On connectivity threshold in random temporal graphs

Viktor Zamaraev
University of Liverpool

Joint work with
Arnaud Casteigts, University of Bordeaux
Michael Raskin, Technical University of Munich
Malte Renken, Technical University of Berlin

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Outline of the talk

1. Motivation for temporal graphs
2. Connectivity problems in temporal graphs
3. Random temporal graphs
4. Connectivity in random temporal graphs

Motivation for temporal graphs

Temporal graph theory: motivation

Real-world complex systems that are modeled with networks:

1. Transportation networks
2. Internet
3. Social networks
4. Mobile-phone networks
5. Food webs
6. Cattle movements network
7. Face-to-face interactions etc.

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Tools	Graph theory

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Model	Networks	Temporal Networks
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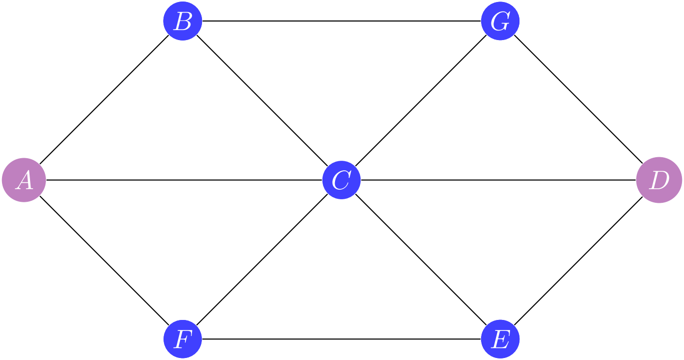
Temporal graph theory: motivation

Example: Static vs Temporal model



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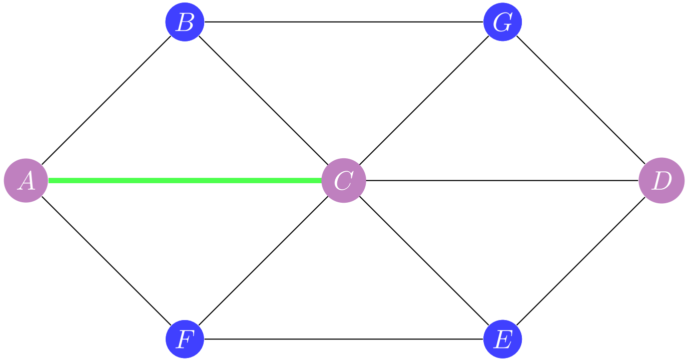

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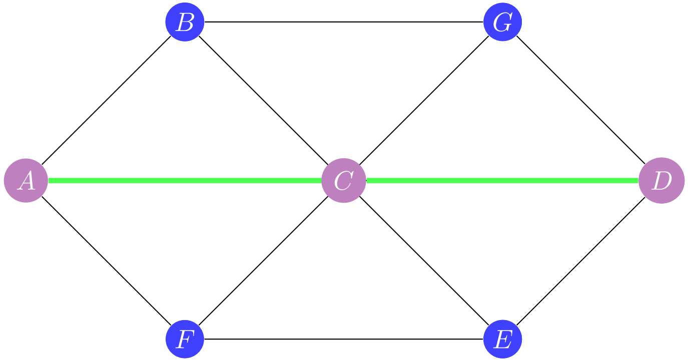

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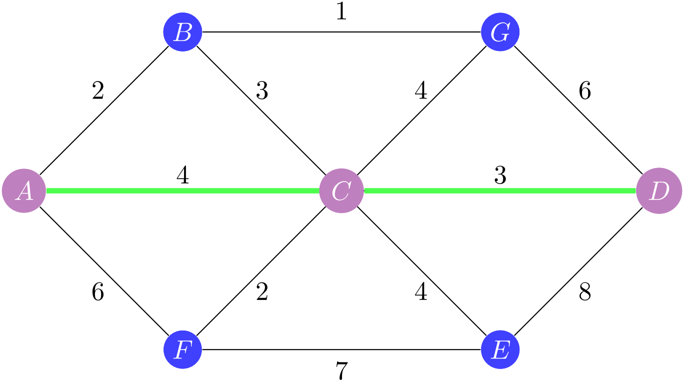

