## **Political Evolution**

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# Abstract

We study a two dimensional opinion model on a predefined agent-agent network structure which defines the possible interactions between agents. When two agents interact they either come to a compromise or ignore each other according to some parameter dependent on distance in opinion space. If the two agents decide to ignore each other then with a given probability they decide to break their connection in the network and rewire this broken link to another agent uniformly across unconnected agents. We also introduce agents with static opinion whom we interpret as media outlets and these media outlets have a defined range over which they can influence agents and pull them towards their political outlook. Thus we have defined a tolerance for both media outlet and individual agent. As we vary this tolerance for both media and agent we see various different results such as the formation of clusters of agents with the same opinion or complete disagreement within society. The resulting degree distributions also vary with the tolerance parameters, rewiring rule and number of media sources. We often see a transition from the initially defined network structure, usually a network with a Poisson distribution for the degree sequence to that with other distributions.

# 1 Introduction

## 1.1 Motivation

Opinions on various issues are not usually static and can be subject to change from various internal and external sources. Such examples of internal sources can be an individuals lack of faith or confidence in their own opinion which could imply that their opinion or political outlook is subject to a diffusion or random walk bounded by the defined space. We consider examples of external sources to be an individuals interactions with neighbours in their social network or interactions with media outlets and political parties. On a short time scale, political parties and media outlets may be modelled as individuals with a static opinion, where their goal is to maximise the number of regular individuals they are connected with at any given time-step. However, over long time-scales media outlets or political parties may see a shift in their outlook as they become more centralist, in an attempt to win votes of viewing figures, or extreme. Media sources and political parties typically have the ability to polarise the opinions of a population with either individuals agreeing or disagreeing with their agenda. Typically, individuals have some qualitative value of trust in their opinion or the opinion of other sources which defines how and who they interact with. This trust parameter could be seen as some interaction radius in which individuals will seek to compromise on an issue or ignore each other depending on a distance function in opinion space and tolerance/trust.

The model that we construct in section 2 defines the opinion space to be the two dimensional unit box. Often, the political spectrum is defined as [0, 1] the unit interval when being close to zero, left wing, is interpreted as being a communist and close to one, right wing, a fascist. Our interpretation of a two dimensional opinion space is to assign one axis to be the usual political outlook and the other axis to represent an individuals economic outlook. For example, an individual may see themselves as a liberal on political policy and at the same time hold conservative views about economic policy. An argument for modelling a political outlook as two dimensional can be found in [1], called Grid-group theory.

The most simple models consider the system of agents to have equal tolerance which implies that all areas of the opinion spectrum behave in the same way and we expect to find a number of opinion clusters dependent of the value of this tolerance. We see the results of this in section 3.3. Introducing opinion dependent tolerance parameters could result in more interesting dynamics. A simple example of tolerance in this form is to allow an area of opinion space to become more intolerant relative to the remaining opinion space. By doing this we may expect to see a shift in the mean opinion of agents towards this intolerable region and an area of opinion space devoid of any agents dependent on how we initialise tolerance parameters of this form. We discuss the results of this process in section 3.4. Another implementation of opinion dependent parameters can be such that the further away you are from the central opinion the more intolerant you become. Again we discuss these results in section 3.5.

In our model we allow in the dynamics for a link in the network to be broken with some probability dependent on the tolerance discussed previously and the distance of the two agents in opinion space. We also restrict that the number of links in the network is conserved. By allowing for this restriction, we ensure that our dynamics don't allow the network to become either the complete network, where every individual is connected to every other, or the empty network, where there exists no links between individuals. Once we break a link in the network we must then define a rewiring rule which is possibly dependent on the opinion space. If we consider the case when individuals i and j break their link, we rewire this link from i or j to an unconnected individual, either preferentially according to distance in opinion space or randomly.

Our goal in this project is to understand how two dimensional opinions evolve in time. We are looking to investigate the formation of clusters different opinion and will also study how the underlying opinion dynamics change the degree distribution of the social network. Next, we attempt to understand the role of the noise process in the dynamics of opinion formation and what if affects it has on the opinion distributions and on the network structure. The role of external sources such as media outlets can have a huge impact on the formation of opinions, so we aim to understand how the inclusion of media sources with fixed opinion change the structure of the opinion distributions.

We begin this paper with a short introduction and discussion into various existing models, see section 1.2. Then move on to the definition of our opinion model in section 2 and outline the parameters in section 2.1. In section 3 we show the results of simulations, building in complexity throughout the section. Lastly a discussion of the results and possible extensions to the project in section 4.

