

## 1 Special structures in hard optimisation problems

Given a hard optimisation problem, we are interested in identifying conditions on underlying data so that the problem can be solved efficiently. The paper below is an example of a possible research outcomes in this well established research area of combinatorial optimisation.

Having identified special conditions, the next step would be to find out whether the conditions can be easily recognised.

For the problem in the paper below, it is not for example clear whether the recognition problem can be solved efficiently if the set of points is not partitioned into two sub-sets.

The objective of the project would be an investigation of special solvable cases of hard optimisation problems and design and implementation of algorithms for recognising special cases and for possible use of the recognition algorithms as approximation algorithms in the general case.

The paper below is suggested as a starting point for the research in this area. One possible way to proceed could be a computational investigation of cases where the low bounds based on special cases can identified good approximate solutions.

Interests in algorithms and abilities to code and implement algorithms would be a strong advantage.

# The Shoelace TSP: the case of very old shoes

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**Abstract.** We consider the NP-hard bipartite traveling salesman problem (BTSP). We show that the technique developed for the analysis of the TSP can successfully be used to characterize some solvable cases of the BTSP. In particular, we present a polynomial time algorithm that decides whether there exists a renumbering of the cities such that the resulting distance matrix allows one to write the optimal tour implicitly without further analysis of input data. Our results generalize some previously published solvable cases of the BTSP known also as the shoelace problem.

**Keywords:** Bipartite traveling salesman problem, shoelace problem, polynomially solvable case, relaxed Monge matrix, recognition algorithm, pick-and-place robot.

**Keywords.**

## 1 Introduction

In the *travelling salesman problem* (TSP) the objective is to find for a given  $n \times n$  distance matrix  $C = (c_{ij})$  a cyclic permutation  $\tau$  of the set  $\{1, 2, \dots, n\}$  that minimizes the sum  $c(\tau) = \sum_{i=1}^n c_{i\tau(i)}$ . In the maximization version of the TSP (MaxTSP), one is interested in finding the *longest* tour.

The cyclic permutations are also called *tours*, the elements of  $\{1, 2, \dots, n\}$  are called *cities* or *points*, and  $c(\tau)$  is referred as the length of the permutation  $\tau$ . The set of all permutations over set  $\{1, 2, \dots, n\}$  is denoted by  $\mathcal{S}_n$ . For  $\tau \in \mathcal{S}_n$ , we denote by  $\tau^{-1}$  the *inversion* of  $\tau$ , i.e. the permutation for which  $\tau^{-1}(i)$  is the predecessor of  $i$  in the tour  $\tau$ , for  $i = 1, \dots, n$ . We also use a cyclic representation of a cyclic permutation  $\tau$  in the form

$$\tau = \langle i, \tau(i), \tau(\tau(i)), \dots, \tau^{-1}(\tau^{-1}(i)), \tau^{-1}(i), i \rangle.$$

In the *bipartite travelling salesman problem* (BTSP) the set of  $n = 2k$  cities is partitioned into two subsets: set  $K_1 = \{1, 2, \dots, k\}$  of blue cities and set  $K_2 = \{k + 1, k + 2, \dots, n\}$  of white cities. Any feasible tour in the BTSP has to

alternate blue and white cities. The objective is to find the shortest tour with this special structure. The set  $\mathcal{T}_n$  of all feasible tours for the BTSP can formally be defined as

$$\mathcal{T}_n = \{\tau \in \mathcal{S}_n \mid \tau^{-1}(i), \tau(i) \in K_2 \text{ for } i \in K_1; \tau^{-1}(i), \tau(i) \in K_1 \text{ for } i \in K_2\}.$$

By  $C[K_1, K_2]$  we denote the  $k \times k$  matrix which is obtained from matrix  $C$  by deleting rows with numbers from  $K_2$  and columns with numbers from  $K_1$ . Clearly, the length  $c(\tau)$  of any feasible BTSP tour  $\tau$  is calculated by using only elements from  $C[K_1, K_2]$ . We will say that the inequality  $a \leq b$  is reverse to the inequality  $b \leq a$  (and vice versa).

The BTSP has also drawn attention of researchers ([3, 4, 8, 11]) in particular due to its relevance to pick-and-place robots([1, 2, 17, 19]).

