On the cover time of planar graphs

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Abstract

1. Introduction

Let G be a finite, undirected and connected graph. Fix $v \in V(G)$ and consider the following procedure: starting from v, at each step choose uniformly at random a neighbor of the current vertex, and move to it. This defines a sequence $\{X_k\}_{k=1}^{\infty}$ of vertices of G, called a random walk on G.

Random walks arise naturally in many branches of science. To name a few applications, random walks are used to model the Brownian motion of a dust particle, to describe the movement of an animal in mathematical ecology, or to sample uniformly at random an element from an exponentially large set, with potentially complicated underlying structure. In the beginning of the talk we will examine one such application more closely.

For a starting vertex v and $x \in V(G)$, let $T_x = \min\{k \in \mathbb{N}^* : X_k = x\}$ and set $C_v = \mathbb{E}[\max_x T_x]$, the *cover time* of the random walk starting at v. Let $C_G = \max_v C_v$ be the cover time of the graph G.

One important problem in the study of random walks is finding upper and lower bounds for the cover time C_v. Tight bounds for arbitrary graphs were proved by Feige, and some special classes of graphs (e.g. regular graphs, random graphs) were also studied.

The central point of this talk will be the proof of the following result:

Theorem 1 *Let G be a finite connected planar graph with n vertices and $\Delta(G) \leq M$. Then there is a positive constant $c := c(M)$ such that for every vertex $v \in V$,*

$$cn(\log n)^2 < C_v < 6n^2.$$

Moreover, the result is tight.

2. Quantitative measures of a random walk

Besides the cover time, the proof of Theorem 1 relies on the analysis of several other parameters as well. Let $x, y \in V(G)$ be two distinct vertices. Then we may consider the following:

1. *The hitting time* h_{xy} is the expected number of steps before vertex y is visited, when the random walk starts at x.
2. *The commute time* is defined as $C_{xy} = h_{xy} + h_{yx}$.
3. *The difference time* is defined as $D_{xy} = h_{xy} - h_{yx}$.

We shall study several properties of the above parameters. We shall also outline a connection between C_{xy} and R_{xy}, the effective resistance between x and y in G, which is just the voltage difference between x and y when each edge of G has unit resistance, and $1A$ of current is injected into x and removed from y.

3. Proof of the main result

In the proof of Theorem 1 we will bound the cover time using hitting times. For this we shall need the following result of Matthews.

Lemma 1 *Let G be a finite graph on n vertices. Then for any $v \in V(G)$ and for any $\emptyset \neq V_0 \subset V(G)$ we have that:*

$$H_{|V_0|-1} \min\{h_{uv} : u, v \in V_0, u \neq v\} \leq C_v \leq H_{n-1} \max\{h_{uv} : u, v \in V(G), u \neq v\},$$

where H_k is the k-th Harmonic number.

We shall also need the following lemma.

Lemma 2 *Let G be a connected planar graph with $\Delta(G) \leq M$. Then there exists positive constants $c := c(M)$ and $r := r(M)$ such that for any $W \subset V(G)$ there is a $V' \subset W$ with $|V'| \geq |W|^c$ and $R_{uv} \geq r \log |W|$ for any two distinct vertices $u, v \in V'$.*

A circle packing is an arrangement of circles in the plane such that any two have disjoint interiors, and some are mutually tangent. The proof of Lemma 2 will be an application of the Circle Packing Theorem, first proved by Paul Koebe in 1936.

Theorem 2 (Circle Packing Theorem) *Let G be a finite planar graph. Then there is a circle packing $\{C_v : v \in V(G)\}$ such that $C_v \cap C_u \neq \emptyset$ iff $uv \in E(G)$.*