section 5.3.2, we generalise the birth-rate of the birth-process to depend on time, to correspond to a defect-site moving at a different rate to the background but with the same functional form, e.g.



(a) **(b)**

Figure 7. (Left) Example configurations for the zero-range process (red dashed) and growth process (green lines) with the defect at site 10. (Right) CPU time, scaled by system size, for continuous time birth-process with $r = 0.8$. $L = 2$ - blue, $L = 256$ - green and $L = 512$ - red. For large system sizes CPU time grows faster below the critical density than above. This is due to the growth process utilizing the binary search algorithm less above the critical density.

$u_d(k) = rk$. The birth rates are defied by

$$\alpha_i^d(t) = \alpha g(t) \quad \text{for all} \quad i \in S,$$
$$\alpha_i^x(t) = \alpha \quad \text{for all} \quad x \neq d \text{ and } i \in S.$$

It is easy to see, via the methods in Section 5.2 and 5.3.2 that the generating function and distribution of the defect chain are of the form

$$F(s,t) = e^{-\alpha(s-1)G(t)} \quad \text{and} \quad \mathbb{P}(X_t^d = n) = \frac{(\alpha G(t))^n \, e^{-\alpha G(t)}}{n!} \quad \text{where} \quad G(t) = \int_0^t g(s)ds.$$

To solve for $G(t)$, we again solve the system of equations

$$\phi = \alpha t,$$
$$\frac{\phi}{r} = \alpha G(t),$$

and therefore, $G(t) = t/r$. Unlike the previous Section, we see the intensity function $G(t)$, does not diverge in finite time. Thus, the defect site never gains such an advantage such that it exhibits condensation.

6 Discussion and future work

In this work, we have studied two methods for growing stationary configurations of the zero-range process. The first growth rule was a discrete time process where the probability of adding a particle to a particular site depends on current configuration. It is clear that such a growth process is not necessarily unique and that many processes may give rise to the same measure. We showed that the stationary measure of two coupled zero-range processes gives a valid growth rule and we give sufficient conditions for the conditional measure, $\mu(y|\eta) = \alpha_\eta(y)$, of the coupled dynamics to be stationary. The growth rule $\alpha_\eta(y)$ depended on the underlying dynamics of the zero-range process, and we saw a valid growth rule for the constant jump-rate case did not satisfy equation (13), which was found using totally asymmetric dynamics. To avoid spatial correlations in the growth rule, we should consider the mean-field case, where Λ_L is totally connected. Simulations of our growth process confirm the results shown analytically and also show a great speed-up in computation times compared to MCMC techniques. Finding an explicit solution to equation (22) for general jump-rates is still an interesting problem to solve.

The second method was via pure-birth processes, where the birth-rates were motivated by known growth probabilities, $p_N(\eta, x)$. We first show the one-to-one correspondence between time in the birth-process and the fugacity parameter that parametrises the grand-canonical measure of the zero-range process. Generalising the jump-rates at a defect site, such that the birth-rate became a function both time and position, allowed us to sample from the stationary measure of a zero-range process with a defect site. We found the intensity function, the time-integrated rate, exhibits finite time blow up in the case where condensation is known to exist and does not in the case where condensation is not observed. We present simulation results of the pure-birth process and zero-range process to illustrate the condensate at the defect. Unfortunately, the generalisation did not shed light on a possible solution for equation (22).

The homogeneous zero-range process is attractive for increasing jump-rates and is known to not exhibit condensation. However, in general it is not known if the property of attractiveness implies that a process does not exhibit condensation in homogeneous systems. We may generalise our coupling of the zero-range process to couple the general driven-diffusive models (3) if the exit-rate u is increasing and entry-rate v is decreasing in the number of particles. Interestingly, unlike the zero-range process, the coupled dynamics allow for the extra particle to jump against the model dynamics. Once again, we may find sufficient conditions on the conditional measure for the dynamics to be stationary by using the functional form of the stationary product measure of the process.

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