Size-Ramsey numbers of powers of tight paths

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University College London

joint work with **Alexey Pokrovskiy** and **Liana Yepremyan**

Warwick

March 2021

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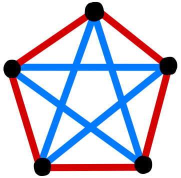
E.g. $K_5 \not\rightarrow K_3$ , but $K_6 \rightarrow K_3$.

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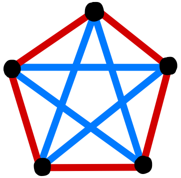
E.g. $K_5 \not\rightarrow K_3$ , but $K_6 \rightarrow K_3$.

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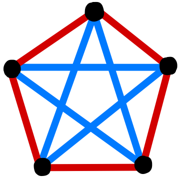
E.g. $K_5 \not\longrightarrow K_3$  , but $K_6 \longrightarrow K_3$. So $r(K_3) = 6$.

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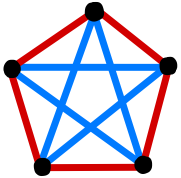
E.g. $K_5 \not\to K_3$ , but $K_6 \to K_3$. So $r(K_3) = 6$.

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Equivalently, $r(H) = \min\{|G| : G \to H\}$.

#vs in G

Size-Ramsey numbers

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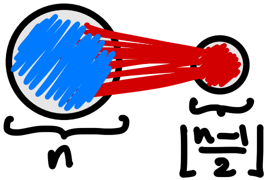
path of length n ↗



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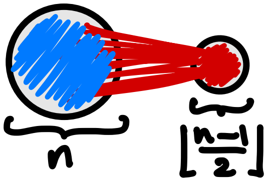

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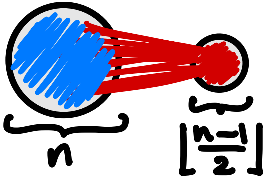

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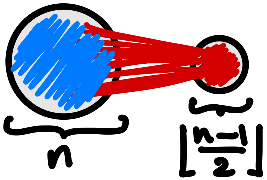

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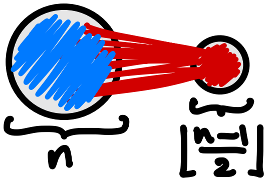

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Best bounds: $3.75n \lesssim \hat{r}(P_n) \leq 74n$.

Bal–DeBiasio '20 ↗ ↖ Dudek–Prałat '17

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$\forall \ell$: $\hat{r}(P_n^\ell) = O(n)$.

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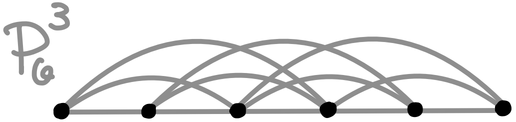

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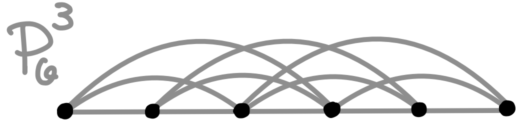

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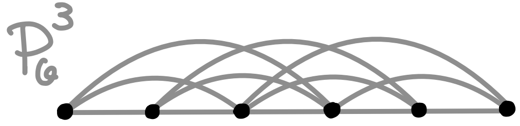

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Han-Jenssen-Kohayakawa-Mota-Roberts '20: $\forall \ell, s$: $\hat{r}_s(P_n^\ell) = O(n)$.

Bounded degree trees

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The above results do not generalise to bounded degree graphs.

Rödl-Szemerédi '00: there is a family $\{H_n\}$ where H_n is an n-vx graph with max deg 3 and $\hat{r}(H_n) = \Omega(n(\log n)^{1/60})$.

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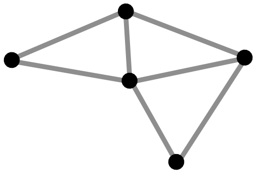

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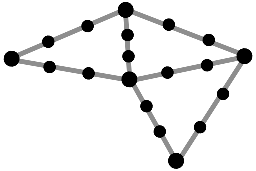

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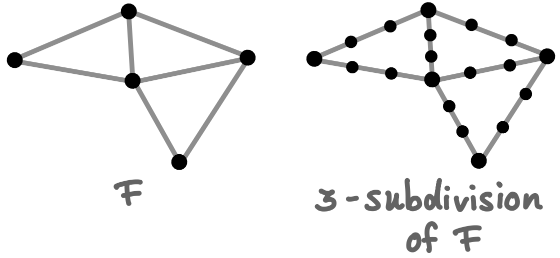

Draganić-Krivelevich-Nenadov '20:

- $\ast$ $\forall s, \Delta, q$: for every q-subdivision H of a graph with max degree $\leq \Delta$ s.t. $|H| = n$: $\hat{r}_s(H) = O(n^{1+1/q})$.

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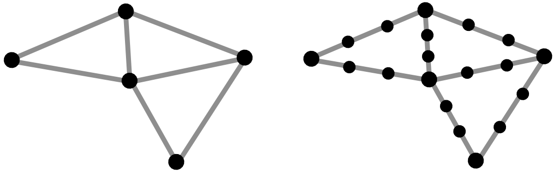

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- ∀s,Δ,q: for every q-subdivision H of a graph with max degree ≤ Δ s.t. |H|=n: $\hat{r}_s(H)=O(n^{1+1/q})$.

- ∀s,Δ ∃c: for every L-subdivision H of a graph with max degree ≤ Δ s.t. |H|=n and $L \geq c \cdot \log n$: $\hat{r}_s(H)=O(n)$.