#### **1.2** Current Literature

#### 1.2.1 Linear Voter model

The linear voter model [2], [3] is a interacting particle system (IPS), which is a Markov chain where the state space has a particular structure, defined on a lattice (or network)  $\Lambda = \{0, \ldots, L\}$ . The state space is given by the set of all configurations

$$\eta = \{\eta(x) | x \in \Lambda\} \in S = \{0, 1\}^{L}$$

that is each agent in the network has a binary opinion at a given time.

The dynamics of the IPS is given by the local continuous-time transition rates:

$$\eta \to \eta^x$$
 with rates  $c(\eta, \eta^x)$ .

The (liner) voter model has rates

$$c(\eta, \eta^x) = \sum_{y-x} q_{x,y}(\eta(x)(1-\eta(y)) + (1-\eta(x))\eta(y)) \quad \text{for all} \quad x \in \Lambda.$$

#### 1.2.2 Properties of the voter model

- Symmetric under relabelling opinions
- Has two absorbing states  $\eta = 0, 1$

- $\pi(t) \to \alpha \delta_{\underline{0}} + (1-\alpha)\delta_{\underline{1}}$  where  $\alpha \in [0,1]$  depends on initial conditions
- For infinite  $\Lambda$  other stationary distributions may exist, where both opinions persist

#### 1.2.3 The Deffuant model

The Deffuant model, Kozma and Barrat [5] and [6], is a model which N agents are given continuous opinion  $\phi \in [0, 1]$ . At each time-step, two neighbouring agents are chosen at random. If their opinions are within some tolerance,  $|\phi(i, t) - \phi(j, t)| \leq d$  then the two agents interact and compromise on their opinion, according to the rule

$$\phi(i, t+1) = \phi(i, t) + \mu(\phi(j, t) - \phi(i, t))$$
  
$$\phi(j, t+1) = \phi(j, t) - \mu(\phi(j, t) - \phi(i, t))$$

where  $\mu \in [0, \frac{1}{2}]$  is a convergence parameter. For simplicity they take the convergence parameter  $\mu = \frac{1}{2}$ 

There is an underlying network structure which defines the possible interactions each agent may have. They make the network structure dynamic by introducing a probability of breaking a link in the network. That is, at each time-step an agent and its neighbour is chosen at random and with probability w an attempt to break the connection is made; if  $|\phi(i,t) - \phi(j,t)| \ge d$ , the link (i,j) is removed and a new link is created. With probability 1-w the opinions evolve according the previous interaction rule. The authors also introduce a parameter  $p \in [0,1]$ , such that with probability p they rewire the link from agent i to another agent k and with probability 1-p rewiring from agent j.

#### **1.2.4** Properties of the Deffuant model

The dynamics of the Deffuant model are heavily dependent on the parameter p, the probability of breaking a connection between two individuals. In the case when p = 0, the network is static, where they see a fragmented phase for very small tolerance parameters. The fragmented phase is characterized by a lack of extensive-size clusters. This fragmented phase is not seen in the case when  $p \neq 0$  since rewiring allows small groups of agents with similar tolerance and opinion to form. They also find non-trivial correlations between the size of a group and average degree in the case where the network is dynamic. Depending on the value of p this average degree is either a sub or super linear function of group size.

#### 1.2.5 The Compromise Process

The model analysed by Ben-Naim, Krapivsky and Redner [7]. Each agent is initially assigned an opinion  $x \in [-\Delta, \Delta]$  from some specified distribution. Randomly selected pairs undergo sequential interactions. Such interactions are restricted to agents whose opinion lies below a threshold that is set to unity without loss of generality. When two agents with opinion  $x_1$  and  $x_2$  interact, they both acquire the average opinion

$$(x_1, x_2) \to (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2))$$
 if  $|x_2 - x_1| < 1$ ,

if  $|x_2 - x_1| > d$  no interaction occurs.

Let P(x,t) denote the proportion of agents that have opinions in the range [x, x + dx] at time t. This distribution evolves according to the rate equation

$$\frac{\partial}{\partial t}P(x,t) = \int \int_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1,t) P(x_2,t) [\delta(x - \frac{x_1 + x_2}{2}) - \delta(x - x_1)]$$

The first two moments,  $M_0$  and  $M_1$ , are conserved under dynamics of this form where  $M_k = \int dx x^k P(x, t)$ . The authors restrict themselves to flat distributions on the interval  $[-\Delta, \Delta]$  and their goal was to determine the final state  $P_{\infty}(x) = P(x, \infty)$ .

