Effective evolution equations from many body quantum dynamics

Benjamin Schlein, University of Cambridge

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Based on joint works with
L. Erdös, A. Michelangeli, I. Rodnianski, H.-T. Yau

I. Introduction.

N-Boson System: described by a wave function

$$\psi_N \in L^2(\mathbb{R}^{3N}, \mathrm{d}x_1 \dots \mathrm{d}x_N) \quad \text{symmetric w.r.t. permutations}$$

Probabilistic interpretation:

$$|\psi_N(x_1, \dots, x_N)|^2 = \text{probability density} \quad \Rightarrow \quad \|\psi_N\| = 1$$

The dynamics is governed by the Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \qquad \Rightarrow \qquad \psi_{N,t} = e^{-iH_N t} \psi_N$$

H_N is the Hamiltonian of the system. For example,

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} + V_{\text{ext}}(x_j) \right) + \sum_{i<j}^{N} V(x_i - x_j)$$

Macroscopic Dynamics: in typical systems, $N \simeq 10^3 - 10^{23} - \dots$

$\Rightarrow$ Impossible to solve the Schrödinger equation.

For practical purposes, it is enough to describe the macroscopic evolution, resulting from averaging over the particles.

Mean-field systems: every particle interacts very weakly with all other particles. Consider Hamiltonian

$$H_N = -\sum_{j=1}^{N} \Delta_{x_j} + \kappa \sum_{i<j}^{N} V(x_i - x_j)\,.$$

in the regime

$$N \gg 1, \qquad \kappa \ll 1, \qquad \text{so that} \quad \kappa_0 = N\kappa \simeq 1$$

Kinetic and potential energy are $O(N)$; macroscopic evolution described by an effective nonlinear one-particle equations.

Self-Consistent Evolution: consider evolution of a factorized state,

$$\psi_{N,0}(\mathbf{x}) = \prod_{j=1}^{N} \varphi(x_j) \qquad (\mathbf{x} = (x_1, \ldots, x_N)).$$

If factorization is preserved in time,

$$\psi_{N,t}(\mathbf{x}) \simeq \prod_{j=1}^{N} \varphi_t(x_j)$$

we may replace interaction by an effective one-particle potential

$$\kappa \sum_{i\neq j}^{N} V(x_i - x_j) \simeq \kappa \sum_{i\neq j}^{N} \int \mathrm{d}x_i \, V(x_i - x_j)|\varphi_t(x_i)|^2 \simeq N\kappa \, (V * |\varphi_t|^2)(x_j)$$

The orbital φ_t must solve the self-consistent Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + \kappa_0 \, (V * |\varphi_t|^2)\varphi_t \, .$$

Reduced Densities: define the density matrix

$$\gamma_{N,t} = |\psi_{N,t}\rangle\langle\psi_{N,t}| \qquad \Big(\text{kernel: } \gamma_{N,t}(\mathbf{x};\mathbf{y}) = \psi_{N,t}(\mathbf{x})\overline{\psi}_{N,t}(\mathbf{y})\Big)$$

as the orthogonal projection onto $\psi_{N,t}$.

For $k = 1, .., N$, the reduced k-particle density matrix is given by

$$\gamma^{(k)}_{N,t} = \text{Tr}_{k+1,\dots,N}\, \gamma_{N,t} \qquad \text{acting on } L^2(\mathbb{R}^{3k})$$

$\gamma^{(k)}_{N,t}$ is an operator on $L^2(\mathbb{R}^{3k})$ with kernel

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = \int d\mathbf{x}_{N-k}\, \gamma_{N,t}(\mathbf{x}_k,\mathbf{x}_{N-k};\mathbf{x}'_k,\mathbf{x}_{N-k})\,,$$

with $\mathbf{x}_k = (x_1,\dots,x_k)$, $\mathbf{x}_{N-k} = (x_{k+1},\dots,x_N)$, $\text{Tr}\,\gamma^{(k)}_{N,t} = 1$.

