Permutation Patterns

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October 13, 2020

A quick introduction

Permutation Patterns (PP)

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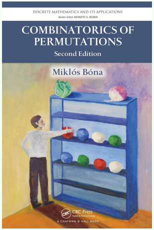
The origins of the area

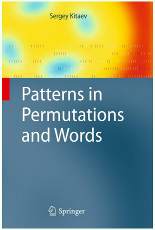
The introduction of the area is traditionally attributed to Donald Knuth and in particular to exercises on pages 242–243 in his first volume of "The Art of Computer Programming" in **1968**, while the first systematic study of pattern avoidance was done by Rodica Simion and Frank W. Schmidt in **1985**.

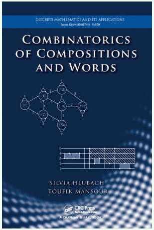
A quick introduction

Literature

There are several survey papers on PP, and the books







Also, Permutation Patterns appear in the **2015** Handbook of Enumerative Combinatorics (the chapter "Permutation classes" by Vincent Vatter), and there are ca **2,000 research papers** related to the subject (just an educated guess!).

Organisation of the talk

- Introduction to classical, vincular, consecutive, bivincular, mesh, frame, marked mesh, quadrant marked mesh, and partially ordered patterns

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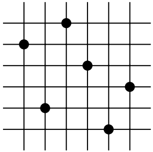
- Questions of interest
- Open problems in selected research directions related to
  - classical patterns;
  - consecutive patterns;
  - Wilf-equivalence for patterns of length 4;
  - distribution of mesh patterns;
  - algorithmic aspects;
  - bijective questions on partially ordered patterns;
  - crucial and bicrucial permutations;
  - graph representations.

We can view a permutation as a permutation diagram:

526413 = 

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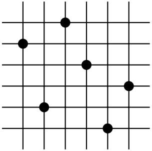
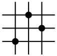
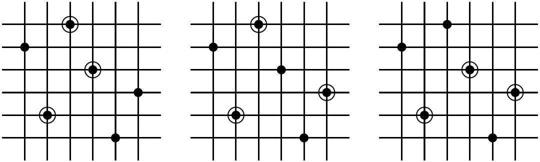
526413 = 

There are 3 occurrences of the pattern 132 =  in the permutation:



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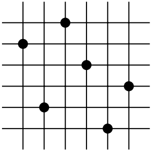
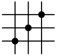
526413 = 

The same permutation avoids the pattern 123 = .

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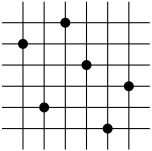
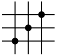
526413 = 

The same permutation avoids the pattern 123 = .

It is a well-known result (proved many times) that the pattern 123 is Wilf-equivalent to the pattern 132, that is, for any fixed permutation length the number of **123-avoiding permutations** is equal to that of **132-avoiding permutations**.

Vincular patterns = generalized patterns

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- Requirements for some elements to be adjacent (consecutive)

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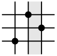
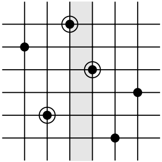
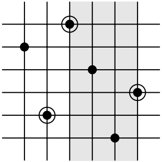
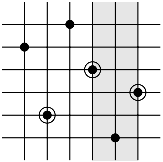
The pattern $1\underline{32}$ =  occurs in the permutation 526413:



occurrence



non-occurrence



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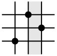
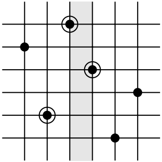
The pattern $1\underline{32}$ =  occurs in the permutation 526413:



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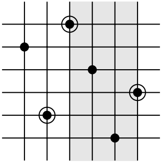

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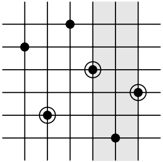

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Vincluar patterns were introduced by Babson and Steingrímsson in **2000** to express Mahonian statistics as combinations of permutation patterns. Vinculum is latin for bond.