Hypergraphs

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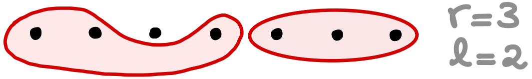

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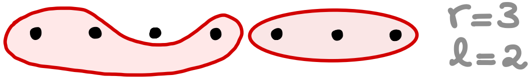

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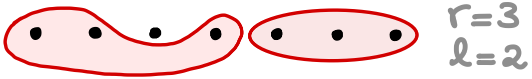

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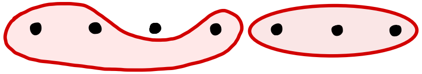

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Thm (L., Pokrovskiy, Yepremyan '21+). $\forall s, \ell, \Delta\ \exists c$: for every H on n vs which is the L-subdivision, where $L \geq c \log n$, of a graph with max deg $\leq \Delta$:

$$\hat{r}_s(H^\ell) = O(n).$$

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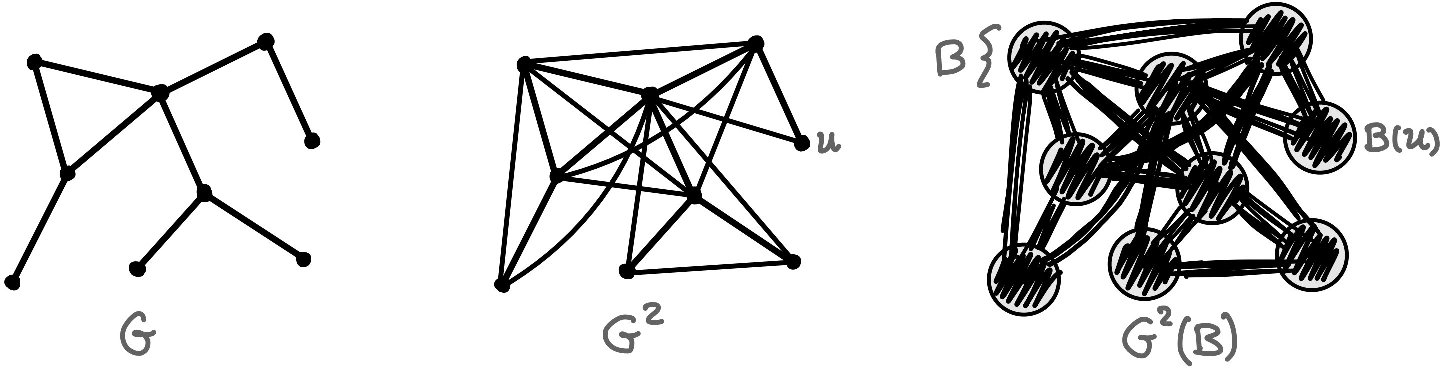

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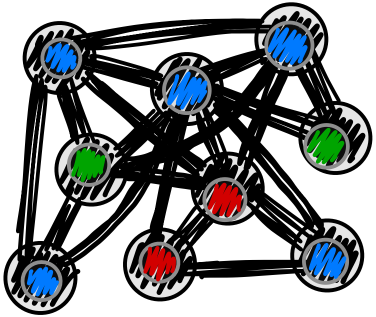

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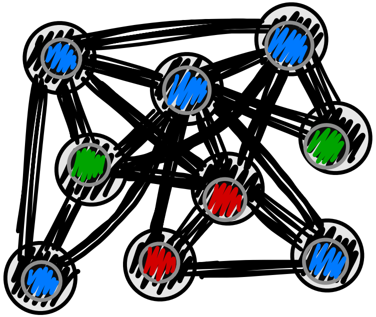

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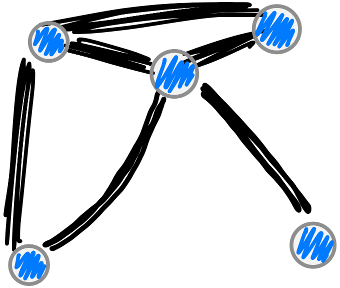

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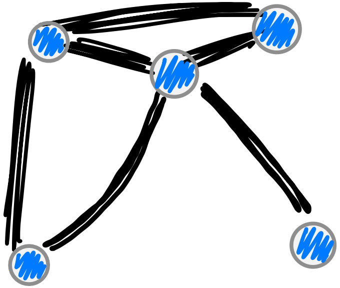

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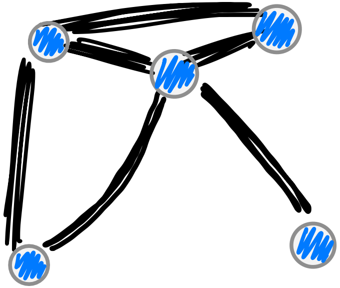

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If fail, aim to exploit the sparsity of blue edges...

New ingredient: stronger Ramsey lemma

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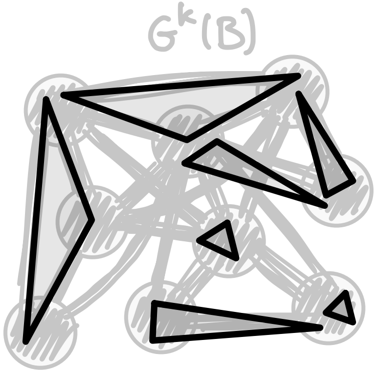

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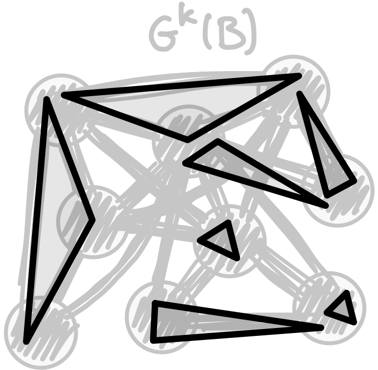

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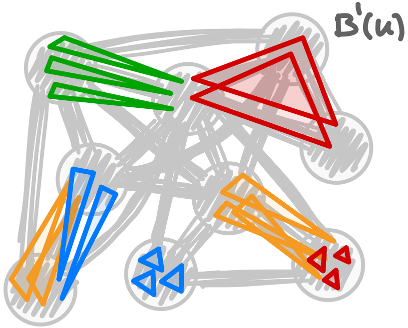
Lemma. H hypergraph, $\Delta(H)=O(1)$. $B \gg b$.
For every s-colouring of $H(B)$ $\exists B'(u) \subseteq B(u)$ with $|B'(u)|=b$ s.t. in $\bigcup B'(u)$ if $|e \cap B'(u)| = |f \cap B'(u)|$ $\forall u$ then e & f have the same colour.

