A note on the random greedy triangle-packing algorithm

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Abstract. The paper considers the random greedy algorithm for triangle-packing. This stochastic graph process begins with the complete graph on n vertices and repeatedly removes the edges of a triangle, one at a time, where each triangle removed is chosen uniformly at random from the collection of all remaining triangles. This stochastic process terminates once it arrives at a triangle-free graph. The authors show that with high probability the number of edges in the final graph is at most $O\big(n^{7/4}\log^{5/4}(n)\big)$.

1 Overview

Several random variables are introduced and tracked through the evolution of the process. Roughly speaking, these variables are 'close' to martingales and can be 'approximated' by pairs of sub- and supermartingales. Then the concentration of the variables under consideration is essentially shown using variants of the Azuma-Hoeffding martingale inequality.

1.1 Random variables

Let $G(i)$ denote the graph after i triangles have been removed and by $E(i)$ we denote its edge set. An important variable is

$$Q(i) \ := \ \#\text{ of triangles in } G(i) \ .$$

Furthermore, as we shall see, it turns out to be useful to 'understand' the codegrees in $G(i)$. So, for all $\{u, v\} \in \binom{[n]}{2}$ we set

$$Y_{u,v}(i) \ := \ |\{x \in [n] \ : \ xu, xv \in E(i)\}| \ .$$

1.2 Their values

What values should the above random variables roughly have? It is 'widely believed' that $G(i)$ should behave similarly to the uniform random graph on n vertices and $\binom{n}{2} - 3i$ edges, which in turn shares many properties with binomial random graph $G_{n,p(i)}$ with edge-density

$$p(i) \ := \ 1 - \frac{6i}{n^2} \ .$$

Based on this random graph intuition we expect

$$Y_{u,v}(i) \ \approx \ p^2(i)n \qquad \text{and} \qquad Q(i) \ \approx \ \frac{p^3(i)n^3}{6} \ . \tag{1}$$

1.3 Expected one-step changes

Observe that with $Y_{u,v}(i)$ we can express the one-step change of $Q(i)$ as follows: if the $(i+1)$-th triangle taken is abc then

$$Q(i) - Q(i+1) \; = \; Y_{a,b}(i) + Y_{b,c}(i) + Y_{a,c}(i) - 2 \; .$$

With this in mind, after conditioning on the first i steps, the expected one-step change of $Q(i)$ is given by

$$\begin{aligned} \mathbb{E}[Q(i+1) - Q(i) \mid G(i)] \; &= \; - \sum_{xyz \in Q(i)} \frac{Y_{xy}(i) + Y_{xz}(i) + Y_{yz}(i) - 2}{Q(i)} \\ &= \; 2 - \frac{1}{Q(i)} \sum_{xy \in E(i)} \big(Y_{xy}(i)\big)^2 \; , \end{aligned} \tag{2}$$

where we also have $3Q(i) = \sum_{xy \in E(i)} Y_{xy}(i)$. But why is this useful? Well, $Q(i)$ can essentially be 'transformed' to sub- and supermartingales by adding or subtracting appropriate functions. Then (modulo some technical details) the concentration follows from Azuma-Hoeffding type martingale inequalities.

We remark that the same strategy also applies to $Y_{u,v}(i)$; but here the details are more involved.

1.4 Main Result

Loosely speaking, the main technical result states that during the first

$$i^* := \frac{1}{6}n^2 - \frac{5}{3}n^{7/4} \log^{5/4}(n) \tag{3}$$

steps the approximations given by (1) hold whp. Note that $Q(i^*) = \omega(1)$, and thus the process has not terminated yet; so whp the final graph of the random process has

$$\binom{n}{2} - 3i^* = O(n^{7/4} \log^{5/4}(n))$$

edges, which implies their main result.

1.5 Remark

In the current proof the bound (3) is 'limited' by the way in which the concentration of $Y_{u,v}(i)$ is shown. In fact, the authors claim that with a more sophisticated analysis (and introducing more random variables) they are able to reduce the final number of edges to roughly $O(n^{1.65+o(1)})$. Finally, it is widely believed that whp the final graph has $n^{3/2+o(1)}$ edges (Joel Spencer has offered \$200 for a resolution of this question).