The BTSP is NP-hard, moreover, there is no a constant factor approximation algorithm for the BTSP, unless  $P = NP$  ([11]). The characterization of polynomially solvable cases is one of the recognized directions for research of NP-hard problems. While there are quite a lot of solvable cases for the TSP (see [12],[5],[15]), we aware of only two papers [14, 20] published on the solvable cases of the BTSP.

Halton [14] considered the BTSP as a shoelace problem, in which cities represent the eyelets of shoes and the objective is to find an optimal shoe lacing strategy that minimizes the length of the shoelace. In Halton's model the eyelets can be viewed as the points on the Euclidean plane: blue points  $\{1, 2, \dots, k\}$  have coordinates  $(0, d), (0, 2d), \dots, (0, kd)$  and white points  $\{k+1, k+2, \dots, n\}$  have coordinates  $(a, d), (a, 2d), \dots, (a, kd)$ , correspondingly. Halton proved that in this case the tour

$$\tau^* = \langle 1, k+1, 2, k+3, 4, k+5, 6, \dots, 7, k+6, 5, k+4, 3, k+2, 1 \rangle$$

is the shortest tour in  $\mathcal{T}_n$  (see Figure 1).

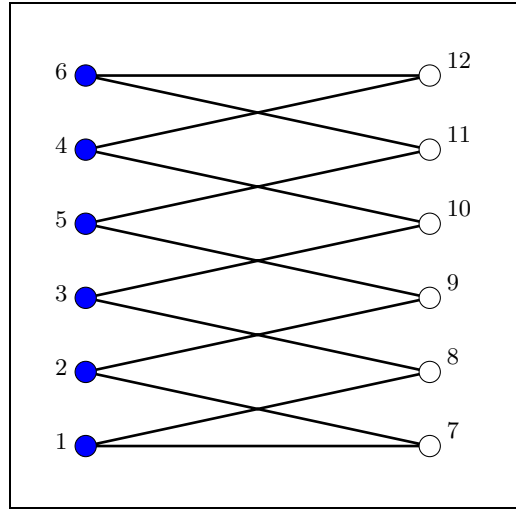
Misiurewicz [20] argued that Halton's case is only a crude approximation of the real situation. He noticed that, to prove the optimality of  $\tau^*$ , it is sufficient to require that the inequalities

$$c_{ij} + c_{\ell m} \leq c_{im} + c_{\ell j} \tag{1}$$

hold for all  $1 \leq i \leq \ell \leq k$  and  $k+1 \leq j \leq m \leq n$ . (In fact Misiurewicz assumed strong inequalities in (1).) Now the points-eyelets are not necessarily have to be on two parallel lines (see Figure 2), and therefore the shoelace problem can be solved "even for old shoes".

In this note we show that the technique developed for the analysis of the TSP can successfully be used to further investigate solvable cases of the BTSP. In Section 2 we shortly review some of the well solved cases of the TSP which are relevant to the BTSP. We generalize results of Halton and Misiurewicz and characterize a new polynomially solvable case of the BTSP. If we use the shoelace analogy, then in our case eyelets may have indeed very peculiar locations. We

hope this justifies the title of the paper: the case of very old shoes. In Section 3 we represent an algorithm for recognizing this solvable case independently of the initial numbering of the points.



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	5	5	5	5	5	5	40	40	40	40	40	40
Y coordinate	10	17	24	31	38	45	10	17	24	31	38	45

**Fig. 1.** Points with their coordinates and an optimal BTSP tour: An illustration to Halton's [14] optimal lacing.

## 2 Polinomially solvable TSP cases and the BTSP

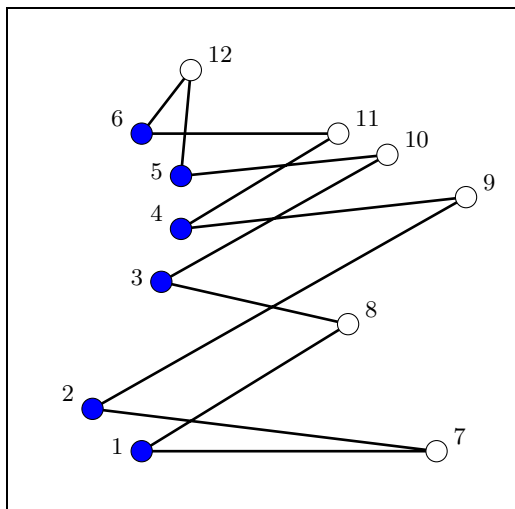
A reader, who is familiar with the literature on combinatorial optimization, may have already recognized in inequalities (1) the notorious Monge structure (see [6] for further references).