#### **1.2.6** Properties of the compromise process

For  $\Delta < \frac{1}{2}$  all agents can interact and the second moment obeys the equation  $\dot{M}_2 + \frac{M_0M_2}{2} = M_1^2$ . Using  $M_1 = 0$  they find  $M_2(t) = M_2(0)e^{-\frac{M_0t}{2}}$  with  $M_0 = 2\Delta$ . All agents approach the centre opinion and the system reaches consensus with  $P_{\infty}(x) = M_0\delta(x)$ . For larger values of  $\Delta$  the final distribution of opinions is a sequence of non-interacting cluster, that is to say  $P_{\infty} = \sum_{i=1}^{p} m_i \delta(x - x_i)$  Integrating the rate equation the authors found that the number of clusters grow while they increase  $\Delta$  as a series of bifurcations and there exists three types of clusters: major clusters, minor clusters and a central cluster.

There exists a variety of other opinion models, both continuous and discrete. The models discussed thus far usually lead to consensus and is usually dependent on network structure. To explain persistent opinion fluctuations in society agents with static opinion are introduced. Accemoglu et al. [8] show that with the introduction of stubborn agents, often referred to as zealots, the dynamics of the stochastic gossip process never lead to consensus [9].

## 2 Our Model

We define the opinion space to be the  $[0,1]^2$  box (unit box in two dimensions) and we consider a variation of the compromise process and Deffuant model. The set of agents will be defined as A with |A| = N, where N is the number of agents. All agents  $i \in A$  are initially uniformly assigned an opinion  $\phi_i(0) \in [0,1]^2$ .

Also all agents  $i \in A$  are assigned a tolerance parameter  $R_i \in [0, \sqrt{2}]$  which defines the radius that an agent will consider changing their opinion when interacting with another agent. There is also an underlying network structure which defines the possible interactions between agents. Throughout this paper we consider both static and dynamic networks.

When an agent  $i \in A$  interacts with  $j \in A$ , if  $d(\phi_i(t), \phi_j(t)) \leq R_i$  then,

$$\phi_i(t') = f_i(\phi_i(t), \phi_j(t)) + \eta_i$$

where  $\eta_i \sim \mathcal{N}(0, \epsilon_i), \epsilon_i > 0$  accounts for noise.

If  $d(\phi_i(t), \phi_j(t)) > R_i$  then with probability  $b_i \in [0, 1]$ , the link between agent *i* and *j* is broken. To conserve the number of links in the network, we choose to rewire this broken link either from agent *i* or agent *j* to another unconnected agent in the system. We rewire with equal probability across unconnected agents.

We also define a set of media outlets to be M, with |M| being the number of media outlets in the system. Similar to the agent-agent dynamics there exists a agent-media network which encodes the possible interactions a media source will have with the population. Unlike the agent-agent dynamics agents  $i \in A$  are not allowed to influence media outlet  $z \in M$ . The update rule for agent-media dynamics is;

If  $d(\phi_i(t), \phi_z(t)) \leq R_z$  then,

$$\phi_i(t') = f_i(\phi_i(t), \phi_z(t)) + \eta_i$$

where  $R_z \in [0, \sqrt{2}]$  is another tolerance parameter, which defines the reach the media has on an individual in the population.

#### 2.1 The Parameters

- For each  $i \in A$  we have  $R_i \in [0, \sqrt{2}]$  defining the tolerance for each agent.
- For each  $i \in A$  we have  $\epsilon_i > 0$  defining the variance in the normally distributed noise term.
- For each  $z \in M$  we have  $R_z \in [0, \sqrt{2}]$  defining the reach a media outlet has on an agent.
- For each  $i, j \in A$  we have  $b_i \in [0, 1]$  defining the probability of breaking a link between i and j in the network conditional on agent j being out of tolerance range.

## 2.2 The Algorithm

To simulate the model, we first build the network using a graph sampling algorithm [4], which takes a decreasing degree sequence and the number of nodes as an input and outputs a random network. We then uniformly consider each link in the network and test if each agent is within tolerance. If they are within their respective tolerance range, then we update the opinions according to the rule defined in section 2.

We also must test that the degree sequence we generate is also graphical. A sequence is called graphical if it can be the degree sequence of some graph. To test the graphicality of a sequence, we use the Erdös-Gallai theorem, taking the formal statement from [4] and [10], which states a non-increasing sequence  $\mathcal{D} = \{d_0, d_1, \ldots, d_{N-1}\}$  is graphical if and only if their sum is even and, for all  $0 \leq k < N - 1$ :

$$\sum_{i=0}^{k} d_i \le k(k+1) + \sum_{i=k+1}^{N-1} \min\{k+1, d_i\}$$

We define one system sweep to be the number of iterations needed to have considered every link in the network at most once.