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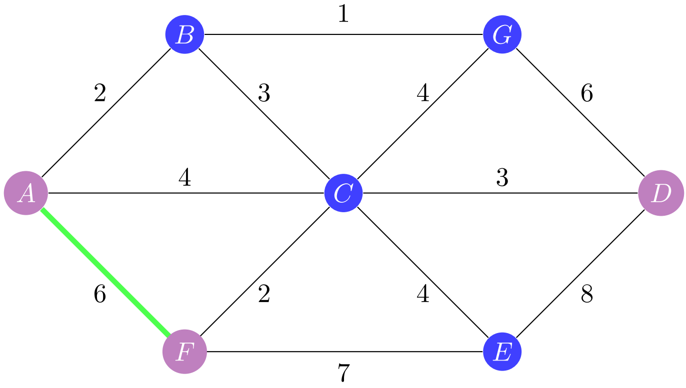

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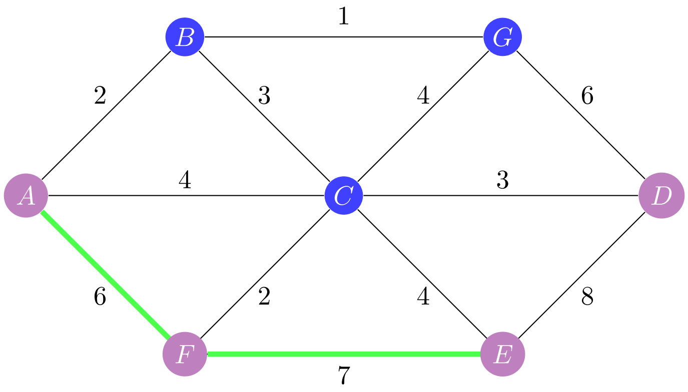

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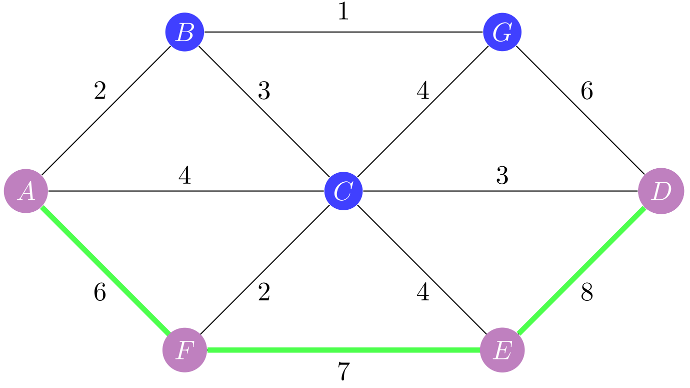

Temporal graphs

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying graph and
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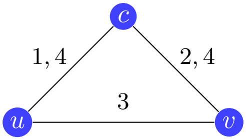

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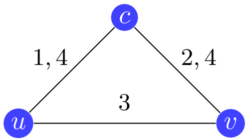

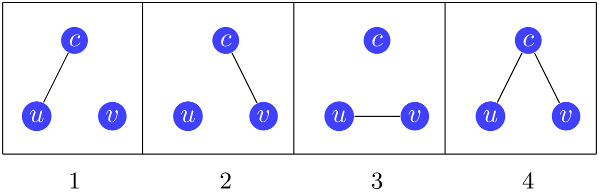

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A temporal graph (G, λ) is *simple* if

1. every edge e has exactly one timestamp, i.e. $|\lambda(e)| = 1$; and
2. all edges have pairwise different timestamps, i.e. $\lambda(e_1) \neq \lambda(e_2)$ for all $e_1, e_2 \in E(G), e_1 \neq e_2$.

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2. simple temporal graphs are also known as edge-ordered graphs (**Chvátal, Komlós, 1971**).

Connectivity in temporal graphs

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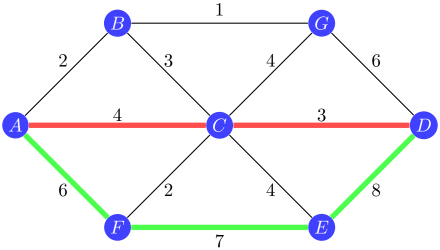

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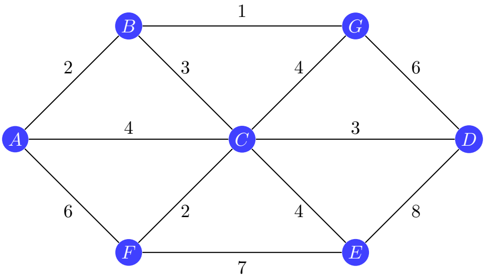

Not temporally connected: there is no temporal path from D to B.

