On the Dependent Random Choice Technique
(based on the survey paper of Jacob Fox and Benny Sudakov)

Andrzej Dudek
Carnegie Mellon University

In this talk we discuss a simple and yet surprisingly powerful method called the *dependent random choice technique*. Roughly speaking it says that every sufficiently dense graph contains a large set of vertices U so that every small subset of U has many common neighbors. This result has attracted a considerable amount of attention and has been studied by several researchers including Gowers, Kostochka, Rödl, and Sudakov.

Formally, the technique is based on the following two results.

Key Lemma *Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with $|V| = n$ and average degree $d = 2|E(G)|/n$. If there is a positive integer t such that*

$$\frac{d^t}{n^{t-1}} \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a,$$

then V contains a subset U of size at least a such that every r vertices in U have at least m common neighbors.

Embedding Lemma *Let $H = (A \cup B, F)$ be a bipartite graph in which $|A| = a$, $|B| = b$, and the vertices in B have degree at most r. If G is a graph with a vertex subset U such that $|U| = a$ and all subsets of U of size r have at least $a + b$ common neighbors, then H is a subgraph of G.*

Clearly, the Key Lemma gives a setup for the Embedding Lemma. We illustrate this with two results from Extremal Graph Theory and one from Additive Combinatorics.

Recall that for a graph H and positive integer n, the *Turán number* $ex(n, H)$ denotes the maximum number of edges of a graph with n vertices that does not contain H as a subgraph. A fundamental problem in extremal graph theory is to determine or estimate $ex(n, H)$. Turán determined these numbers for complete graphs. Furthermore, the asymptotic behavior of Turán numbers for graphs of chromatic number at least 3 is given by a well known result of Erdős, Stone, and Simonovits. For bipartite graphs H, however, the situation is considerably more complicated. We present the following tight result.

Theorem 1 (Füredi, 1991; Alon, Krivelevich and Sudakov, 2003) *If $H = (A \cup B, F)$ is a bipartite graph in which all vertices in B have degree at most r, then $ex(n, H) \leq cn^{2-1/r}$, where $c = c(H)$ depends only on H.*

The second example comes from Ramsey Theory. Recall that for a graph H, the *Ramsey number* $r(H)$ is the minimum positive integer N such that every 2-coloring of the edges of the complete graph on N vertices contains a monochromatic copy of H. Determining or estimating Ramsey numbers is one of the central problems in combinatorics. Here we consider the Ramsey number of the cube. The *r-cube* Q_r is the r-regular graph with 2^r vertices whose vertex set consists of all binary vectors $\{0, 1\}^r$ and two vertices are adjacent if they differ in exactly one coordinate. Burr and Erdős conjectured that $r(Q_r)$ is linear in the number of vertices of the r-cube. Although this conjecture has drawn a lot of attention, it is still open. The first polynomial bound was given by Shi. Here we show a very short proof of this result using the dependent random choice technique.

Theorem 2 $r(Q_r) \le |V(Q_r)|^3$.

In the remaining time of the talk we discuss the Balog-Szemerédi-Gowers theorem from Additive Combinatorics. Let A and B be two sets of integers. Define the *sumset* $A + B = \{a + b : a \in A, b \in B\}$. For a bipartite graph G with parts A and B and edge set E, define the *partial sumset* $A +_G B = \{a + b : a \in A, b \in B, (a, b) \in E\}$. In many applications in Additive Combinatorics, instead of knowing $A + A$ one only has access to a dense subset of this sum. For example, can we draw a useful conclusion from the fact that $A +_G A$ is small for some dense graph G? One can show that in this case $A + A$ can still be very large. Fortunately, we are still able to draw a useful conclusion thanks to the following result.

Balog-Szemerédi-Gowers Theorem *If A and B are two sets of size n, G has at least cn^2 edges and $|A +_G B| \le Cn$, then one can find $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge c'n$, $|B'| \ge c'n$ and $|A' + B'| \le C'n$, where c' and C' depend on c and C only.*

The original proof of Balog and Szemerédi used the Regularity Lemma and gave a tower-like dependence between the parameters. Gowers' approach gives a much better bound and it is based on the following graph-theoretic lemma, which is also of independent interest.

Lemma *Let G be a bipartite graph with parts A and B of size n and with cn^2 edges. Then it contains subsets $A' \subseteq A$ and $B' \subseteq B$ of size at least $cn/8$, such that there are at least $2^{-12}c^5n^2$ paths of length three between every $a \in A'$ and $b \in B'$.*

An essential ingredient in the proof of the above result is the following variant of the Key Lemma.

Key Lemma ("linear" version) *Let G be a bipartite graph with parts A and B and $e(G) = c|A||B|$ edges. Then for any $0 < \varepsilon < 1$, there is a subset $U \subseteq A$ such that $|U| \ge c|A|/2$ and at least a $(1 - \varepsilon)$-fraction of the ordered pairs of vertices in U have at least $\varepsilon c^2|B|/2$ common neighbors in B.*