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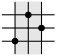
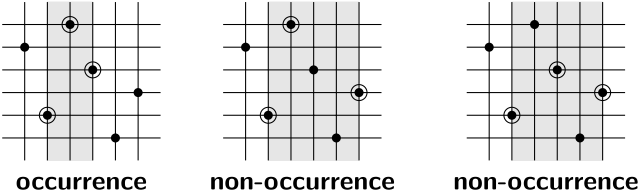
The pattern $\underline{132}$ =  occurs in the permutation 526413:



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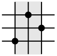
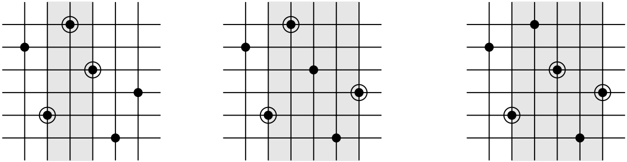
The pattern $\underline{132}$ =  occurs in the permutation 526413:



occurrence **non-occurrence** **non-occurrence**

Approaches in the literature to study **consecutive patterns** range from symbolic method, spectral approach, symmetric functions approach, and cluster method to inclusion-exclusion arguments and homological algebra; see [S. Kitaev. **Patterns in permutations and words.** Springer, 2011].

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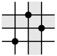
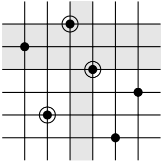
The pattern  does **not** occur in the permutation 526413:



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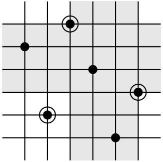

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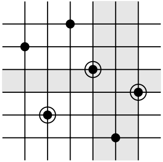

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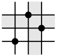
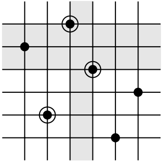
The pattern  does **not** occur in the permutation 526413:



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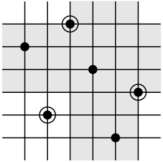

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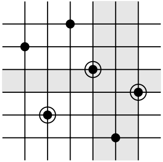

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Bivincular patterns were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in **2010** in the studies related to interval orders counted by the Fishburn numbers.

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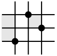
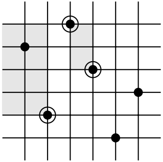
The pattern  occurs in the permutation 526413:



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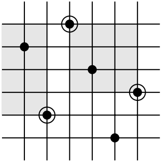

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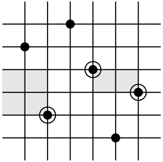

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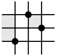
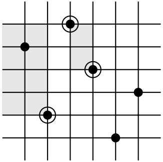
The pattern  occurs in the permutation 526413: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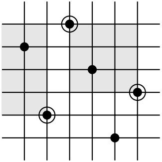
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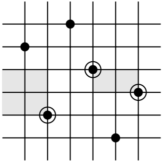
occurrence

Mesh patterns were introduced by Brändén and Claesson in **2011** to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns.

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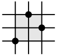
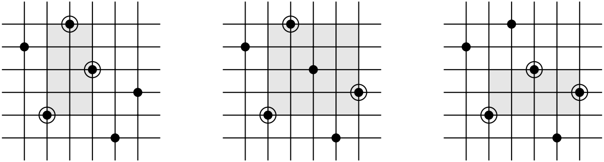
The pattern $\boxed{132}$ =  occurs in the permutation 526413:



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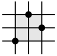
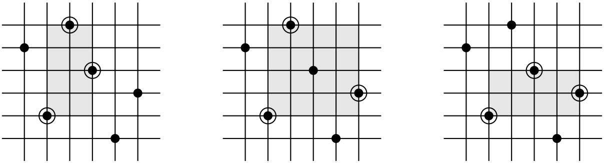
The pattern $\boxed{132}$ =  occurs in the permutation 526413:



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Frame patterns were introduced by Avgustinovich, Kitaev and Valyuzhenich in **2012**. They have several interesting properties. Enumeration of $\boxed{123}$ is still not solved!