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Question (Kempe, Kleinberg, and Kumar, 2000)

Given a temporally connected temporal graph (G, λ) on n vertices, does there exist a spanning subgraph G' of G with $O(n)$ edges such that the temporal graph (G', λ') is also temporally connected, where λ' is the restriction of λ on the edges of G'?

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2. there are minimal temporally connected temporal graphs on n vertices with $\Theta(n^2)$ edges (**Axiotis, Fotakis, 2016**).

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On the positive side.

Any simple temporal clique (K_n, λ) has a temporally connected spanning subgraph with $O(n \log n)$ edges (**Casteigts, Peters, Schoeters, 2018**).

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What are random temporal graphs?

Random temporal graphs

Random simple temporal graphs

Definition (probability space $\mathcal{D}_{n,p}$)

A random simple temporal graph (G, λ) in $\mathcal{D}_{n,p}$ is obtained by

1. first sampling G from $\mathcal{G}_{n,p}$;
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For which values of p a random simple temporal graph from $\mathcal{D}_{n,p}$ is temporally connected a.a.s.?

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Definition (equivalent probability space $\mathcal{F}_{n,p}$)

A random simple temporal graph (G, λ) in $\mathcal{F}_{n,p}$ is obtained by

1. first sampling G from $\mathcal{G}_{n,p}$;
2. and then sampling the timestamps $\lambda(e), e \in E(G)$ uniformly and independently from $[0, 1]$.

From Temporal connectivity to Temporal source

Observation

Let (G, λ) be simple temporal graph, x be a vertex in (G, λ), and let $t \in [0, 1]$ be such that

1. *every vertex in (G, λ) reaches x before time t; and*
2. *x reaches every vertex in (G, λ) after time t.*

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Definition (temporal source (sink))

A vertex x is a *temporal source* (resp. *temporal sink*) in a temporal graph (G, λ) if every vertex in (G, λ) can be reached from x (resp. can reach x) via temporal path.

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Let $(G, \lambda) = \mathcal{F}_{n,p}$ and let

1. (G_1, λ_1) is the subgraph of (G, λ) spanned by the edges with timestamps < 0.5; and
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1. if x is a temporal sink in (G_1, λ_1) and a temporal source in (G_2, λ_2), then (G, λ) is temporally connected;
2. both (G_1, λ_1) and (G_2, λ_2) can be seen as elements of $\mathcal{F}_{n,p/2}$.

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Lemma

Let $p = p(n)$. If $\gamma_{n,p/2} \to 1$ as $n \to \infty$, then a random temporal graph from $\mathcal{F}_{n,p}$ is temporally connected a.a.s.

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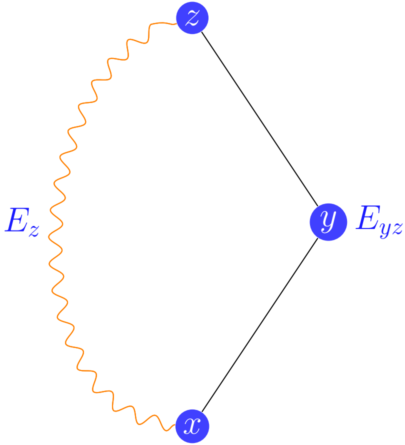

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$$\gamma_{n,p} = 1 - \mathbb{P}\left(\bigcup_{z \neq x} \overline{E}_z\right) \geq 1 - \sum_{z \neq x} \mathbb{P}(\overline{E}_z) = 1 - (n-1)p_1$$

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Lemma

$\gamma_{n,p} = 1 - o(1)$, when $p = 2\sqrt{\log n / n}$.