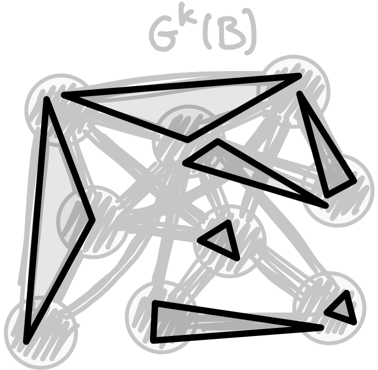
the clique corresponding to u



New ingredient: stronger Ramsey lemma

Consider $K_r(G^k(B))$.

the r-cliques in $G^k(B)$



some edges of $K_3(G^k(B))$

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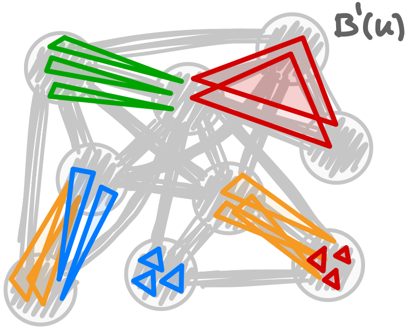

Proof. Apply a Ramsey-type result to each "edge-type". Each $B(u)$ is involved in $O(1)$ applications, so won't shrink too much. □

Looking for tight walks

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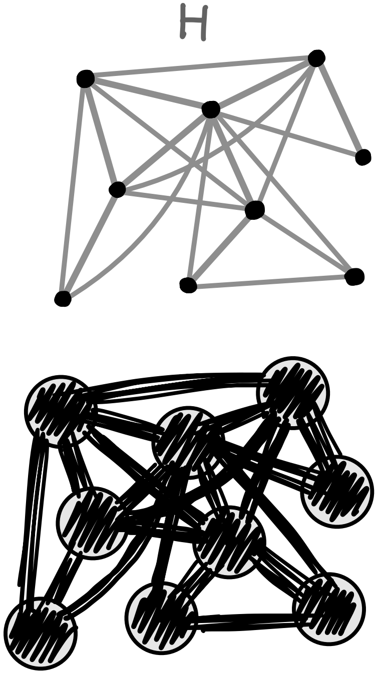

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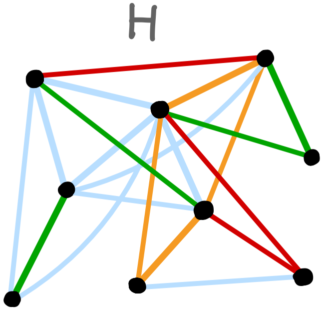

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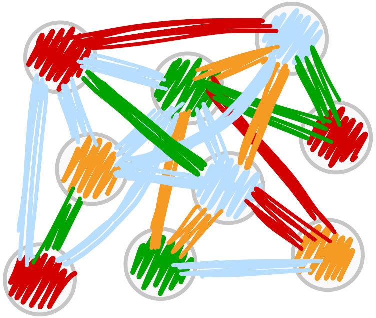

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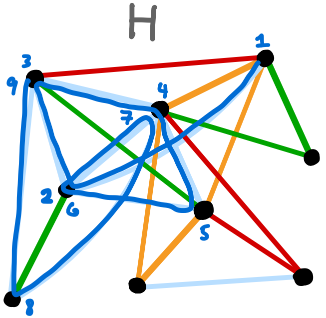

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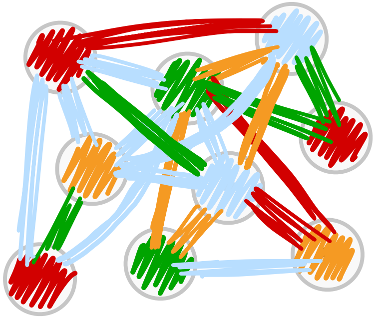

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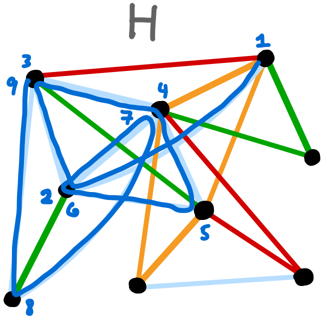

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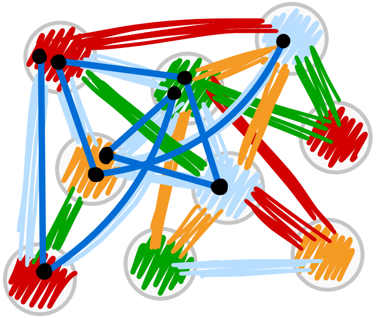

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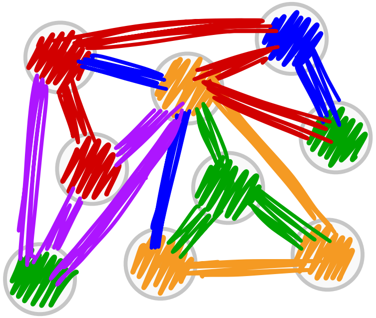
Sketch of our proof for $r=2$

Consider an s-colouring of $G^k(B)$.



By Ramsey lemma from previous slide, may assume that the vs in $B(u)$ are twins.

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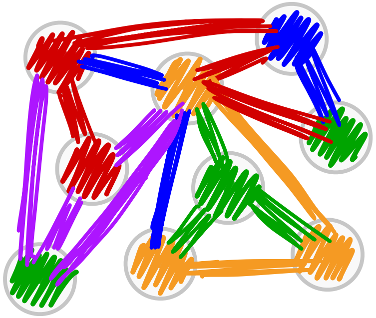

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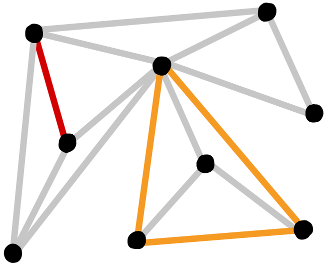
Define auxiliary colouring of G^k:



- colour uv by c if ∃"short" c-coloured ℓ^{th} power of a path starting with ℓ vs in $B(u)$ and ending with ℓ vs in $B(v)$.
- ow, colour uv grey.