For every k-particle observable $J^{(k)}$:

$$\langle\psi_{N,t}, (J^{(k)}\otimes 1^{(N-k)})\,\psi_{N,t}\rangle = \text{Tr}(J^{(k)}\otimes 1^{(N-k)})\gamma_{N,t} = \text{Tr}\, J^{(k)}\gamma^{(k)}_{N,t}$$

II. System of Gravitating Bosons (Boson Stars)

Consider system of N non-relativistic gravitating bosons. In appropriate units, the Hamilton operator is given by

$$H_N = -\sum_{j=1}^{N} \Delta_{x_j} - G \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

where G is the gravitational constant.

In the regime characterized by

$$N \gg 1, \quad \text{and} \quad \lambda := NG \simeq 1$$

it makes sense to consider macroscopic dynamics generated by

$$H_N = -\sum_{j=1}^{N} \Delta_{x_j} - \frac{\lambda}{N} \sum_{i<j} \frac{1}{|x_i - x_j|}$$

in the limit $N \to \infty$.

In 2000, Erdős-Yau proved that, if

$$\psi_{N,t} = e^{-iH_N t}\varphi^{\otimes N}, \qquad \gamma_{N,t}^{(k)} = \mathrm{Tr}_{k+1,\dots,N}\, |\psi_{N,t}\rangle\langle\psi_{N,t}|$$

then, for any $k \in \mathbb{N}$ and $t \in \mathbb{R}$ fixed,

$$\mathrm{Tr}\, \left| \gamma_{N,t}^{(k)} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \right| \to 0 \qquad \text{as } N \to \infty$$

where φ_t solves the Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t - \lambda \left(\frac{1}{|.|} * |\varphi_t|^2 \right) \varphi_t$$

with initial data $\varphi_{t=0} = \varphi$.

In other words, for any k-particle observable $J^{(k)}$, we have

$$\langle \psi_{N,t}, (J^{(k)} \otimes 1^{(N-k)})\psi_{N,t}\rangle \to \langle \varphi_t^{\otimes k}, J^{(k)} \varphi_t^{\otimes k}\rangle \quad \text{as } N \to \infty$$

In 2007, in a joint work with I. Rodnianski, we established that for every $k \in \mathbb{N}$ there exist a constant C_k such that

$$\text{Tr} \left| \gamma^{(k)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \right| \lesssim \frac{e^{C_k|t|}}{\sqrt{N}} \, .$$

Observe that explicit bounds are important; they justify the study of the limit $N \to \infty$.

Recently, Knowles-Pickl improved this result to more singular potentials. Moreover, they showed that, if the solution of the nonlinear equation scatters, the control is uniform in $t \in \mathbb{R}$.

III. Semi-relativistic system of gravitating bosons.

To take into account relativity effects, consider Hamiltonian

$$H_N = \sum_{j=1}^{N} \sqrt{1 - \Delta_{x_j}} - G \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

Again, we are interested in the regime where

$$N \gg 1, \qquad \text{and} \qquad \lambda = NG \simeq 1$$

Hence we write

$$H_N = \sum_{j=1}^{N} \sqrt{1 - \Delta_{x_j}} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

and we consider the limit $N \to \infty$.

We expect dynamics of factorized data to be approximated by the semi-relativistic Hartree equation

$$i\partial_t \varphi_t = \sqrt{1 - \Delta}\, \varphi_t - \lambda \left(\frac{1}{|.|} * |\varphi_t|^2 \right) \varphi_t$$

Depending on value of $\lambda > 0$, we observe two different situations.

Subcritical case: for $\lambda \leq \lambda_{\text{cr}} = 2/\pi$, potential energy is controlled by kinetic energy. Hence

$$H_N = \sum_{j=1}^N \sqrt{1-\Delta_{x_j}} - \frac{\lambda}{N} \sum_{i<j}^N \frac{1}{|x_i - x_j|} \geq 0$$

Moreover, the nonlinear Hartree equation

$$i\partial_t \varphi_t = \sqrt{1-\Delta}\, \varphi_t - \lambda \left(\frac{1}{|.|} * |\varphi_t|^2 \right) \varphi_t$$

is globally well-posed (Lenzmann, 2005).