## 3 Results

## 3.1 Conservation of the mean opinion

If we consider the dynamics as described in section 2, we can show that the mean opinion vector is a conserved quantity, assuming there exists no media outlets. Let  $\Phi(t) = (\phi(1, t), \ldots, \phi(N, t))$  be the N by 2 array of opinions. Then  $\mathbb{E}[\Phi(t)] = \frac{1}{N} [\sum_{i=1}^{N} \phi_x(i, t), \sum_{i=1}^{N} \phi_y(i, t)]$ , where the subscript x and y denotes the dimension of the opinion. We consider the change after one event. In a single event, we either break the link in the network or update the opinion of the two chosen agents. If we break the link, then the agents opinion does not change, keeping the mean opinion fixed. In the latter case we can consider change in each component of the mean vector separately.

Consider only the x component and let i and j be the interacting agents then we have

$$\phi_x(i,t') = \phi_x(i,t) + \frac{1}{4}(\phi_x(j,t) - \phi_x(i,t))$$
  
$$\phi_x(j,t') = \phi_x(j,t) - \frac{1}{4}(\phi_x(j,t) - \phi_x(i,t)).$$

Calculating the mean opinion after this update gives us

$$\mathbb{E}_{x}(\Phi(t')) = \frac{1}{N-2} \sum_{k|k\neq i,j}^{N} \phi_{x}(i,t') + \phi_{x}(i,t) + \frac{1}{4}(\phi_{x}(j,t) - \phi_{x}(i,t)) + \phi_{x}(j,t) - \frac{1}{4}(\phi_{x}(j,t) - \phi_{x}(i,t))$$
$$\mathbb{E}_{x}(\Phi(t')) = \frac{1}{N-2} \sum_{k|k\neq i,j}^{N} \phi_{x}(i,t) + \frac{1}{2}(\phi_{x}(i,t) + \phi_{x}(j,t))$$
$$\implies \mathbb{E}_{x}(\Phi(t')) = \mathbb{E}_{x}(\Phi(t)).$$

This also holds for relabelling of x with y. Therefore the mean is conserved.

#### 3.2**Convergence** Time

To calculate the time to convergence,  $t_{convq}$ , we consider one system sweep as defined in section 2.2. During the sweep we calculate the number of times no update was possible due to the tolerance parameters and whether the distance between two agents was smaller than some small value, c. We set  $c = 10^{-5}$  and we interpret c as some convergence distance. If the sum of these two numbers is equal to the number of iterations in one sweep then we say the system has become static and output  $t_{convg}$  as the number of sweeps before the system becomes static. To state this formally, we say the system is static if;

$$\frac{1}{2}\sum_{i=0}^{N} deg(i) = \sum_{sweep} \mathbb{I}_{d(\phi_i(t),\phi_j(t)) > R_i}(d(\phi_i(t),\phi_j(t))) + \sum_{sweep} \mathbb{I}_{d(\phi_i(t),\phi_j(t)) \le c}(d(\phi_i(t),\phi_j(t)))$$

where deg(i) is the degree of agent *i* in the agent-agent network and

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

We show measurements of the convergence time on both static and dynamic networks in the case with no noise and no media sources.



tolerance parameter on a static network taking 50 Averages

(a) Convergence time rescaled by system size for varying (b) Convergence time rescaled by system size for varying tolerance parameter on a dynamic network, probability of breaking link is 1 taking 50 Averages

#### Figure 1.

We see a qualitative difference in convergence times when comparing the results on a static network with the dynamic network case. This change is due tot the formation of clusters in the case when b = 1 not seen for b = 0. This is explicitly seen in figures 2b and 2c.

#### Agents with equal tolerance 3.3

#### **Opinion distributions** 3.3.1

Allowing for agents with equal tolerance enables us to simplify our model so that we have one control parameter, R. Where for all agents  $i \in A$ , we have  $R_i = R$ . In this subsection we consider the dynamics with no noise,  $\epsilon_i = 0$  for all  $i \in A$ , and no media sources are present. We consider how the properties of the opinion distribution<sup>1</sup> change when we vary the tolerance parameter R. The results we show are the variance of opinions, maximal cluster size<sup>2</sup> and the number of opinion clusters<sup>3</sup>.



Figure 2. Properties of the opinion distribution for both static and dynamic networks, b = 0 and b = 1. Figure 2a is the variance of the opinions, two lines are plotted for each parameter b one for each opinion axis. Figure 2b is the size of the maximal cluster as a function of tolerance R. Figure 2c is the number of opinion clusters on a log-log scale. The error bars presented are of length one standard deviation about the mean.

