On graphs that do not contain the cube and related problems.

R. Pinchasi and M. Sharir
presented by Matthew White

The *Turan number* of a graph G is the maximum number of edges in a graph on n vertices that does not contain G. For graphs G with $\chi(G) = r$, a celebrated theorem of Erdős and Stone [2] states that the Turan number of G is $\left(\frac{r-2}{r-1} + o(1)\right)\binom{n}{2}$. For bipartite graphs, this result only tell us that the Turan number is $o(n^2)$.

The 3-dimensional cube Q is the graph with vertex set $\{0,1\}^3$ where two vertices are adjacent if they differ in exactly one coordinate, it can be seen that this graph is bipartite. Erdős and Simonovits [1] proved that the Turan number of Q is $O\left(n^{8/5}\right)$. This talk is on a paper of Pinchasi and Sharir [3] giving an alternative, simpler proof of this result.

It is immediate that the following theorem is sufficient to prove the result.

Theorem 1. *A bipartite graph G with vertex classes of order m and n, that does not contain Q has $O\left(m^{4/5}n^{4/5} + mn^{1/2} + nm^{1/2}\right)$ edges.*

A configuration is a 6-tuple (a, b, c, d, u, v) of vertices which form two C_4 with a common edge uv and no other common vertex. Note that, if for a vertex disjoint pair of edges ab and cd, there are disjoint pairs uv and $u'v'$ such that (a, b, c, d, u, v) and (a, b, c, d, u', v') are configurations, we have a copy of Q. The proof proceeds by counting configurations, in particular making multiple use of the following form of the Cauchy-Schwarz inequality

$$\sum_{j=1}^{i} a_j \leq i^{1/2}\left(\sum_{j=1}^{i} a_j^2\right)^{1/2}.$$

The proof can be generalised as follows. Let $Q_{k,m}$ be the bipartite graph formed by taking the disjoint union of two $K_{k,m}$, and adding a matching between the two sets of size k and the two sets of size m, so that $Q = Q_{2,2}$. Then the following result may be proved.

Theorem 2. *Let $2 \leq k \leq m$ be positive integers, and let G be a graph on n vertices which does not contain a copy of $Q_{k,m}$, and also does not contain a copy of $K_{k+1,k+1}$. Then G has at most $O\left(n^{\frac{4k}{2k+1}}\right)$ edges.*

I will prove the first theorem, and look at the proof of the second.

References

[1] Erdős, P., and Simonovits, M. Some extremal problems in graph theory. *Combinatorial Theory and Its Applications 1* (1970), 377–390.

[2] Erdős, P., and Stone, A. On the structure of linear graphs. *Bull. Am. Math. Soc. 52* (1946), 1087–1091.

[3] Pinchasi, R., and Sharir, M. On graphs that do not contain the cube and related problems. *Combinatorica 25* (2005), 615–623.