

# How to estimate the volume of convex bodies

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Our aim in this series of talks is to see how one can estimate the volume of a convex body  $K \subseteq \mathbb{R}^n$ , efficiently. The body will be given to us as an oracle. This means that for every  $x \in \mathbb{R}^n$  we can enquire whether  $x$  belongs to  $K$  or not and get a correct answer. Our aim here is to find a good approximation for the volume  $\text{Vol}(K)$  of  $K$ . Ideally, given  $\varepsilon > 0$ , we would like to find an  $\varepsilon$ -approximation of  $\text{Vol}(K)$ , i.e. a real number  $V$  such that

$$(1 - \varepsilon) \text{Vol}(K) \leq V \leq (1 + \varepsilon) \text{Vol}(K).$$

Moreover we would like to compute  $V$  quickly. In particular, we would like our algorithm to take polynomially many steps in  $n$  and  $1/\varepsilon$ .

As it stands we obviously have no chance to fulfil our aim. The body might be “too far off to infinity” and for any element  $x \in \mathbb{R}^n$  we enquire for, we might get the answer that  $x$  does not belong to  $K$ . We will therefore also require that the body is “well-guaranteed” meaning that we will be given real numbers  $0 < r < R$  and we will be guaranteed that  $B(0, r) \subseteq K \subseteq B(0, R)$ , where  $B(x, t)$  denotes the closed ball of radius  $t$  centred at  $x$ .

We will see that every deterministic algorithm which can  $\varepsilon$ -approximate the volume of  $K$  must take exponentially many steps. Not all is lost however. We will allow ourselves to err a little, and switch our attention from deterministic to randomised algorithms. We introduce a new parameter  $\eta > 0$  and we would be happy if with failure probability at most  $\eta$ , we can  $\varepsilon$ -approximate  $\text{Vol}(K)$  in polynomially many steps. The breakthrough here occurred by Dyer, Frieze and Kannan [1] who used the Markov Chain Monte Carlo method to produce such an algorithm.

As an example of how randomisation might help, consider the following approach: Suppose we fix a cube  $Q$  containing  $K$  and take a very fine grid in  $Q$ . Suppose also that we start walking randomly on this grid. I.e. at each step we look at the  $2n$  neighbours and move to one of them uniformly at random. This is a Markov chain and it can be proved by standard methods that the walk converges quite quickly to the uniform distribution. This means that after a small number of steps, given any two points  $x, y$  of the grid, the probability that we will be at  $x$  will be very close to the probability that we will be at  $y$ . (This is not entirely true. The grid has “even” points and “odd” points and at each particular step we know whether we are at an even or at an odd point. However this is not

a main issue and can be fixed quite easily.) This looks very promising as by performing this walk we can essentially pick out random points from  $Q$ . So, having chosen sufficiently many random points from  $Q$ , we can check how many of them lie in  $K$  and declare the ratio to be the volume of  $K$ . Then we can use some concentration inequalities to guarantee that this is an  $\varepsilon$ -approximation with very high probability. There is a catch however. The volume of  $K$  might be exponentially small compared to the volume of  $Q$ . Thus we might need to sample exponentially many points before we even hit a point in  $K$ !

Our aim in this series of talks is to explain the main ideas behind the Dyer-Frieze-Kannan algorithm and also to explain why some more naive approaches like the above fail. As a main reference we used some lecture notes on the topic by Mark Jerrum [2].

## References

- [1] M. Dyer, A. Frieze and R. Kannan, A random polynomial-time algorithm for approximating the volume of convex bodies, *J. Assoc. Comput. Mach.* **38** (1991), no. 1, 1–17.
- [2] M. Jerrum, Volume of a convex body, <http://www.dcs.ed.ac.uk/home/mrj/ETHbook/chap6.ps>