On the measure of intersecting families, uniqueness and stability

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An elegant and concise argument due to Hoffman uses Fourier analysis to obtain an upper bound on the independence number of regular graphs via spectral methods. This argument lies at the heart of the technique used by Friedgut, who manipulates and enriches it to solve a problem in extremal set theory, which is seemingly unconnected at first sight. Friedgut proves the following theorem.

Theorem 1. *Let $t, k, n > 0$ be integers and let $\gamma > 0$ such that $\gamma n < k < \big((t+1)^{-1} - \gamma\big)n$. Let $\mathcal{F} \subset \binom{[n]}{k}$ be an t-intersecting family. If $|\mathcal{F}| \geq (1-\varepsilon)\binom{n-t}{k-t}$ then there exists a set $B \in \binom{[n]}{t}$ such that for $\mathcal{B} := \{X \in \binom{[n]}{k} : B \subset X\}$ we have $|\mathcal{F} \setminus \mathcal{B}| < c\varepsilon\binom{n}{k}$ where $c = c(\gamma)$.*

Spectral graph theory studies the connections between the properties of a graph G and the eigenvalues and eigenvectors of its adjacency matrix $A(G)$. The matrix $A(G)$ is symmetric and thus has real eigenvalues. Moreover, we can choose a set of orthonormal eigenvectors $u_1, \dots, u_m$ corresponding to these eigenvectors. Hoffman used this set of eigenvectors in order to derive an upper bound on the independence number of a d-regular graph G in terms solely of the largest and smallest eigenvalues of $A(G)$ as follows. Let f be the characteristic vector of a largest independent set in G. Then f can be written as a linear combination of $u_1, \dots, u_m$. Now, applying basic tools from Fourier analysis in order to estimate the coefficients in such an expansion and relate them to the eigenvalues of $A(G)$ results in Hoffman's bound. This bound is not always optimal, but for some 'well-behaved' graphs it is sharp.

Extremal set theory investigates how large families of sets of a certain type can be. The Erdős-Ko-Rado (EKR) theorem is one of the most fundamental results in this area. We say that a set family $\mathcal{F}$ is *t-intersecting* if for every pair of sets $X, Y \in \mathcal{F}$, we have $|X \cap Y| \geq t$. Let $\mathcal{F}^*(k, n, t)$ be the family of all k-sets in $[n]$ containing $[t]$. The EKR theorem states that $\mathcal{F}^*(k, n, t)$ is extremal, i.e. largest, among all t-intersecting families of k-sets if n is sufficiently large. This naturally raises several questions.

(i) Which size is 'sufficient' for n?

(ii) Is $\mathcal{F}^*(k, n, t)$ the only extremal family, up to isomorphism? (*Uniqueness*)

(iii) Does every t-intersecting k-set family which is almost as large as $\mathcal{F}^*(k, n, t)$ look similar to $\mathcal{F}^*(k, n, t)$? (*Stability*)

(iv) What is a meaningful analogue of this result for set families that are not restricted to k-sets?

The first of these questions resisted attempts to be settled for some twenty years. Frankl conjectured that $n > (t+1)(k-t+1)$ is large enough, and showed that this is best possible. This was finally proven by Wilson. The second and third questions proved to be even harder. Friedgut's eventual positive answer is precisely Theorem 1.

For the last question one could easily be tempted to ask for an extremal t-intersecting family of sets with arbitrary set sizes. However, an answer to this question is neither closely related to the EKR theorem, nor is it particularly interesting. On the one hand, no t-intersecting family $\mathcal{F}$ in $2^{[n]}$ may contain both a set and its complement, so $|\mathcal{F}| \leq 2^{n-1}$. On the other, the family of all subsets of $[n]$ of size at least $\frac{n+t}{2}$ is t-intersecting, and if $t^2 \ll n$ it has size $\big(1-o(1)\big)2^{n-1}$.

The reason why this first attempt to approach question (*iv*) fails is very simple: there are many more sets of size $n/2$ than of size pn with $p \neq 1/2$. One way to shift from favouring sets of size $n/2$ to favouring sets of size pn with $0 < p < 1$ is to consider the *product measure* μ_p on the Boolean cube $2^{[n]}$. To a set $S \subset [n]$ as this measure assigns weight

$$\mu_p(S) = p^{|S|}(1-p)^{n-|S|}\,.$$

And indeed, it turns out that this leads to a meaningful analogue of the EKR theorem for non-fixed set sizes: What is the maximum μ_p-measure of a t-intersecting family in $2^{[n]}$? The conjecture was that for $p < 1/(t+1)$ again the family of all subsets of $[n]$ containing $[t]$ is extremal. Dinur and Safra proved this in the case $t = 2$.

Friedgut proves in this paper the extremal and corresponding uniqueness and stability results for all t (for $0 < p < 1/(t+1)$). Surprisingly, Theorem 1 then follows as a corollary from these results in the product measure setting.

So the main work in Friedgut's paper goes into the proofs of the uniqueness and stability results under the product measure. The idea of this proof is based on Hoffman's argument, which also lets us complete the path to the spectral methods from the beginning. For obtaining a connection between Hoffman's result and t-intersecting families, define the *t-disjointness graph* D on $2^{[n]}$ to be the graph whose vertex set is $2^{[n]}$, and whose edges are those between pairs of sets whose intersection is smaller than t. An independent set in this graph is simply a t-intersecting family. Accordingly, we are interested in upper bounds of independent sets, which is what Hoffman's bound provides. However, there obviously are a couple of problems: Firstly, we are not looking for an independent set in D with maximal cardinality but one with maximal μ_p-measure. Secondly, the graph D is not regular; so the Hoffman bound does not apply. And thirdly, it is not clear how to compute the eigenvalues of D. Friedgut deals with these inconveniences by replacing the adjacency matrix $A(D)$ by a different matrix A' which does not only capture information about adjacencies between vertices S and T in D but also encodes the size of $S \cap T$. This is achieved by using values from certain polynomial rings as matrix entries. By performing the Fourier analysis from Hoffman's proof also over these polynomial rings, Friedgut proves the product measure version of the EKR theorem.

A sophisticated analysis of this proof then reveals that the extremal family is up to isomorphism *unique*. For proving *stability*, Friedgut uses another advantage of the spectral method: a family with nearly extremal measure must behave (spectrally) very similarly to a genuine extremal family. Using this and a recent deep result due to Kindler and Safra, Friedgut concludes the proof of his result.

In the talk we will provide an overview of the results from extremal set theory that are connected to Friedgut's result. We will introduce the relevant concepts and methods from spectral graph theory and Fourier analysis, give the proof of Hoffman's bound, and explain how Friedgut adapts this idea to his setting. Here, we shall ignore many of the technical difficulties and rather try to convey the ideas. Finally, we outline how this implies Theorem 1.