We see from figure 2 that there exists a transition from uniformly distributed opinions to consensus<sup>4</sup>. We also see that the initial uniform distribution of opinions persists for longer on a static network as shown by the variance of the opinions, size of maximal cluster and number of opinion clusters. By allowing for the network to become dynamic we see opinion clusters almost immediately after R becomes non-zero. Also, when b = 1, the interacting agents reach consensus for smaller tolerance compared to the dynamics on a static network.

#### 3.3.2 Degree Distributions

In this section, we show the resulting degree distributions of the agent-agent networks for the various tolerance parameters using a uniform rewiring rule.

<sup>&</sup>lt;sup>1</sup>We compute the histogram of the opinions allowing for N grid spaces, where N is the number of agents. We make this choice because we initialise the system uniformly. Therefore on average, each grid space should be filled by one agent. Also the size of each grid space is much smaller that the tolerance values we consider, apart from R = 0, agents sharing the same grid space can be seen as in the same opinion cluster.

<sup>&</sup>lt;sup>2</sup>The maximal opinion cluster is the grid space with the most number of agents

<sup>&</sup>lt;sup>3</sup>The number of opinion clusters is the number of non-zero grid spaces in the opinion histogram. Notice the maximal number of opinion clusters is the number of agents, N.

<sup>&</sup>lt;sup>4</sup>All agents with the same opinion



Figure 3. Final degree distributions for 15 tolerance parameters on a network with 1000 interacting agents when initialised with a Poisson distributed degree sequence with mean 10 we also take 100 averages and error bars are of length of one standard deviation. We neglect to show the distributions for larger values of tolerance since the network structure shows little change from that of the original. We see that for tolerance values in the range of 0.06 - 0.10 there exists an increasing number of agents with lower degree consistent with that of fragmentation in the network structure. Otherwise the degree distribution appears to be conserved. Error bars are of length one standard deviation about the mean.

#### 3.4 The effect of non-uniform tolerance parameters: One in tolerant region

In the previous sections, we had only considered tolerance values which are equal across the population, that is for all agents  $i \in A$   $R_i = R$ . We now consider opinion dependent tolerance parameters. For simplicity, we introduce a large tolerance parameter,  $R_{\text{large}}$ , and a small tolerance parameter,  $R_{\text{small}}$ . Agents with opinion  $\phi_i \in [0, 0.5]^2$  will be assigned tolerance  $R_i = R_{\text{small}}$ , otherwise they are assigned tolerance  $R_i = R_{\text{large}}$ . We call the  $[0, 0.5]^2$  region of opinion space the intolerant region, since agents in this region are less likely to change their opinion compared to the other agents in the opinion space. The tolerance of each agent now becomes time dependent. This is because an agent who was initially outside the  $[0, 0.5]^2$  region can eventually fall into the  $[0, 0.5]^2$  region and this agent will change their tolerance from  $R_{\text{large}}$  to  $R_{\text{small}}$  and vice versa.

Setting up the model in this from allows us to consider the dynamics on the  $(R_{\text{small}}, R_{\text{large}})$  – plane with the restriction that  $R_{\text{large}} \ge R_{\text{small}}$ .



#### 3.4.1 Properties of the opinion distributions

Figure 4. Scatter graphs of 100 simulations of bias tolerance parameters on a dynamic network, b = 1. Each figure has  $R_{\text{small}} = 0.02$ . From left to right  $R_{\text{large}} = 0.2, 0.4, 0.5$  The colours of such images represent different simulation results.

We see from figure 4 that the majority of agents get drawn to this intolerant region of opinion space. There also exists a separation in the system around the intolerant region where no agents may remain in limit of large simulation time.

Figure 5 shows various resulting properties of the opinion distribution. We again compare the results for those on a static network, the right column, and on a dynamic network, the left column. Similar to section 3.3.1, the dynamics go through a transition from uniform opinions to consensus. For combinations of  $R_{\text{large}}$  to  $R_{\text{small}}$ , there appears to be three phases a high variance (small maximal cluster), mid variance (mid maximal cluster) and small variance (high maximal cluster). Similar to the previous section, the shift to consensus occurs for smaller combinations of parameters while the network is dynamic, b = 1 compared a static network, b = 0.





**Figure 5.** The figures in the left column are associated with the dynamics on a dynamic network, b = 1, while the figures on the right are associated with the dynamics on the static network, b = 0. Since the opinions are two dimensional, we show the  $\sqrt{o_x^2 + o_y^2}$  where  $o_i$  is the *i*th component of the observable stated in the title of the respective figures. The diagonal line is exactly equivalent to section 3.3. Figure 5g and 5h show the logarithm of the number of opinion clusters.