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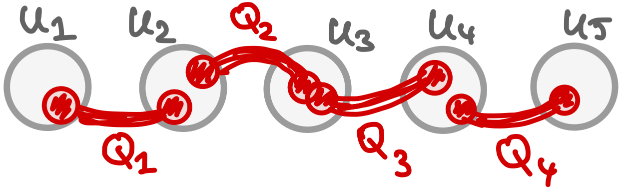

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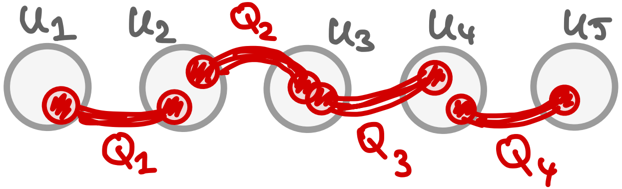
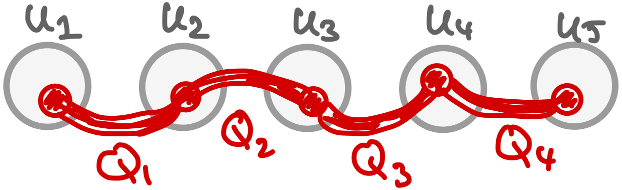


Because all vs in $B(u_i)$ are identical, may assume that the last ℓ vs in Q_i are the first vs in Q_{i+1}.



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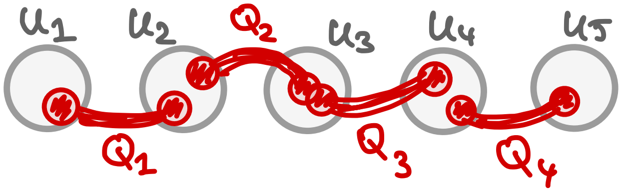

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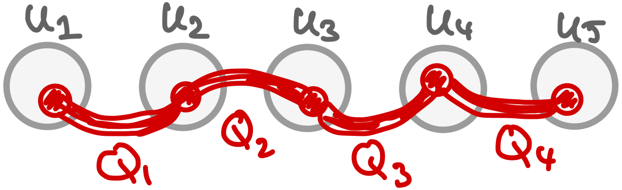


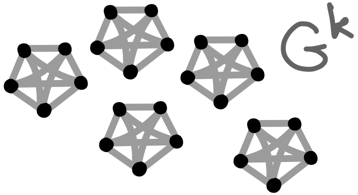
$\Rightarrow \exists$ ℓ^{th} power of a red walk on n vs, with few repetitions.

(If $v \in Q_i$ then $\mathrm{dist}(u_i, v) = O(1)$.
This can happen for $O(1)$ u_i's.) □

Many grey cliques

Lemma. If there is no non-grey mono P_n in the auxiliary colouring, then there are disjoint grey K_t's that cover most vs.

← large constant



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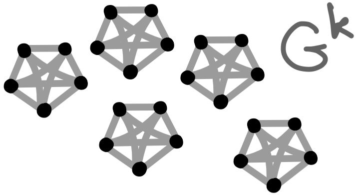

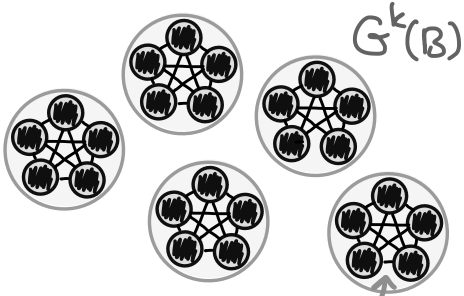

no short mono ℓ^{th} powers of paths between small blobs

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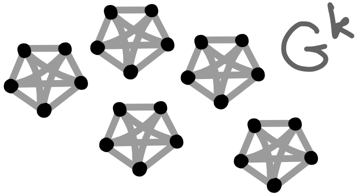

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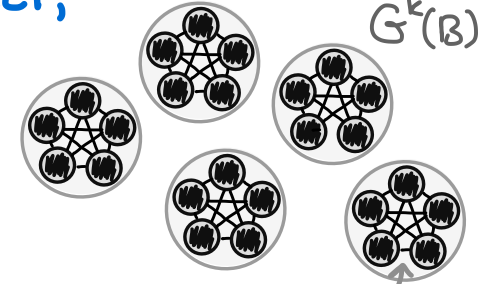

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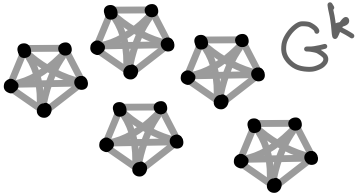

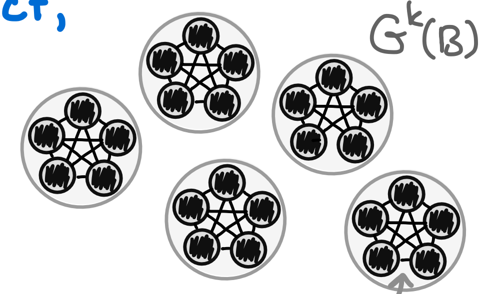

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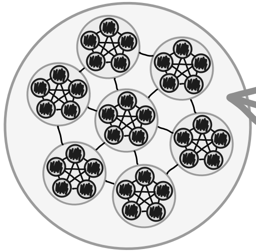

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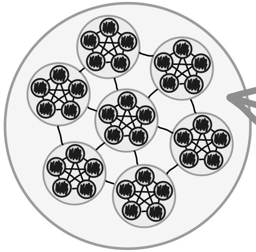

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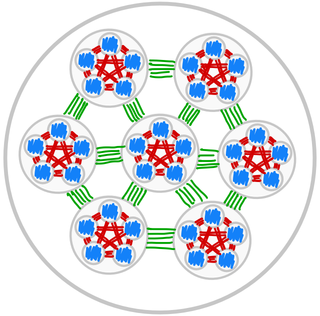