An  $n \times n$  matrix  $C = (c_{ij})$  is called a *Monge matrix* if it satisfies the following conditions for all indices  $i, j, m, \ell \in \{1, \dots, n\}$  with  $i < \ell$  and  $j < m$ :

$$c_{ij} + c_{\ell m} \leq c_{im} + c_{\ell j}. \quad (2)$$

Clearly, system (1) is a relaxation of (2), because it defines Monge inequalities only for items from  $C[K_1, K_2]$ .

Supnick [21] showed that the TSP with a symmetric Monge matrix is solved by the tour  $\pi_1^* = \langle 1, 3, 5, \dots, 6, 4, 2, 1 \rangle$ , while the MaxTSP on Monge matrices is solved by the tour  $\sigma^* = \langle 1, n, 2, n-2, 4, n-4, \dots, n-3, 3, n-1, 1 \rangle$ . Notice that



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	11	6	13	15	15	11	41	32	44	36	31	16
Y coordinate	10	14	26	31	36	40	10	22	34	38	40	46

**Fig. 2.** An illustration to Misiurewicz’s [20] optimal lacing for irregular shoes; an instance with the Euclidean distance matrix.

if the white points in the shoelace problem were numbered in the reverse order, i.e.  $i \in K_2$  were renumbered by  $n + k + 1 - i$ , then the permutation  $\tau^*$  identified by Halton would become the Supnick permutation  $\sigma^*$ . We mention this fact here in order to stress that the BTSP seems to have something in common with the MaxTSP.

Another well-known polynomially solvable case is the TSP with Kalmanson distance matrices. A symmetric  $n \times n$  matrix  $C$  is called a *Kalmanson matrix* if it fulfils the *Kalmanson conditions*

$$c_{ij} + c_{lm} \leq c_{il} + c_{jm}, \tag{3}$$

$$c_{im} + c_{jl} \leq c_{il} + c_{jm}, \quad \text{for all } 1 \leq i < j < \ell < m \leq n. \tag{4}$$

Kalmanson [16] showed that the TSP with a Kalmanson matrix is solved by the tour  $\pi_2^* = \langle 1, 2, 3, \dots, n - 2, n - 1, n, 1 \rangle$ , while an optimal tour for the MaxTSP can be found among  $n/2$  specially structured tours containing among them the tour  $\tau^*$ .

The generalization of Supnick and Kalmanson matrices is the class of Demidenko matrices. A symmetric matrix  $C = (c_{ij})$  is called a *Demidenko matrix* if

$$c_{ij} + c_{lm} \leq c_{il} + c_{jm}, \quad \text{for all } 1 \leq i < j < \ell < m \leq n. \tag{5}$$

Demidenko [10] showed that an optimal tour for the TSP with a Demidenko matrix can be found in  $O(n^2)$  time. The MaxTSP with a Demidenko matrix is NP-hard (see [9]), however, for a special subclass of these matrices the longest tour can be found in the set  $\mathcal{T}_n$  of all feasible tours for the BTSP (tours of type (I) in [9]).

**Proposition 21** (from Theorem 4.1 in [9])

Let  $C$  be a symmetric  $n \times n$  Demidenko matrix,  $n = 2k$ , that additionally fulfils the conditions

$$c_{ik} + c_{k+1,j} \leq c_{k+1,k} + c_{ij}, \text{ for } i \in K_1 \setminus \{k\}, j \in K_2 \setminus \{k+1\}, \quad (6)$$

then there exists an optimal MaxTSP tour which belongs to the set  $\mathcal{T}_n$ .

The problem of finding an optimal MaxTSP tour in  $\mathcal{T}_n$  remains NP-hard, however, it becomes trivial, if a distance matrix has the following special structure:

**Proposition 22** (from Theorem 4.1 in [9])

Let  $C$  be a symmetric  $n \times n$  matrix,  $n = 2k$ , that fulfils the conditions

$$c_{1,k+1} + c_{ij} \geq c_{1j} + c_{i,k+1}, \quad i = 2, \dots, k, j = k+2, \dots, n \quad (7)$$

$$c_{p+1,k+p} + c_{ij} \geq c_{p+1,j} + c_{i,k+p}, \quad i = p+2, \dots, k, j = k+p+1, \dots, n \quad (8)$$

$$c_{p,k+p+1} + c_{ij} \geq c_{pj} + c_{i,k+p+1}, \quad i = p+1, \dots, k, j = k+p+2, \dots, n \quad (9)$$

$$p = 1, \dots, k-2,$$

then the tour  $\tau^*$  is a tour of maximum length in  $\mathcal{T}_n$

It is easy to see that conditions (7)-(9) is a relaxation of the Kalmanson conditions (4). Therefore, the TSP with a Kalmanson matrix that fulfils (6) has  $\tau^*$  as the tour of maximum length. Any Supnick matrix fulfils inequalities (6) and the inequalities which are reverse to (7)-(9). Therefore, if points  $i \in K_2$  are renumbered by  $n+k+1-i$ , then by Propositions 21 and 22 permutation  $\sigma^*$ , which is obtained from  $\tau^*$  by the renumbering, is a solution to the MaxTSP with a Supnick matrix. (Notice that the renumbering does not affect (6).) This comment explains the relationship between the TSP and the MaxTSP with a Supnick matrix.

In the prove of Proposition 22 in [9] the well-known tour-improvement technique is used: starting from an arbitrary tour  $\tau$ , a sequence of tours  $\tau_1, \tau_2, \dots, \tau_T$  is constructed, with  $\tau_1 = \tau$  and  $\tau_T = \tau^*$  such that

$$c(\tau_1) \leq c(\tau_2) \leq \dots \leq c(\tau_T).$$

The inequalities (7)-(9) are used to prove the relationship  $c(\tau_i) \leq c(\tau_{i+1})$ .

If inequalities (7)-(9) are reversed, then it can be proved in a similar way that the tour  $\tau^*$  is the shortest tour in  $\mathcal{T}_n$ . We formulate this result as the following

**Theorem 23** Let  $C$  be a symmetric  $n \times n$  matrix,  $n = 2k$ , that fulfils conditions

$$c_{1,k+1} + c_{ij} \leq c_{1j} + c_{i,k+1}, \quad i = 2, \dots, k, \quad j = k + 2, \dots, n \quad (10)$$

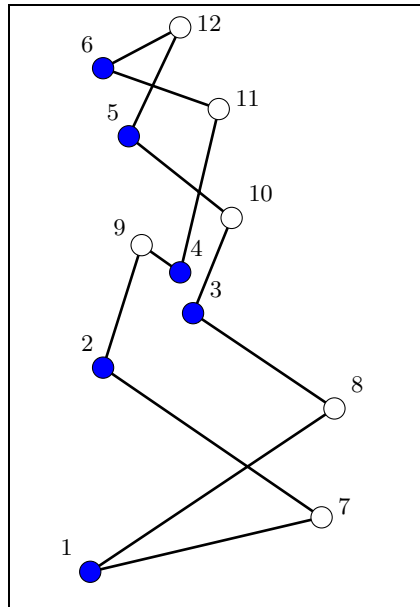
$$c_{p+1,k+p} + c_{ij} \leq c_{p+1,j} + c_{i,k+p}, \quad i = p + 2, \dots, k, \quad j = k + p + 1, \dots, n \quad (11)$$

$$c_{p,k+p+1} + c_{ij} \leq c_{pj} + c_{i,k+p+1}, \quad i = p + 1, \dots, k, \quad j = k + p + 2, \dots, n \quad (12)$$

$$p = 1, \dots, k - 2,$$

then the tour  $\tau^*$  is a tour of minimum length for the BTSP with  $C$  as the distance matrix.

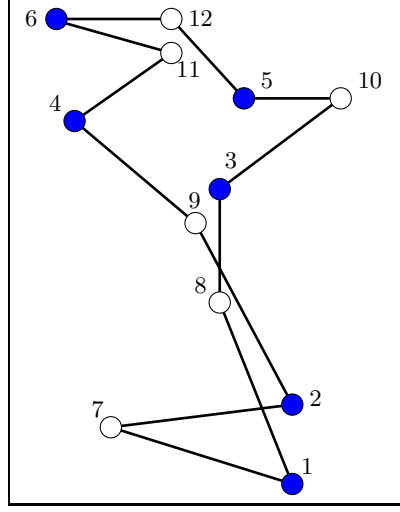
System (10)-(12) is just a further relaxation of Monge inequalities (2) and their relaxation (1). On Figures 3 and 4 we show two instances of the BTSP with the Euclidean distance matrices that satisfy (10)-(12) but violate some of Misiurewicz inequalities (1).