**Figure 6.** Final degree distributions for 8 combinations of small and large tolerance parameters with 2000 interacting agents when initialised with a Poisson distributed degree sequence with mean 10 where we takes 100 averages and error bars are of length one standard deviation about the mean. In all cases there is a promotion of agents with degree 1 and there also exists a small number agents with a large degree.



Figure 7. Properties of the resulting degree distribution, b = 1 when initialised with a Poisson degree sequence with mean 10 and 2000 interacting agents. Figure 7a shows the variance of the distribution. Figure 7b shows the range in the degree distributions, where the range is defined as the maximal degree minus the minimal degree. Figure 7c shows the degree with the maximum number of agents, where we see a transition from values of around 1 and 2 to values near 10 the initial average degree.

# 3.5 The effect of non-uniform tolerance parameters: More extreme the more intolerant you become

Our second implementation for agents with non equal tolerance parameters is to consider dynamics where an agents tolerance is a function of distance from the central opinion [0.5, 0.5]. For simplicity, we consider a linear function of distance, and for a realistic implementation, we allow the agents furthest away from the central opinion to be the most intolerant. Therefore we define the tolerance of agent  $i \in A$  at time t to be;

$$R(\phi(i,t)) = M\left(1 - \sqrt{2}d(\phi(i,t), 0.5)\right)$$

This definition of tolerance defines a new parameter M, which is the tolerance value at the centre of opinion space. We visualise these tolerance parameters in figure 8.



Figure 8. Heat map of tolerance values where the more extreme an agent is the less tolerant this agent becomes.

In this set up of the model, agents with an initial opinion of [0,0],[0,1],[1,0], and [1,1] will always have zero tolerance. Therefore, consensus can never be achieved if we start with atleast two agents with opinion previously mentioned, and their opinion is non-equal. However, if we allow for large enough M and have no agents with opinion at the edge of the box, consensus may eventually be reached.



#### 3.5.1 Properties of the opinion distributions

Figure 9. 9a shows the variance of the opinions as a function of M. 9b is the size of the largest opinion cluster. 9c is the number of opinion clusters on a log-log scale. Taking 50 averages with error bars of one standard deviation.

Figure 9b shows how the largest opinion cluster grows for increasing M. In the case of the static network, once we see a largest opinion cluster forming, the growth rate of this cluster with respect to M is greater than that for the dynamic network. The increase in size of the largest cluster, on a dynamic network, appears constant. Once again we see the formation of opinion clusters for smaller parameter values when the network is dynamic, compared to the static network. Figure 9c shows the decrease in the number of opinion clusters as we vary M, the number of clusters falls quicker for the dynamic network. However, compared to the previous sections, more opinion clusters persist for higher parameter values. In this case, for large M there still exists regions of opinion space which are intolerant, located around the perimeter of the opinion space, causing there to be more opinion clusters. For resulting degree distributions see appendix, 5.

#### 3.6 Sensitivity to Noise

The dynamics of the model are sensitive to the strength of the noise parameter  $\epsilon_i$ . We show this in figure 10a. The transition from disconcensus to consensus increases in tolerance as we increase the strength of this noise parameter. In this section, we revert back to the case when  $R_i = R$  for all agents.

What we see from figure 10a and figure 10b is an increase in noise will cause the agents to become increasingly extreme in opinion. For small tolerance, this increase in extremism is not seen because the noise term is implemented in the opinion update rule, which for small R on a static network, b = 0 this update rule is rarely used.



Figure 10. Figure 10a: Heat map of the variance for varying noise and tolerance parameters. Considering we have two dimensional opinions to visualise the variance, we plot  $\sqrt{(var(\phi_x))^2 + (var(\phi_y))^2}$ , b = 0. Figure 10b shows an example opinion distribution to show why there exists a larger variance region shown in 10a. Here the parameters are  $\epsilon_i = 0.009$  and R = 0.22.

## 3.7 Media Dynamics

Media outlets are considered as agents with static opinion. One control parameter we have for the media outlets are the tolerance parameters  $R_z$ . We can vary this parameter to model how the opinion distribution changes for media sources with increase influence on the population. That is how the distribution changes for increasing  $R_z$ .

We place two media sources at [0.25, 0.25] and [0.75, 0.75].



Figure 11. Opinion distributions for increasing media influence taking 100 ensemble averages. Each agent  $i \in A$  has tolerance  $R_i = 0.2$ . The media sources are places at [0.25, 0.25] and [0.75, 0.75]. Each media source can connect with every regular agent. We observe a transition of the opinion clusters, for small  $R_z$  we see the formation of two clusters centred at the media outlets. While increasing  $R_z$  three clusters appear, one at the midpoint of the media sources and the other two again centred at the media outlets. For larger  $R_z$  we see one opinion cluster.