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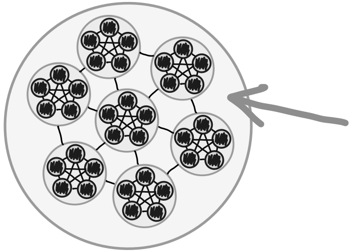

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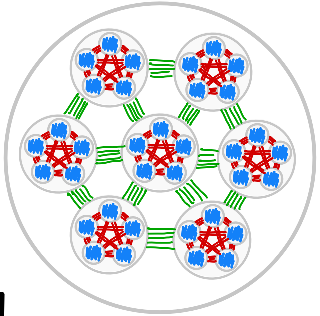
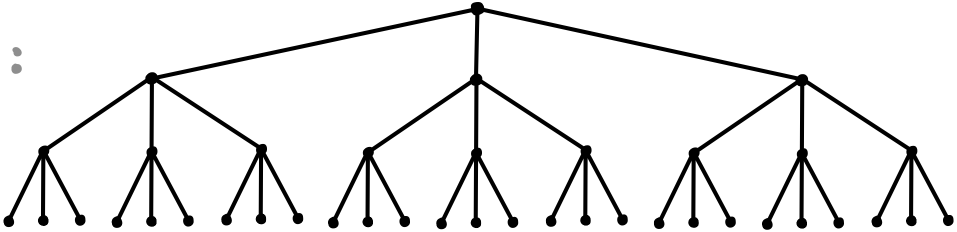


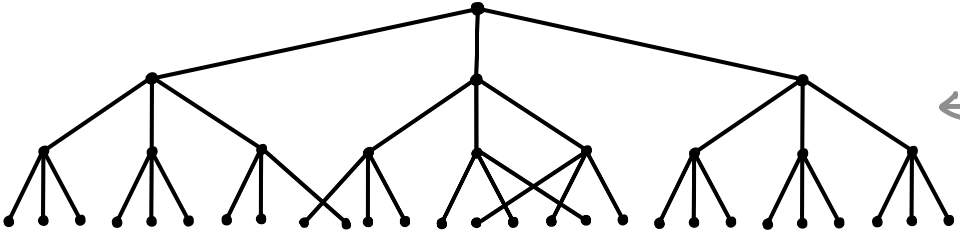
After $\leq s+1$ iterations, find required power of a walk. $\square$

Ordered trees

A **d-ary ordered tree of height h** is a complete d-ary tree of height h, along with an ordering of its leaves obtained from a planar drawing of the tree with all leaves on a line.

ordered 3-ary tree of height 3:
(leaves ordered left-to-right)

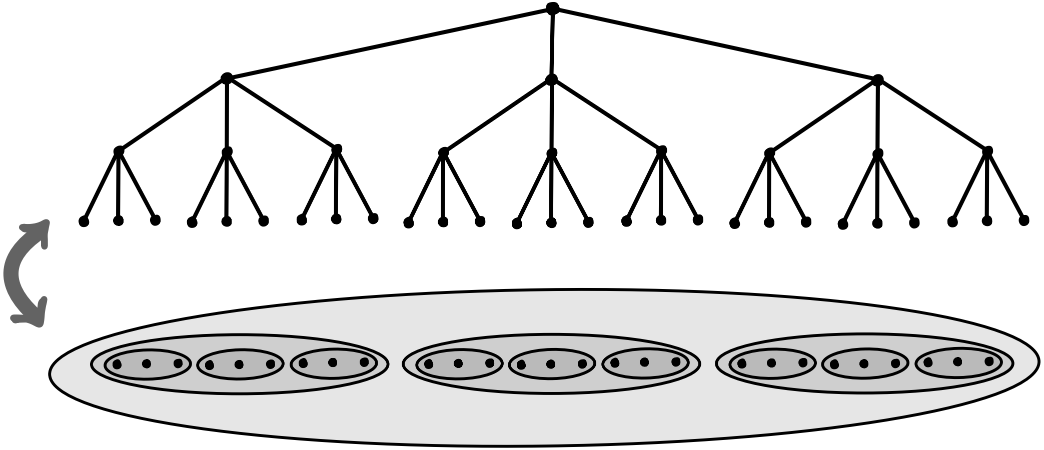




← not an ordered tree

Using ordered trees to model leveled blobs

We model "h-level blobs" by the leaves of ordered d-ary trees of height h,



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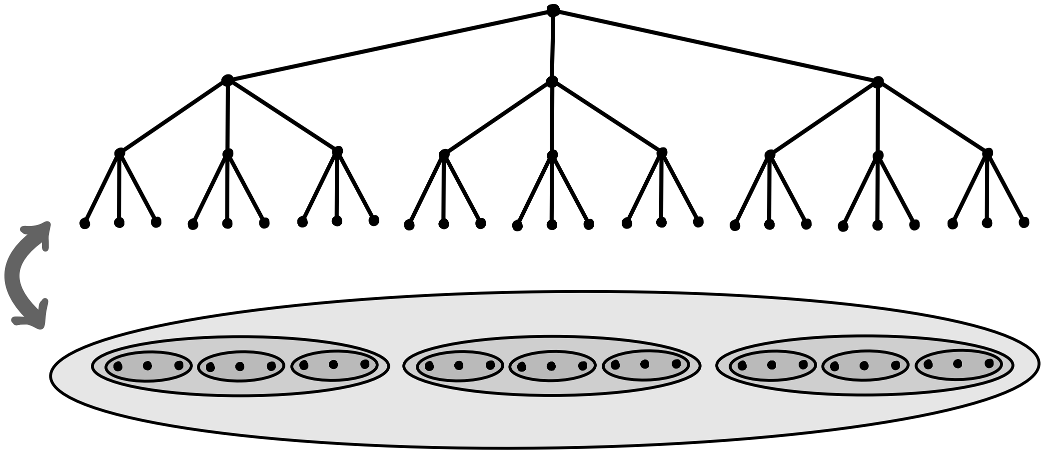