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	5	6	13	12	8	6	23	24	9	16	15	12
Y coordinate	6	21	25	28	38	43	10	18	30	32	40	46

**Fig. 3.** Instance 1 of the BTSP with relaxed Monge structure in the Euclidean distance matrix.

System (10)-(12) contains  $\Theta(n^3)$  inequalities, however it can be checked in  $O(n^2)$  time as shown in the proposition below.



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	22	22	16	4	18	2	7	16	14	26	12	12
Y coordinate	5	12	31	37	39	46	10	21	28	39	43	46

**Fig. 4.** Instance 2 of the BTSP with relaxed Monge structure in the Euclidean distance matrix.

**Proposition 24** *Inequalities (10)-(12) can be checked in  $O(n^2)$  time.*

*Proof.* To simplify notations, we consider an *asymmetric*  $k \times k$  sub-matrix  $A$  of  $n \times n$ , ( $n = 2k$ ), matrix  $C$ :  $A = C[K_1, K_2]$ . System (10)-(12) can be rewritten then as

$$a_{11} + a_{st} \leq a_{1t} + a_{s1}, \quad 1 < s, t \leq k \quad (13)$$

$$a_{p,p-1} + a_{st} \leq a_{pt} + a_{s,p-1}, \quad s = p+1, \dots, k; t = p, \dots, k; \quad (14)$$

$$a_{p-1,p} + a_{st} \leq a_{p-1,t} + a_{sp}, \quad s = p, \dots, k; t = p+1, \dots, k; \quad (15)$$

$$p = 2, 3, \dots, k-1.$$

We claim that system (13)-(15) is equivalent to the following system of  $2(k-1)(k-2) + 1$  inequalities:

$$a_{11} + a_{22} \leq a_{12} + a_{21}; \quad (16)$$

$$a_{p,p-1} + a_{sp} \leq a_{p,p} + a_{s,p-1}, \quad (17)$$

$$a_{p,p-1} + a_{s,p+1} \leq a_{p,p+1} + a_{s,p-1}, \quad s = p+1, \dots, k; \quad (18)$$

$$a_{p-1,p} + a_{pt} \leq a_{pp} + a_{p-1,t}, \quad (19)$$

$$a_{p-1,p} + a_{p+1,t} \leq a_{p+1,p} + a_{p-1,t}, \quad t = p+1, \dots, k; \quad (20)$$

$$p = 2, 3, \dots, k-1.$$



Indeed, it can be easily checked that the set of inequalities (16)-(20) is a proper subset of system (13)-(15). In particular, inequalities (14) and (15) with  $p = k - 1$  are in (16)-(20). So what we need to prove is that (13)-(15) follow from (16)-(20). Consider  $p^* \leq k - 1$ , and assume that (14)-(15) are satisfied for all  $p \geq p^*$ . Inequalities (14) with  $s = p^*$  and  $s = p^* + 1$ , and inequalities (15) with  $t = p^*$  and  $t = p^* + 1$  are in (16)-(20). The inequalities for  $s > p^* + 1$  and  $t > p^* + 1$  follow immediately from (16)-(20) and the following straightforward algebraic rearrangements:

$$\begin{aligned} & a_{p^*, p^*-1} + a_{st} - a_{p^*t} - a_{s, p^*-1} = \\ & (a_{p^*, p^*-1} + a_{s, p^*+1} - a_{p^*, p^*+1} - a_{s, p^*-1}) + (a_{p^*, p^*+1} + a_{st} - a_{p^*t} - a_{s, p^*+1}); \\ & a_{p^*-1, p^*} + a_{st} - a_{p^*-1, t} - a_{s, p^*} = \\ & (a_{p^*-1, p^*} + a_{p^*+1, t} - a_{p^*-1, t} - a_{p^*+1, p^*}) + (a_{p^*+1, p^*} + a_{st} - a_{p^*+1, t} - a_{s, p^*}). \end{aligned}$$

Inequalities (13) follow from (14),(15), (16) and the following simple transformation:

$$\begin{aligned} a_{11} + a_{st} - a_{1t} - a_{s1} &= (a_{11} + a_{22} - a_{12} - a_{21}) + \\ & (a_{12} + a_{st} - a_{1t} - a_{s2}) + (a_{21} + a_{s2} - a_{22} - a_{s1}). \end{aligned}$$

It completes the proof of the proposition.