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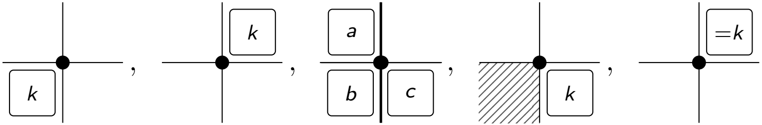

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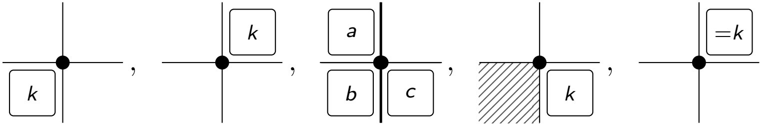

Marked mesh patterns were introduced by Úlfarsson in **2011** to give an alternative description of Schubert varieties defined by inclusions, Gorenstein Schubert varieties, 123-hexagon avoiding permutations, Dumont permutations and cycles in permutations. **Quadrant marked mesh patterns** were introduced by Kitaev and Remmel in **2012**.

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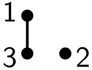
- For example, the POP $p =$ 

occurs five times in the permutation 41523, namely, as the subsequences 412, 413, 452, 453, and 523. Clearly, avoiding p is the same as avoiding the patterns 312, 321 and 231 at the same time.

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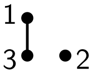
- For example, the POP $p =$ 

occurs five times in the permutation 41523, namely, as the subsequences 412, 413, 452, 453, and 523. Clearly, avoiding p is the same as avoiding the patterns 312, 321 and 231 at the same time.
- Any classical pattern of length k corresponds to a k-element chain.

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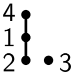
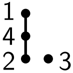
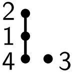
- For example, the simultaneous avoidance of the patterns 3214, 3124, 2134, and 2143 considered, up to trivial bijections, in [Defant. Stack-sorting preimages of permutation classes, 2018, https://arxiv.org/abs/1809.03123.] is nothing else but the avoidance of the POP 

, which suggests natural directions of research to study the avoidance of the POPs 

, 

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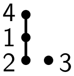
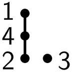
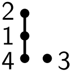
Partially ordered patterns

- POPs provide a uniform notation for several combinatorial structures such as peaks, valleys, modified maxima and minima, *p*-descents in permutations, and others. Also, POPs provide a convenient language to deal with larger sets of permutation patterns.
- For example, the simultaneous avoidance of the patterns 3214, 3124, 2134, and 2143 considered, up to trivial bijections, in [Defant. Stack-sorting preimages of permutation classes, 2018, https://arxiv.org/abs/1809.03123.] is nothing else but the avoidance of the POP 

, which suggests natural directions of research to study the avoidance of the POPs 

, 

, etc.
- POPs were studied in the context of permutations, words and compositions in the literature.

What to do with these patterns?

For permutations of length n, we are interested in the following questions:

- [Avoidance] How many permutations avoid a pattern? Set of patterns? Exact formulas? Generating functions? Wilf-equivalence?

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- [Algorithmic] How hard is it to find an occurrence of a pattern?
- [Topology] Based on the fact that the set of all permutations forms a poset with respect to pattern containment
- And some others ...

The pattern 1324

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The number a_n of n-permutations avoiding 1324 is known for all lengths ≤ 50, and a chronology of lower and upper bounds for the $\lim_{n\to\infty} \sqrt[n]{a_n}$ is as follows (for references see "Bevan et al.: A structural characterisation of Av(1324) and new bounds on its growth rate, *Europ. J. Comb.* (2020)"):

	Lower	Upper
2004: Bóna		288
2005: Bóna	9	
2006: Albert et al.	9.47	
2012: Claesson et al.		16
2014: Bóna		13.93
2015: Bóna		13.74
2015: Bevan	9.81	
2017: Bevan et al.	10.27	13.5

Consecutive patterns

6 conjectures are presented in "Nakamura. Computational approaches to consecutive pattern avoidance in permutations. PU.M.A. (2011)". Here are 5 of them (starting with a **2001** conjecture by Elizalde and Noy; $s_n(P)$ is the # of n-permutations avoiding a pattern, or a set of patterns P):