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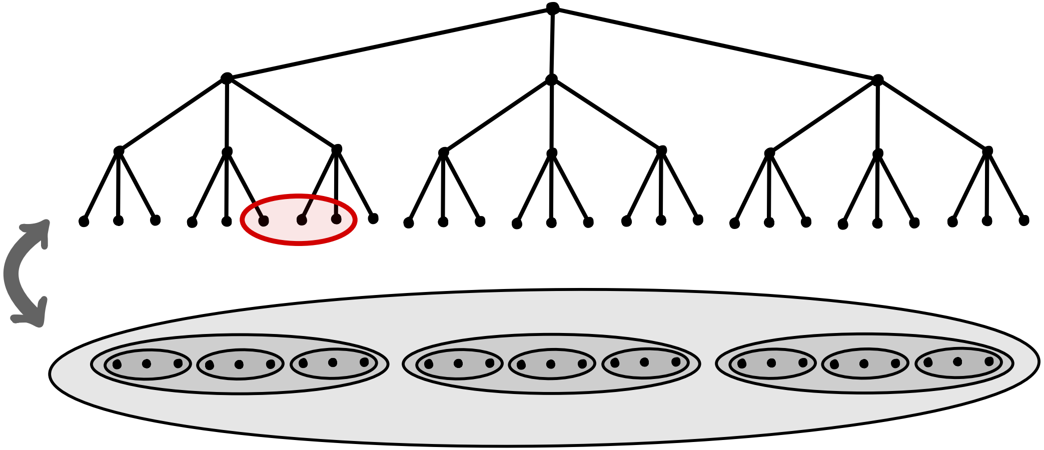

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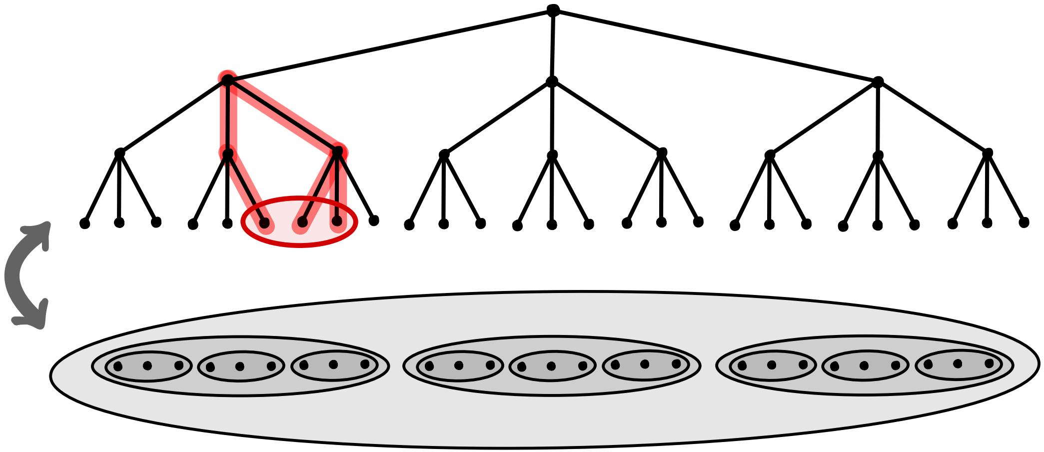

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In step j:

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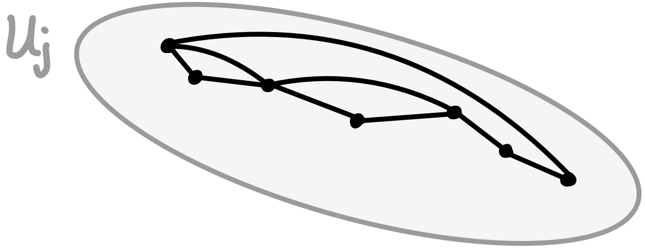

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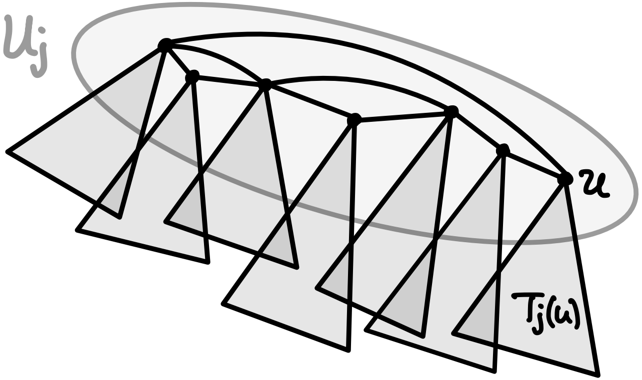

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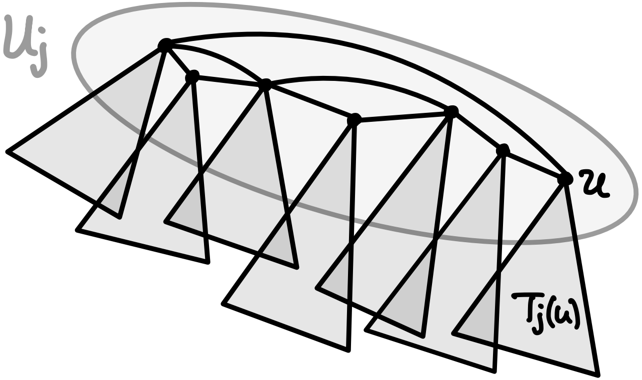

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  (in the r-graph whose edges are r-sets of leaves whose roots are cliques in G^{k_j}).