Figure 11 shows the transition between two clusters of opinions each centred on the media outlets towards one cluster centred at the mid point of the media outlets. During this transition three opinion clusters coexist with the two clusters centred on the media outlets reducing in size as we increase  $R_z$ .

## 4 Discussion

#### 4.1 Results discussion

In section 3.1 we showed that the mean is conserved when we consider the system with equal tolerance values,  $R_i$  across the population and there exits no media or noise term. This was because each interaction either broke the link in the network or changed the opinion of two agents in such away that the sum of the opinions before and after was the same. We see in section 3.4 that if we relax the assumption where tolerance is equal across the population, then the mean opinion will no longer be conserved. This is explicitly shown in 5b and 5a where the dynamics contained a region of opinion space which was intolerant compared with the overall opinion space. The shift in the mean is due to the interactions now allowing one agent to change opinion while the other remains static and the shift in mean is towards the intolerant region. In this case there are only three possibilities of interaction between an agent outside the intolerant region and one inside: both change opinion, no one changes opinion and the agent with the larger tolerance will change opinion, and in this latter case moves towards the intolerant region.

In section 3.2, we show convergence times for systems of various sizes. In the case of a static network  $t_{\text{convg}}$ , rescaled by system size, peaks around some critical value of the tolerance,  $R_i$ . If one compares these convergence times with figure 1a with figure 2a, this peak in convergence times is where a transition occurs from a large opinion variance to a small opinion variance. It also decreases in tolerance with increasing system size. So, we may expect for an infinite system this transition to consensus to occur at  $R \neq 0$ .

Relaxing the assumption that the network is static, the convergence times showed a qualitative difference to the static network dynamics shown in 1b. In fact, the convergence times decreased as we increased that parameter  $R_i$  for a system size N = 1000 while for N = 100 the convergence times eventually decreased. The increase in convergence time on a dynamic network results from the formation of opinion clusters at lower tolerance, see figure 2b. We investigate how the degree distributions change for varying tolerance, shown in figure 3, for the most part the initial Poisson structure is largely conserved. For extremely small values of  $R_i$ , this process is effectively only a rewiring process on the network allowing only the opinion distribution to vary slightly from its initial position. There exists a range of tolerance,  $R_i = 0.06 \sim 0.14$  where we see an increase in the number of agents with small degree and fewer agents with a large degree. This shows that there are a small number of agents gaining a large proportion of other agents while the majority of agents have a small degree, they have few neighbours.

In section 3.4, we relax the assumption that tolerance,  $R_i$  is equal across the population. We considered how the dynamics change when we have a region of opinion space more intolerant to the rest of opinion space. Figure 4 shows examples of the resulting opinion distributions. For the tolerance parameters shown in figure 4, we see a region of opinion space that is devoid of any agents, this is because no agent may cross this region since they are always outside of tolerance. The size of this region without agents appears to be of length  $R_{\text{large}}$ . Also the majority of the agents lie on the edge of this intolerant region. The images in 4 are all realisations on a dynamic network.

Figure 5 shows the mean and variance of the opinions for both a static network and dynamic network. We see a shift in the mean towards the more intolerant region for all combinations of  $R_{\rm small}$  and  $R_{\rm large}$  which are not equal as discussed previously. This transition from opinion distributions with higher variance to states with lower variance is still present, see figures 5c and 5d, in the dynamics with bias tolerances, this suggests that clusters of opinions are still forming. However, they are forming with a bias towards this intolerant region. Figure 5e and 5f are the sizes of the largest cluster formed for each combination of  $R_{\rm small}$  and  $R_{\rm large}$ , the peak of the opinion distribution. We see from these figures that consensus is achieved

for smaller combinations of tolerance  $R_{\text{small}}$  and  $R_{\text{large}}$  when we allow for b = 1 compared to b = 0. This result was also seen in section 3.3.1. Allowing for the network to become dynamic makes the transition from uniform opinions to consensus appear for smaller combinations of parameters since we now allow for interactions to occur that were usually excluded by non-existing links.

Again, we discuss how the degree distributions change for different combinations of  $R_{\rm small}$  and  $R_{\rm large}$ . Figure 6 shows the resulting distributions for 8 combinations of  $R_{\rm small}$  and  $R_{\rm large}$  as shown by the legends of the figures. Similar to the case for equal tolerance, the network sees an increase in the number of agents with a degree equal to 1. Figure 7c shows the degree which has the maximum number of agents associated with it, the peak of the degree distribution. We see a sharp transition from degree sequences with low maximal degree to a maximal degree defined by the initial networks, degree of 10, 11 or 12. This suggests that if we have a region of opinion space that is extremely intolerant compared to the other regions, then the social network fragments and most agents trust a small number of other agents.