- $s_n(\underline{12\cdots k}) \geq s_n(p)$ for all p of length k and for all n (settled asymptotically by Elizalde in **2013**);
- $s_n(\underline{12\cdots(k-2)k(k-1)}) \leq s_n(p)$ for all p of length k and for all n (settled asymptotically by Elizalde in **2013**);
- $s_n(\underline{12\cdots k}, \underline{23\cdots k1}) \geq s_n(B)$ for all $B \in \binom{S_k}{2}$ and for all n;
- $s_n(\underline{12\cdots(k-2)k(k-1)}, \underline{12\cdots(k-3)(k-1)k(k-2)}) \leq s_n(B)$ for all $B \in \binom{S_k}{2}$ and for all n;
- $s_n(\underline{12\cdots k}, \underline{23\cdots k1}, \underline{k12\cdots(k-1)}) \geq s_n(B)$ for all $B \in \binom{S_k}{3}$ and for all n.

Wilf-equivalence for patterns of length 4

For references for this slide see Baxter, Shattuck. Some Wilf-equivalences for vincular patterns, *Journal of Combinatorics* (2015)

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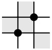
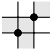
- For **classical patterns** there are 3 Wilf-equivalent classes given by 1234, 1342, and 1324.
- For **consecutive patterns** there is only one equivalence not due to symmetry: $\underline{2341} \equiv \underline{1342}$.
- For **vincular patterns** the two yet non-proved equivalences are given by the following conjectures.
  Conjecture. $\underline{23}14 \equiv 1\underline{23}4$
  Conjecture. $\underline{14}23 \equiv 2\underline{14}3$

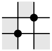
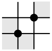
In "Kitaev, Zhang. Distributions of mesh patterns of short lengths, *Adv. Appl. Math.* (2019)" the **distributions** for **27** out of **65** patterns considered by Hilmarsson et al are given:

Nr.	Repr. p	Distribution	Nr.	Repr. p	Distribution
1		Non-inversions given by (1); [14, p. 21]	20		Theorem 2.8
3		Conjecture 6.1	21		Theorem 2.9
5		Theorem 2.1	22		Theorem 2.10
8		Theorem 4.1 Unsigned Stirling numbers of the first kind, [13, A132393]	27		Theorem 3.3
9			28		Theorem 3.4
10		Theorem 2.2	30		Theorem 3.5
11		Theorem 2.3	33		Theorem 3.6
12		Theorem 2.4	34		Theorem 3.7
13		Theorem 2.5	36		Theorem 4.3
14		Theorem 4.2 small descents, [13, A123513]	45		Theorem 4.4
15			55		Theorem 3.8
16		Theorem 3.1	56		Theorem 3.9
17		Theorem 3.2	63		Theorem 3.10
18		Theorem 2.6	64		Theorem 3.11
19		Theorem 2.7	65		Theorem 3.13

Distributions of mesh patterns of short length

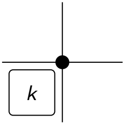
In the case of unknown distributions a number of equidistributions were proved and conjectured; four of the conjectures were proved in "Han, Zeng. Equidistributions of mesh patterns of length two and Kitaev and Zhang's conjectures, (2020)". The remaining conjectures are as follows.

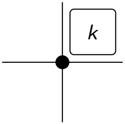
Conjecture. The patterns  and  are equidistributed.

Conjecture. The patterns  and  are equidistributed.

Quadrant marked mesh patterns (QMMPs)

One-line notation for quadrant marked mesh patterns:

$MMP(0,0,k,0) =$ 

, $MMP(k,0,0,0) =$ 

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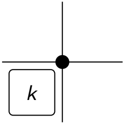
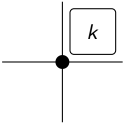
$MMP(0,0,k,0) =$ 

, $MMP(k,0,0,0) =$ 

QMMPs were studied on **permutations**, **alternating permutations**, **123-avoiding permutations**, and **132-avoiding permutations**, and they were linked to r-**Stirling numbers**. A particularly nice result is a refinement of classic enumeration results of André on alternating permutations by showing that the **distribution** of $MMP(0,0,0,1)$ is given by $(\sec(xt))^{1/x}$ on **up-down permutations of even length** and by $\int_0^t (\sec(xz))^{1+\frac{1}{x}}\,dz$ on **down-up permutations of odd length**.