Ramsey lemma for ordered trees

Lemma. T ordered D-ary tree of height h, $D \gg d$.
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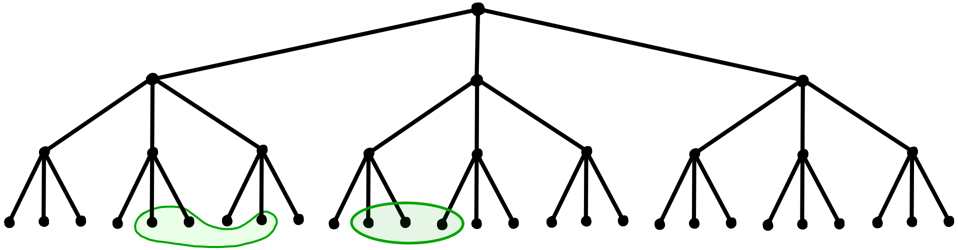

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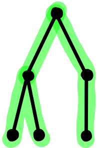

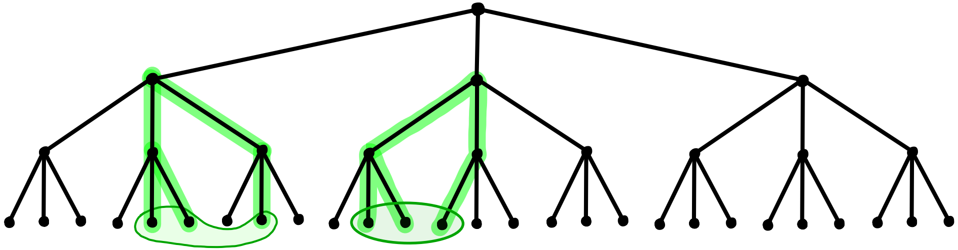

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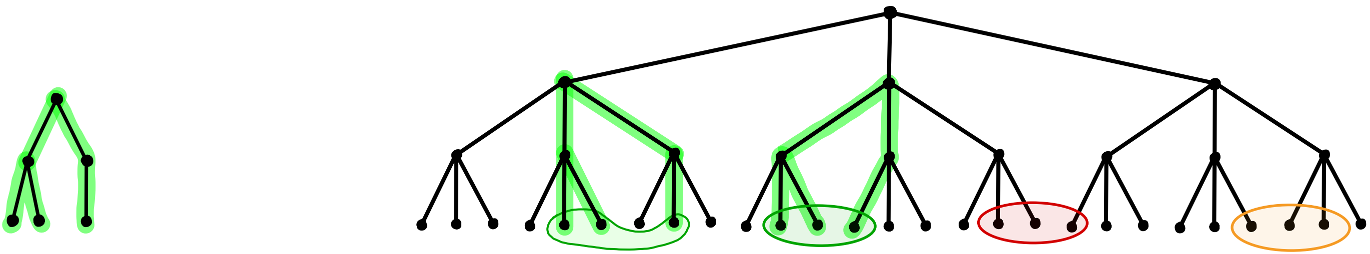
For every s-colouring of r-sets of leaves of T, there is a d-ary subtree $T' \subseteq T$ of height h, s.t. r-sets of leaves of T' corresponding to isomorphic ordered trees have the same colour.



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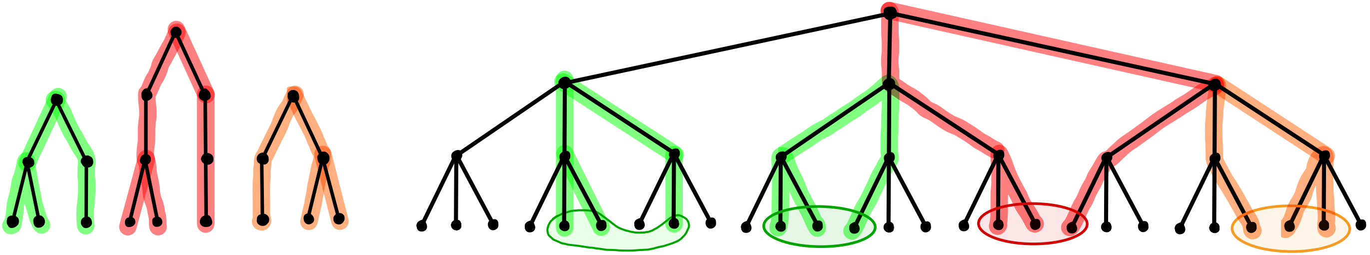

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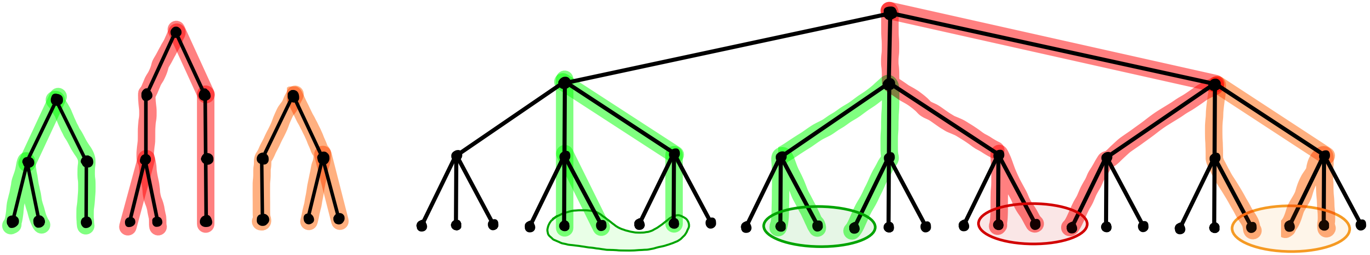

By this lemma:

may assume that edges corresponding to isomorphic ordered forests have the same colour.

Auxiliary colouring

Define auxiliary colouring of $G^{k_{j+1}}$:

colour ↘ ↙ ordered tree on ℓ leaves

* colour uv (c,S) if there is a short c-coloured ℓ^{th} power of path from an S-copy in $T_j(u)$ to an S-copy in $T_j(v)$.