Our second implementation of non-equal tolerances was in section 3.5. We define the tolerance of an agent to be a linear decreasing function of the distance from the centre of opinion space and restricting that the tolerance at the corner of the opinion space was zero. By doing this we implicitly defined a new parameter M, the tolerance of an agent at [0.5, 0.5]. In figure 9b we saw how the size of the largest cluster of opinions changed as a function of M. We see that in the case of the dynamic network this cluster size grows linearly.

In section 3.6 we introduce noise in to the dynamics as defined in section 2. We also revert back to the case when  $R_i$  is equal across the population. Figure 10a tell us that there still exists a transition in the opinion distribution. This transition from no clusters to one cluster grows as we increase the strength of the noise parameter  $\epsilon_i$ . We also see an area in which the variance of opinions is higher than that of a uniform distribution. In figure 10b we see a realisation the opinion distribution in this region. This realisation shows that opinions are forced to the boundary of the box. This can be explained by the implementation of the noise process, if the noise term takes the opinion of an agent outside the defined opinion range we set the opinion at the edge it goes over. In section 4.2 we discuss a different implementation of noise into the dynamics.

In section 3.7 we introduce media sources into the dynamics. We place two media sources at  $z_1 = [0.25, 0.25]$  and  $z_2 = [0.75, 0.75]$  and simulate the model for increasing media tolerance  $R_z$ . Figure 11 shows the formation of opinion clusters centred about the media outlets for small  $R_z$ . As we increase the range these media sources can influence the population we see the emergence of a cluster in the centre of the media sources. This transition goes from 2-3-1 clusters of opinions. There is coexistence of three opinion clusters through this transition. This suggests that media outlets have a large effect on how the distribution of opinions behaves. Media outlets have the power to convince the entire population of agents to their agenda if there tolerance is large enough. In the case of multiple media sources, depending of their respective tolerances, they either share a proportion of the population or they force the the opinion distribution to be between the two sources. In the latter case, no agent shares the media outlets political outlook.

To conclude, we often find a transition from uniformly distributed opinions to consensus in society. We see this explicitly in models with equal tolerance, an area with bias tolerance and the addition of an underlying noise process. We also find that this transition occurs for smaller parameter values, when we allow the network to become dynamic. In all cases discussed, we find the formation of opinion clusters occurs earlier for smaller tolerance values, while the network is dynamic. We develop an implementation of tolerances where consensus may never be achieved. However, clusters still form on the edges of opinion space.

#### 4.2 Future work

We need to run simulations for larger system size while taking a much larger number of ensemble averages.

Currently the noise parameter  $\epsilon_i$  plays too strong a role in the dynamics of opinion formation, as shown in figure 10a. More interesting dynamics could possible be seen if we allow the opinions of agents to act as diffusing particles constrained in the defined opinion space. Such as starting the dynamics from an equilibrium state, i.e. all agents start with the same opinion, and then running the dynamics for an increasing rate of diffusion relative to the rate of our compromise process. One possible result is seeing a sharp change in the structure of the opinion distribution from all agents with the similar opinion too uniform across the opinion space. Other possible results can be seeing the opinion distribution spread out slowly across the opinion space until the compromise process is not seen in the underlying diffusion process.

One may consider how the dynamics change if we were to introduce memory into the system. Currently the next step or iteration in the dynamics only depend on the current state of the system, Markovian (Memoryless). If agents were allowed to remember what their previous opinions were and move with a bias towards or against this previous opinion more interesting dynamics may occur.

We have already seen how media sources, currently defined as agents with static opinion often referred to as zealots, can play a dominant role in the dynamics of opinion formation. Allowing for multiple media sources and relaxing the condition that these sources are static agents, a game could be played where at any given time the media will try to maximise its expected number of unique agents it is influencing while keeping some other agenda or political outlook.

There also exist a plethora of ways to introduce non-equal tolerance parameters across the population. Such as considering increasing or decreasing the tolerance of a agent dependent on how many times this agent has changed opinion. Another example would be to consider increasing an agents tolerance under a successful attempt to change its opinion, argued as the agent becoming more open-minded, and decreasing the tolerance under a unsuccessful attempt, argued as the agent becoming increasingly stubborn and heading towards becoming a zealot. This process on agent tolerance joint with a dynamic network process may lead to more exotic network structures than that seen in this project.

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# 5 Appendix



Figure 12. Final degree distributions for the dynamics defined in section 3.5 for a system of 1000 interacting agents when initialised what a Poission distributed degree sequence with mean 10. We also take 50 averages and error bars are of length one standard deviation about the mean. The yaxis is plotted on a logarithmic scale