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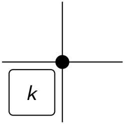
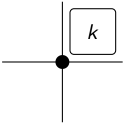
$MMP(0,0,k,0) =$ 

, $MMP(k,0,0,0) =$ 

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Research direction: Study QMMPs on other classes of permutations.

Some algorithmic aspects

Pattern matching problem

The pattern matching problem, also known as the pattern involvement problem, for permutations is to determine whether a given n-permutation π contains a given classical k-pattern p, $k \leq n$.

Abbreviations:

PPM: Permutation Pattern Matching Problem for classical patterns;

$\mathcal{C}$ PPM: PPM for $\mathcal{C}$ $\in${classical, vincular, bivincular, mesh, boxed mesh, consecutive, POP,...}

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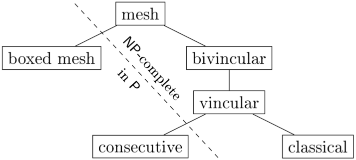
If p is of fixed length, then brute force approach gives a worst case execution time of $O(n^k)$. Albert et al. developed general algorithms whose worst case complexity is considerably smaller than $O(n^k)$ (see Sec 8.2 in "Kitaev. Patterns in Permutations and Words, Springer, 2011." for references). An **unproven observation** is that the algorithms should never be worse than $O(n^{2+k/2} \log n)$ and in some cases they are much better.

Some algorithmic aspects

The following hierarchy of pattern types appear in "Bruner, Lackner. The computational landscape of permutation patterns, PU.M.A. (2013)."



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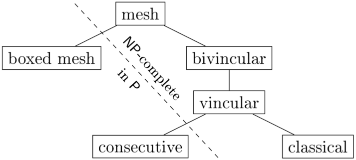

Exponential time algorithms for pattern length k: $O(1.79^n \cdot nk)$ by Bruner and Lackner; $2^{O(k^2 \log k)} \cdot n$ by Guillemot and Marx.

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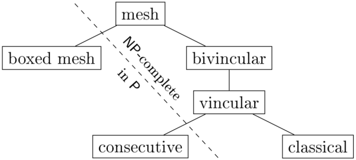

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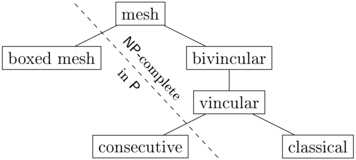

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Research direction: Other subclasses of mesh patterns PPM in P?

Some algorithmic aspects

Permutation pattern avoiders problem (PPA)

Construct all permutations of size $\leq n$ avoiding a given pattern p.

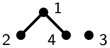
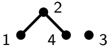
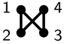
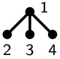
Permutation pattern counting problem (PPC)

Find the number of occurrences of a pattern p in each permutation of size at most n.

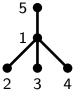
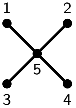
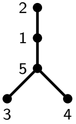
Relevant sources (not to be discussed):

- Garrabrant, Pak. Permutation patterns are hard to count. *Proceedings of the 2016 Annual ACM-SIAM Symposium on Discrete Algorithms*
- Kuszmaul. Fast algorithms for finding pattern avoiders and counting pattern occurrences in permutations. *Math. of Computation* (2017)