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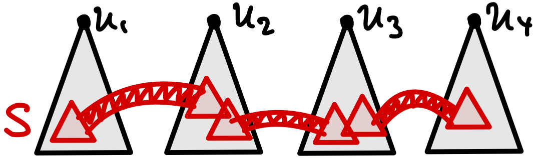

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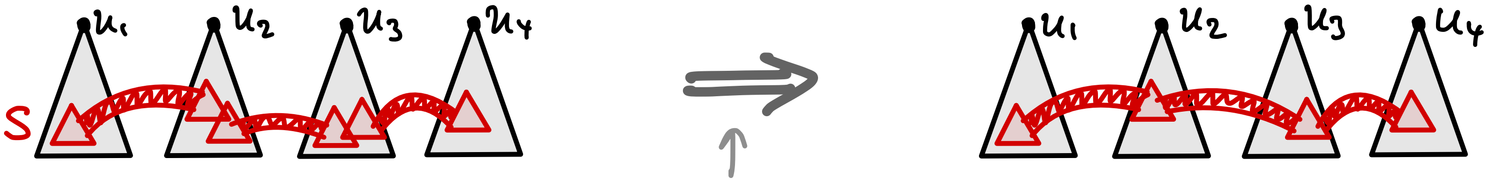

↑ using assumption that edges corresponding to isomorphic ordered forests have same colour

□

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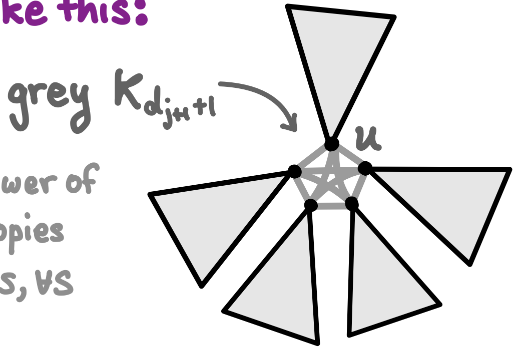

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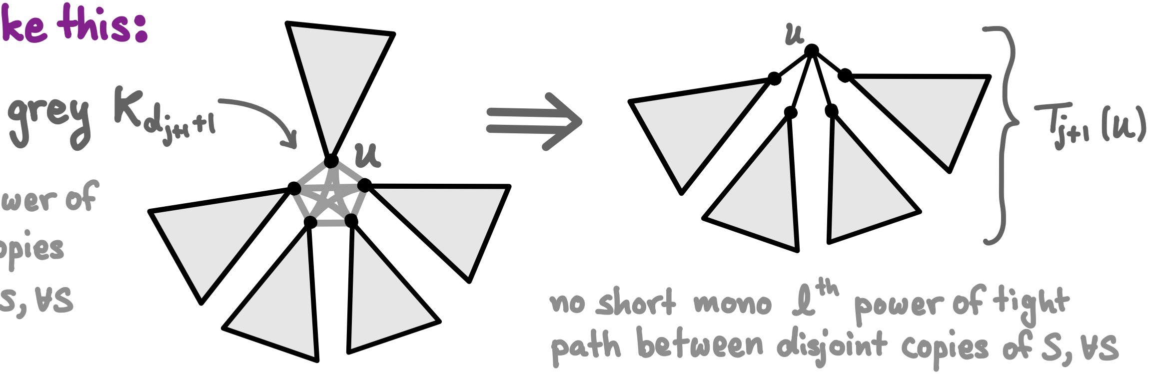

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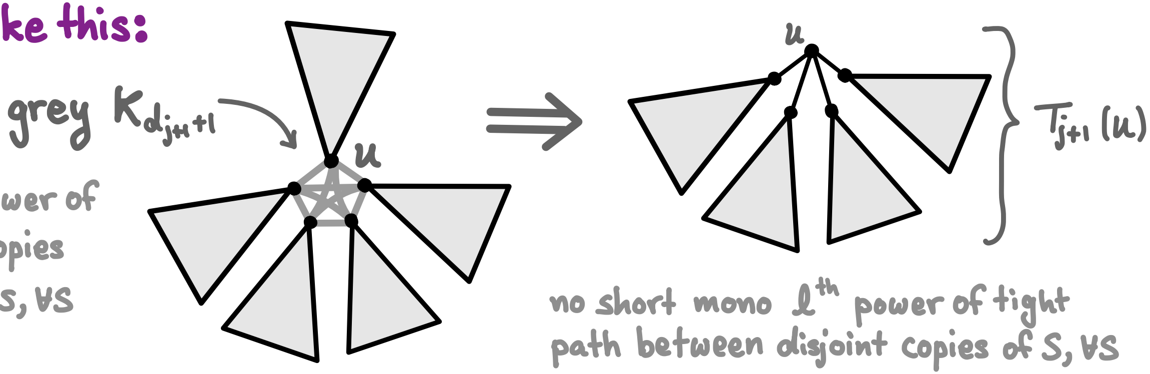

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For every s-colouring of r-sets of leaves $\exists$ mono ℓ^{th} power of tight path on 3ℓ vs starting and ending at disjoint copies of some ordered tree S on ℓ leaves.

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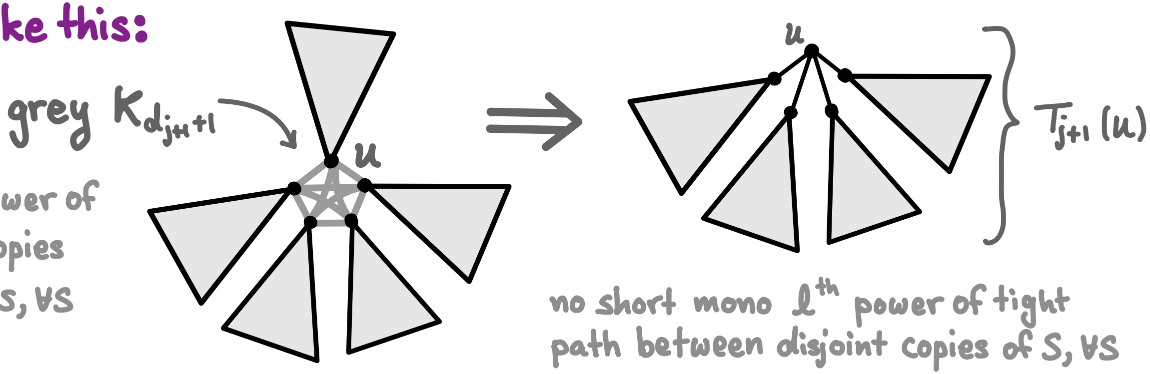

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⇒ After $\leq h$ steps find the required power of a walk. ☐

Open problems

- Best bounds on $\hat{r}(P_n)$: $3.75n \lesssim \hat{r}(P_n) \leq 74n$.

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Thank you for listening!