### 3 Recognition of specially structured matrices

Clearly, the combinatorial structure of a distance matrix depends on the numbering of the rows and columns. Therefore it is naturally to formulate the following *recognition* problem:

Given an  $n \times n$  distance matrix  $C = (c_{ij})$ , does there exist a renumbering of the cities, i.e. a permutation  $\alpha$  of the rows and columns of  $C$ , such that the resulting matrix  $(c_{\alpha(i)\alpha(j)})$  satisfies conditions (10)-(12)?

If we consider matrix  $A = C[K_1, K_2]$ , then the recognition problem above is reduced to the problem of finding two permutations: a permutation for permuting rows and a permutation for permuting columns in the *asymmetric* matrix  $A$ :

Given a  $k \times k$  matrix  $A = (a_{ij})$ , does there exist a permutation of rows  $\gamma$  and permutation of columns  $\delta$ , such that the resulting permuted matrix  $(c_{\gamma(i)\delta(j)})$  satisfies conditions (13)-(15)?

The recognition algorithm below is based on the technique developed in [7] for the recognition of a similar relaxed Monge structure in the *symmetric* distance matrix.

**Proposition 31** *Given a  $k \times k$  matrix  $A = (a_{ij})$ , it can be decided in  $O(k^4)$  time whether there exist permutations  $\gamma$  and  $\delta$  such that the permuted matrix  $(a_{\gamma(i)\delta(j)})$  satisfies conditions (13)-(15). If the permutations  $\gamma$  and  $\delta$  exist, then they can explicitly be determined within this time bound.*

*Proof.* First try all  $k$  indices as candidates for the first position in  $\gamma$ . Let  $\gamma(1) = 1$ . An index  $i$  can be placed on the first position in  $\delta$  if and only if the following inequalities hold:

$$a_{1i} + a_{st} \leq a_{si} + a_{1t} \text{ for all } s \neq 1, t \neq i. \tag{21}$$

If there is another candidate  $j$  with the same property, then it follows immediately from (21) that  $a_{1i} + a_{sj} = a_{si} + a_{1j}$ , i.e.  $a_{sj} = a_{si} + d$  for all  $s$ , where  $d = a_{1i} - a_{1j}$  is the constant for fixed  $i$  and  $j$ . Since adding a constant to a row or a column of matrix  $A$  does not affect inequalities (13)-(15), any of the indices  $i$  or  $j$  can be placed on the first position in  $\sigma$ .

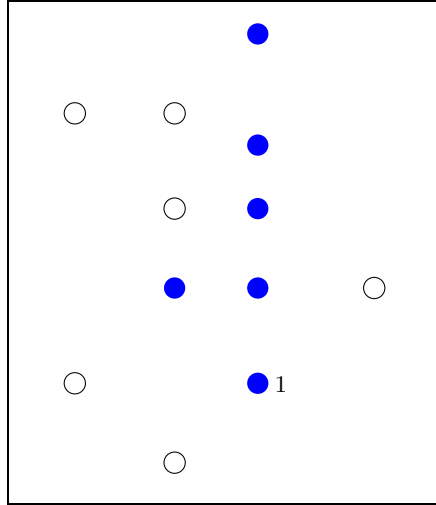
We claim that the candidate  $i$  can be chosen in  $O(k^2)$  time. Indeed, the transformation  $a'_{st} = a_{st} - a_{1t}$ ,  $s = 1, \dots, k$ ,  $t = 1, \dots, k$ , transforms matrix  $A$  into matrix  $A'$  with zeros in the first row. The inequalities (21) are equivalent to  $a'_{st} \leq a'_{si}$  for all  $s, t$  and  $i$ . Clearly, index  $i$  can be found in  $O(k^2)$  time by looking through the indices of maximal elements in rows of  $A'$ .