Open bijective problems on POPs and other patterns

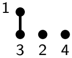
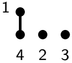
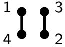
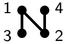
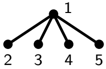
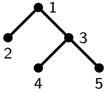
POP	OEIS	Equinumerous structures
	A111281	permutations avoiding the patterns 2413, 2431, 4213, 3412, 3421, 4231, 4321, 4312
	A111282	permutations avoiding the patterns 1432, 2431, 3412, 3421, 4132, 4231, 4312, 4321
	A111277	permutations avoiding the patterns 2413, 4213, 2431,4231, 4321; also, permutations avoiding the patterns 3142, 3412, 3421, 4312, 4321
	A006012	permutations avoiding the vincular patterns $1\underline{32}4$, $1\underline{42}3$, $2\underline{31}4$, $2\underline{41}3$; see [Y. Biers-Ariel. The number of permutations avoiding a set of generalized permutation patterns, *J. Integer Sequences* **20** (2017), Article 17.8.3.]
	A025192	permutations $\pi_1 \cdots \pi_{3n}$ avoiding the patterns 231, 312, 321 and satisfying $\pi_{3i+1} < \pi_{3i+2}$ and $\pi_{3i+1} < \pi_{3i+3}$ for all $0 \leq i < n$. Equivalently, 2-ary shrub forests of n heaps avoiding the patterns 231, 312, 321; see [D. Bevan, D. Levin, P. Nugent, J. Pantone, L. Pudwell, M. Riehl, M. Tlachac. Pattern avoidance in forests of binary shrubs. *Discr. Math. Theor. Comp. Sci.* **18:2** (2016), #8.]

More bijective problems on POPs and other patterns

POP	OEIS	Equinumerous structures
	A054872	permutations avoiding the patterns 12345, 13245, 21345, 23145, 31245, 32145; note that avoiding these patterns is the same as avoiding the POP $\{5>4, 4>1, 4>2, 4>3\}$
	A212198	permutations avoiding the marked mesh pattern M(2,0,2,0); see [S. Kitaev, J. Remmel. Quadrant marked mesh patterns. *J. Integer Seq.* **15(4)** (2012), Art. 12.4.7, 29.]; these permutations are proved to be in bijection with pattern-avoiding involutions $\mathbf{I}_n(>,\neq,>)$; see [M. Martinez, C. Savage. Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations. *J. Integer Seq.* **21** (2018), Article 18.2.2.]
	A224295	permutations avoiding the patterns 12345 and 12354; note that avoiding these patterns is the same as avoiding the POP $\{1>2, 2>3, 3>4, 3>5\}$

All problems on POPs are from "A. Gao, S. Kitaev: On partially ordered patterns of length 4 and 5 in permutations, Electr. J. Combin. **26** (2019)."

Other open bijective problems

POP	OEIS	Equinumerous structures
	A045925	levels in all compositions of $n+1$ with only 1's and 2's
	A214663 A232164	n-permutations for which the partial sums of signed displacements do not exceed 2 Weyl group elements, not containing ...
	A271897	sum of all second elements at level n of the TRIP-Stern sequence corresponding to the permutation triple (e, e, e)
	A052544	compositions of $3n+1$ into parts of the form $3m+1$
	A084509	number of ground-state 3-ball juggling sequences of period n
	A118376	series-reduced enriched plane trees of weight n; also, trees of weight n, where nodes have positive integer weights and the sum of the weights of the children of a node is equal to the weight of the node

Exotic directions I – (bi)crucial permutations

Extensions of permutations

A permutation of length n has $n+1$ extensions to the right (or to the left), e.g. the extensions of 2413 to the right are 3524**1**, 3514**2**, 2514**3**, 2513**4** and 2413**5**.

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Main idea

Given a set of restrictions on permutations (or words), study permutations avoiding the restrictions, but whose **every** extension **to the right** (resp., **and to the left**) contains a prohibition. Such permutations are crucial (resp., bicrucial) with respect to the set of prohibitions.

Exotic directions I – (bi)crucial permutations

(Bi)crucial permutation w.r.t. squares

A square in a permutation is two occurrences of a **consecutive pattern** following each other. E.g. 21**365498**7 contains a square, while 32145 is square-free.

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- **even** length exist and the **smallest** such permutations are of length 32. Conjecture: **arbitrary long** such permutations exist.

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- length $8k+3$ exist for $k = 2, 3$ and they don't exist for $k = 1$. Conjecture: there exist such permutations for any $k \geq 2$.

