Induced Ramsey Type Theorems

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Abstract

We will present some improved results for Ramsey type theorems on graphs with a forbidden induced subgraph. Below we give basic definitions and state main results.

Definition 1 (Homogeneous set)**.** *A subset of vertices of a graph is homogeneous if it is either an independent set (empty subgraph) or a clique (complete subgraph). We use hom(G) to denote the size of the largest homogeneous set in G.*

Definition 2 (Induced subgraph)**.** *A graph H is an induced subgraph of a graph G if $V(H) \subset V(G)$ and two vertices of H are adjacent if and only if they are adjacent in G.*

Definition 3 (k-universal)**.** *A graph is k-universal if it contains all graphs on at most k vertices as induced subgraphs.*

Definition 4 (H-free graph)**.** *A graph is H-free if it does not contain H as an induced subgraph.*

We first state Rödl's theorem on H-free graphs.

Theorem 5. *For each $\varepsilon \in (0, 1/2)$ and graph H there is a $\delta = \delta(\varepsilon, H) > 0$ such that every H-free graph on n vertices contains an induced subgraph on at least δn vertices with edge density at most ε or at least $1 - \varepsilon$.*

The authors provide a better bound on $\delta(\varepsilon, H)$ than given by Rödl's proof. The new result is stated in the following theorem.

Theorem 6. *There is a constant c such that for each $\varepsilon \in (0, \frac{1}{2})$ and graph H on $k \geq 2$ vertices, every H-free graph on n vertices contains an induced subgraph on at least $2^{-ck(\log \frac{1}{\varepsilon})^2} n$ vertices with edge density either at most ε or at least $1 - \varepsilon$.*

Such results on H-free graphs imply that H-free graphs are far from having a uniform edge distribution. One of the questions raised by Chung and Graham on the edge distribution in H-free graphs is based on their following theorem.

Theorem 7. *For a graph G on n vertices the following properties are equivalent:*

1. *For every subset S of G, $e(S) = \frac{1}{4}|S|^2 + o(n^2)$.*
2. *For every fixed k-vertex graph H, the number of labeled copies of H in G is $(1 + o(1))2^{-\binom{k}{2}} n^k$.*

Now one can ask that by how much a graph G can deviate from property 1 assuming a deviation from property 2. The authors answer this question in the following theorem.

Theorem 8. *Let $G = (V, E)$ be a graph on n vertices with at most $(1 - \varepsilon)2^{-\binom{k}{2}}n^k$ labeled induced copies of a k-vertex graph H. Then there is a subset $S \subset V$ with $|S| = n/2$ and $|e(S) - \frac{n^2}{16}| \geq \varepsilon c^{-k} n^2$, where $c > 1$ is an absolute constant.*