Indices for the second position in  $\delta$  and  $\gamma$  can be chosen by applying the same procedure to submatrix  $A[\{1, \dots, k\}, \{1, \dots, k\} \setminus \{i\}]$  with the first row fixed to be 1 and to submatrix  $A[\{2, \dots, k\}, \{1, \dots, k\}]$  with the first column fixed to be  $i$ . This yields an  $O(k^3)$  time complexity for each candidate on the position  $\gamma(1)$  and, therefore, an  $O(k^4)$  overall time complexity.  $\square$

To illustrate the algorithm we consider the BTSP with a rectilinear distance matrix (see Fig. 5) where the distances between points  $i$  and  $j$  are calculated as  $c_{ij} = |x_i - x_j| + |y_i - y_j|$ . We assume here that the first fixed item is 1:  $\gamma(1) = 1$ . Submatrix  $A$  of the distance matrix  $C$  and transformed matrix  $A'$  are shown below:

$$A_{6 \times 6} = \begin{matrix} & 7 & 8 & 9 & 10 & 11 & 12 \\ \mathbf{1} & 10 & 11 & 16 & 13 & 28 & 22 \\ 2 & 11 & 12 & 5 & 12 & 17 & 11 \\ 3 & 16 & 17 & 10 & 7 & 22 & 16 \\ 4 & 21 & 22 & 5 & 12 & 17 & 11 \\ 5 & 25 & 26 & 9 & 16 & 13 & 7 \\ 6 & 32 & 33 & 16 & 23 & 16 & 10 \end{matrix}, \quad A'_{6 \times 6} = \begin{matrix} & 7 & 8 & 9 & 10 & 11 & 12 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & -11 & -1 & -11 & -11 \\ 3 & 6 & 6 & -6 & -6 & -6 & -6 \\ 4 & 11 & 11 & -11 & -1 & -11 & -11 \\ 5 & 15 & 15 & -7 & 3 & -15 & -15 \\ 6 & 22 & 22 & 0 & 10 & -12 & -12 \end{matrix}$$

The indices of maximal elements in all rows are  $\{7, 8\}$ , so any of these columns can be chosen for the first column. Let it be 7:  $\delta(1) = 7$ . To chose a row to be placed on the second position in permutation  $\gamma$ , we use  $5 \times 6$  submatrix of the



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	26	21	26	26	26	26	21	15	21	33	15	21
Y coordinate	12	18	18	23	27	34	7	12	23	18	29	29

**Fig. 5.** Illustration to recognising a rectilinear instance of the BTSP. First item is chosen.

distance matrix:

$$A_{5 \times 6} = \begin{matrix} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12} \\ \mathbf{2} & (11 & 12 & 5 & 12 & 17 & 11) \\ \mathbf{3} & (16 & 17 & 10 & 7 & 22 & 16) \\ \mathbf{4} & (21 & 22 & 5 & 12 & 17 & 11) \\ \mathbf{5} & (25 & 26 & 9 & 16 & 13 & 7) \\ \mathbf{6} & (32 & 33 & 16 & 23 & 16 & 10) \end{matrix}, \quad A'_{5 \times 6} = \begin{matrix} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12} \\ \mathbf{2} & (0 & 1 & -6 & 1 & 6 & 0) \\ \mathbf{3} & (0 & 1 & -6 & -9 & 6 & 0) \\ \mathbf{4} & (0 & 1 & -16 & -9 & -4 & -10) \\ \mathbf{5} & (0 & 1 & -16 & -9 & -12 & -18) \\ \mathbf{6} & (0 & 1 & -16 & -9 & -16 & -22) \end{matrix}$$

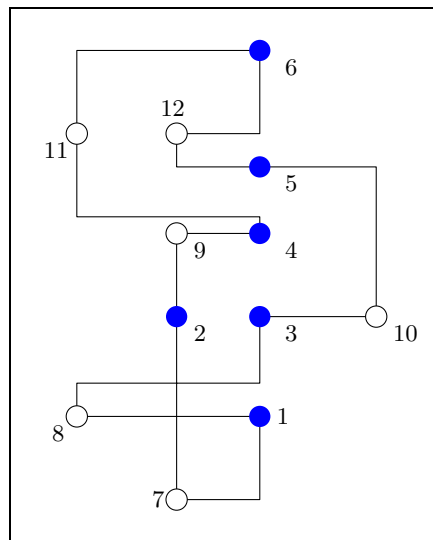
The indices of maximal elements in the columns are:  $\{2, 3, 4, 5, 6\}$ ,  $\{2, 3\}$ ,  $\{2\}$ ,  $\{2, 3\}$  and  $\{2, 3\}$ . The only index that belongs to all sets is 2, so  $\gamma(2) = 2$ . If the intersection of these sets were empty, the choice of the first fixed item, which is currently 1, had to be reconsidered.