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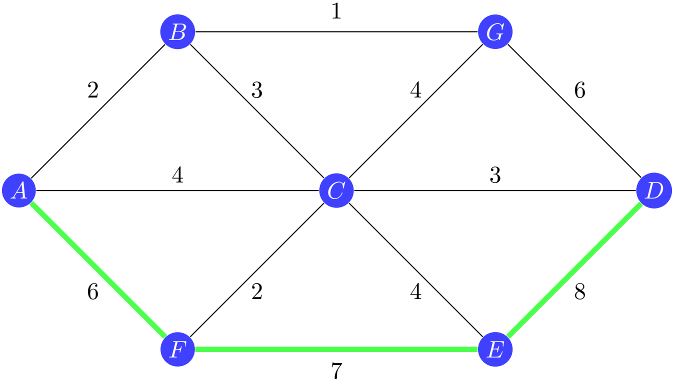

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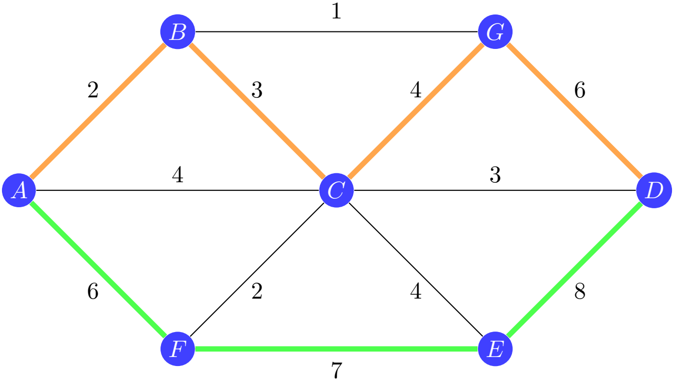

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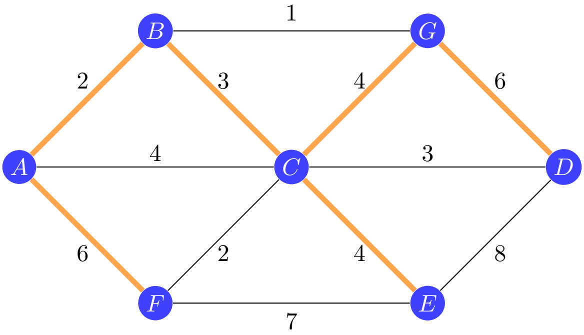
Definition (Foremost tree)

Let (G, λ) be a temporal graph. A temporal spanning subtree (T, λ') of (G, λ) rooted at vertex v is a *foremost tree* for v, if every path in (T, λ') from v to another vertex w is a foremost (v, w)-path in (G, λ).



Temporal source: foremost tree approach

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Let v be a temporal source in a temporal graph (G, λ). Then there exists a foremost tree for v in (G, λ).

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Input: Temporal graph (G, λ); temporal source v in (G, λ).

Output: Foremost tree for v.

1: $T \leftarrow (\{v\}, \emptyset)$
2: **while** $V(G) - V(T) \neq \emptyset$ **do**
3: Let e be an edge that *extends* T and has the minimum timestamp.
4: Add e to T
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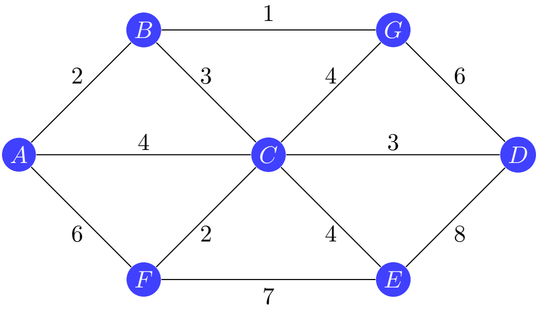

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```
T ← ({v}, ∅)
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```

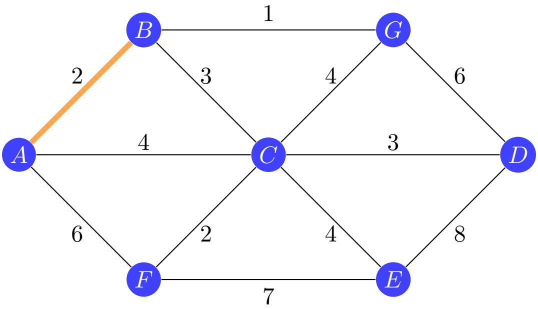

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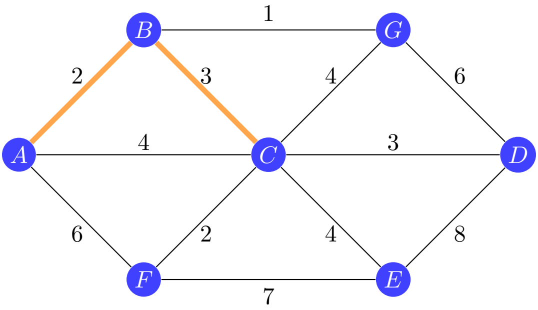