Exotic directions I – (bi)crucial permutations

(Bi)crucial permutation w.r.t. monotone arithmetic patterns

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. Then for a fixed $d \geq 1$, $\pi_i\pi_{i+d}\cdots\pi_{i+(k-1)d}$ is an arithmetic subsequence of length k with difference d, assuming $i \geq 1$ and $i+(k-1)d \leq n$. If an occurrence of a pattern forms an arithmetic subsequence, we refer to the occurrence as an arithmetic occurrence of the pattern. A permutation is (k,ℓ)-anti-monotone if it avoids arithmetically the patterns $12\cdots k$ and $\ell(\ell-1)\cdots 1$.

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A permutation π is (k,ℓ)-crucial (resp. (k,ℓ)-bicrucial) if π is (k,ℓ)-anti-monotone but any extension of π to the right (resp., and to the left) is not (k,ℓ)-anti-monotone. E.g. 216453 is $(3,3)$-crucial, while 73418562 is $(3,3)$-bicrucial.

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From "Avgustinovich et al.: Crucial and bicrucial permutations with respect to arithmetic monotone patterns, *Siberian Electr. Math. Reports* (2012)" we see that there exist **arbitrary long** (k, ℓ)-(bi)crucial permutations, and the minimal length of such permutations is

- for crucial ones: $m(k, \ell) = \max(k, \ell)(\min(k, \ell) - 1)$;

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Open problem: Classify lengths for which (k, ℓ)-crucial and (k, ℓ)-bicrucial permutations exist for $k, \ell > 2$. (Note that no (3,3)-crucial permutation exists of length 9, and thus all (3,3)-crucial permutations of length 8 are (3,3)-bicrucial.)

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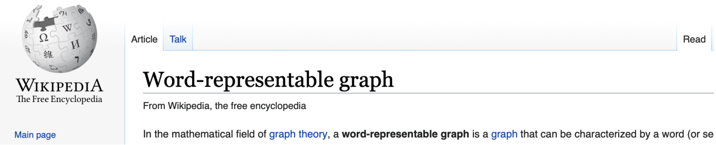
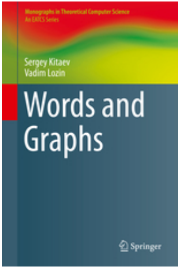
Open problem: Study (bi)crucial permutations for other sets of (arithmetic) prohibitions.

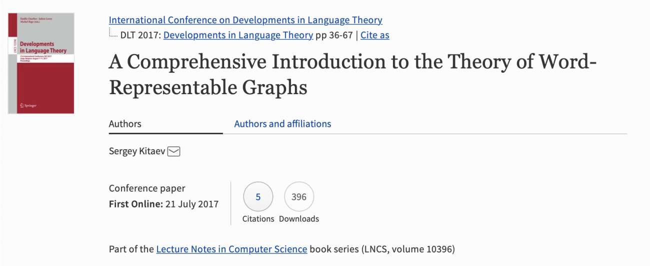
Exotic directions II – graph representations

The best ways to learn about the subject:

- Jones et al.: Representing graphs via pattern avoiding words, *Electr. J. Comb.* (2015)
- Cheon et al.: On k-11-representable graphs, *J. Comb.* (2019)
- Kitaev: Existence of u-representation of graphs, *J. Graph Theory* (2017)







Exotic directions II – graph representations

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k-u-representation of graphs

Let u be a binary pattern. A graph $G = (V, E)$ is *k-u-representable* if there exists a word w over the alphabet V such that w restricted to letters x and y, $x \neq y$, contains **at most** k occurrences of u **if and only if** $xy \in E$. (w **must** contain **each** letter in V.)

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Key questions

For given k and u, is **every** graph k-u-representable? If not, then how do we characterise k-u-representable graphs? k can be thought of as the degree of tolerance.

Exotic directions II – graph representations

A few interesting facts

- If u is of length at least 3 then **any** graph is 0-u-representable.

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- **Every** 0-12-representable graph is a comparability graph.
- **Any** graph is 2-11-representable. A challenging question: Is every graph 1-11-representable?

Acknowledgment

Thank you for your attention!

Any questions?