We proceed with the next (similar) steps and eventually find two permutations for proper numbering of points:  $\gamma = \langle 1, 2, 3, 4, 5, 6 \rangle$  and  $\delta = \langle 7, 8, 9, 10, 11, 12 \rangle$ . Items 7 and 8, as well as items 11 and 12 in  $\delta$  can be permuted, hence generating four pairs of permutations for renumbering the points (given that  $\gamma(1) = 1$ ).

The numbering found and the corresponding optimal BTSP solution are shown on Figure 6.

## 4 Conclusion

In this paper we have characterised a polynomially solvable case of the Bipartite TSP. We have shown that the technique used for the analysis of the TSP can successfully be used for characterization of polynomially solvable cases of the BTSP. We introduced specially structured matrices that allow one to solve the BTSP efficiently and suggested an algorithm to recognize permuted matrices with this special structure. In a way our simple results illustrate the power and beauty of algorithms: an algorithm developed for a problem can be found unexpectedly useful and powerful for a rather different new problem.



**Fig. 6.** Illustration to recognising a rectilinear instance of the BTSP. Numbering of points is found.

## References

1. S. Anily, R. Hassin, The swapping problem, *Networks* **22**, 1992, 11–18.
2. M.J. Atallah, S.R. Kosaraju, Efficient solutions to some transportation problems with applications to minimizing robot arm travel, *SIAM Journal on Computing* **17**, 1988, 419–433.
3. A. Baltz, *Algorithmic and probabilistic aspects of the bipartite traveling salesman problem*, PhD Thesis, University Kiel, 2001.
4. A. Baltz, A. Srivastav, Approximation algorithms for the Euclidean bipartite TSP, *Operations Research Letters* **33**, 4, 2005, 403–410.
5. R.E. Burkard, V.G. Deineko, R. van Dal, J.A.A. van der Veen, G.J. Woeginger, Well-solvable special cases of the TSP: A survey, *SIAM Review* **40** No.3, 1998, 496–546.

6. R.E. Burkard, B. Klinz and R. Rudolf, Perspectives of Monge properties in optimization, *Discrete Applied Mathematics* **70**, 1996, 91–161.
7. R.E. Burkard, V. Deineko, On the traveling salesman problem with a rexed Monge matrix, *Information Processing Letters* **67**, 1998, 231–237.
8. P. Chalasani, R. Motwani, A. Rao, Approximation algorithms for robot grasp and delivery, In *Proceedings of the 2nd International Workshop on Algorithmic Foundations of Robotics*, Toulouse, France, 1996, 347 – 362.
9. V. Deineko, G.J. Woeginger, The maximum traveling salesman problem on symmetric Demidenk matrices, *Discrete Applied Mathematics* **99**, 2000, 413–425.
10. V.M. Demidenko, A special case of traveling salesman problems, *Izv. Akad. Nauk. BSSR, Ser. Fiz.-mat. Nauk* **5**, 1976, 28–32, (in Russian).
11. A. Frank, B. Korte, E. Triesch, J. Vygen, *On the bipartite travelling salesman problem*, Report No. 98866-OR, Research Institute for Discrete Mathematics, University of Bonn, 1998.
12. P.C. Gilmore, E.L. Lawler and D.B. Shmoys, Well-solved special cases, Chapter 4 in [18], 87–143.
13. G. Gutin and A.P. Punnen, *The travelling salesman problem and its variations*, Kluwer Academic Publishers, 2002.
14. J.H. Halton, The shoelace problem, *The Mathematical Intelligencer* **17**, No.4, 1995, 36–41.
15. S.N. Kabadi, Polynomially solvable cases of the TSP, Chapter 11 in [13], 489–583.
16. K. Kalmanson, Edgeconvex circuits and the traveling salesman problem, *Canadian Journal of Mathematics* **27**, 1975, 1000–1010.
17. T. Leipälä, O. Nevalainen, Optimization of the movements of a component placement machine, *European Journal of Operational Research* **38**, 1989, 167–177.
18. E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, *The Traveling Salesman Problem*, Wiley, Chichester, 1985.
19. C. Michel, H. Schroeter, A. Srivastav, Approximation algorithms for pick-and-place robots, *Annals of Operations Research*, **107**, 2001, 321–338.
20. M. Misiurewicz, Lacing irregular shoes, *The Mathematical Intelligencer* **18**, 1996, 32–34.
21. F. Supnick, Extreme Hamiltonian lines, *Annals of Math.* **66**, 1957, 179–201.