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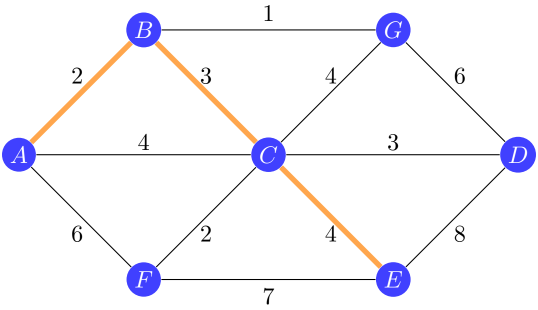

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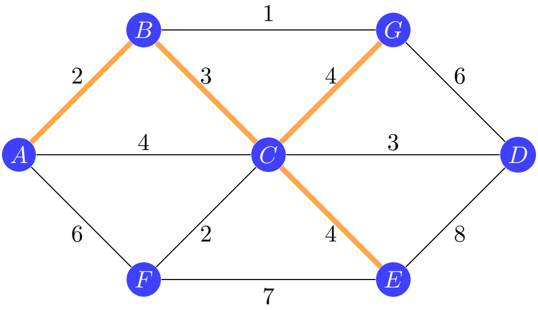

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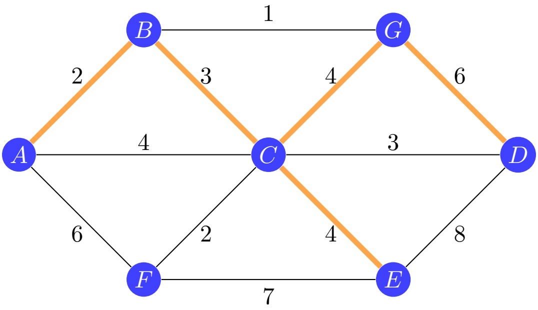

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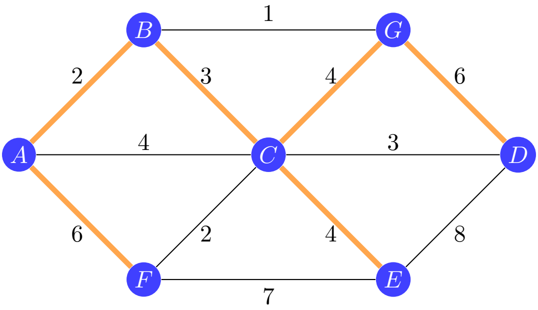

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The following model is equivalent to $\mathcal{F}_{n,p}$:

1. sample a random graph (K_n, λ) from $\mathcal{F}_{n,1}$;
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Idea:

1. Run FOREMOST TREE algorithm from an arbitrary vertex in a random graph $\mathcal{F}_{n,1}$.
2. Analyze the smallest value p such that all edges of the constructed tree have timestamps at most p.

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Outline of the analysis:

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   2. Y concentrates around its expected value $E[Y]$.

Temporal source: foremost tree approach

Theorem (Sharp threshold for Temporal Source)

There exists a function $\varepsilon(n) = o(1)$ *such that an arbitrary vertex in a random temporal graph* $\mathcal{F}_{n,p}$ *is*

1. *a temporal source a.a.s. if* $p > \frac{2\log n}{n}(1+\varepsilon(n))$*; and*
2. *not a temporal source a.a.s. if* $p < \frac{2\log n}{n}(1-\varepsilon(n))$*.*

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Theorem (Threshold for Temporal Connectivity)

There exists a function $\varepsilon(n) = o(1)$ *such that a random temporal graph* $\mathcal{F}_{n,p}$ *is*

1. *temporally connected a.a.s. if* $p > \frac{4 \log n}{n}(1 + \varepsilon(n))$*; and*
2. *not temporally connected a.a.s. if* $p < \frac{2 \log n}{n}(1 - \varepsilon(n))$*.*

Thank you for your attention!