In 2005, in a joint work with A. Elgart, we proved that, if

$$\psi_{N,t} = e^{-iH_N t} \varphi^{\otimes N}, \qquad \text{and} \qquad \gamma^{(k)}_{N,t} = \text{Tr}_{k+1,\dots,N}\, |\psi_{N,t}\rangle\langle\psi_{N,t}|$$

$$\Rightarrow \quad \text{Tr} \left| \gamma^{(k)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \right| \to 0 \qquad \text{as } N \to \infty$$

Supercritical case: if $\lambda > \lambda_{\text{cr}} = 2/\pi$, H_N is not bounded below.

$$\inf_{\|\psi\|=1} \left\langle \psi, \left(\sum_{j=1}^{N} \sqrt{1-\Delta_{x_j}} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \right) \psi \right\rangle = -\infty$$

Moreover, there exist solutions of the Hartree equation

$$i\partial_t \varphi_t = \sqrt{1-\Delta}\,\varphi_t - \lambda \left(\frac{1}{|.|} * |\varphi_t|^2 \right) \varphi_t$$

which blow-up in finite time, in the sense that

$$\|\varphi_t\|^2_{H^{1/2}(\mathbb{R}^3)} = \left(\int \mathrm{d}x \left| (1-\Delta)^{1/4} \varphi_t \right|^2 \right) \to \infty \qquad \text{as } t \to T^-$$

for some $T < \infty$ (Fröhlich-Lenzmann, 2006).

These solutions are supposed to describe the phenomenon of stellar collapse (Chandrasekhar's theory).

Question: is there a relation between the linear many body evolution and the nonlinear dynamics in the supercritical regime?

Mathematical problem: the operator

$$H_N = \sum_{j=1}^{N} \sqrt{1 - \Delta_j} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

has no self-adjoint realization if $\lambda > \lambda_{\text{cr}} = 2/\pi$.

We introduce therefore regularized Hamiltonians

$$H_N^{\alpha} = \sum_{j=1}^{N} \sqrt{1 - \Delta_{x_j}} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j| + \alpha}$$

where $\alpha = \alpha(N) > 0$ is such that $\alpha(N) \to 0$ as $N \to \infty$.

Physically, the cutoff is justified because, on very short length scales, the gravitational interaction is regularized by other forces.

For arbitrary $\alpha(N) > 0$, the Hamiltonian H_N^{α} is bounded below, and generates a one-parameter group of unitary evolutions.

Theorem 1 [Michelangeli-S., 2009]: Choose $\varphi \in H^2(\mathbb{R}^3)$. Let

$$\psi_{N,t} = e^{-iH_N^\alpha t}\varphi^{\otimes N} \qquad \text{and} \qquad \gamma_{N,t}^{(k)} = \text{Tr}_{k+1,\dots,N}\, |\psi_{N,t}\rangle\langle\psi_{N,t}|$$

Let φ_t be the solution of

$$i\partial_t\varphi_t = \sqrt{1-\Delta}\,\varphi_t - \lambda\left(\frac{1}{|.|} * |\varphi_t|^2\right)\varphi_t\,.$$

Fix $T > 0$, and assume that

$$\kappa := \sup_{|t|\leq T} \|\varphi_t\|_{H^{1/2}(\mathbb{R}^3)} < \infty\,.$$

Then for every $k \in \mathbb{N}$, there exists a constant C, depending on k, T and κ such that

$$\text{Tr}\left|\gamma_{N,t}^{(k)} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \leq \frac{C}{\sqrt{N}} \to 0 \qquad \text{as } N \to \infty$$

for all $|t| \leq T$.

In words: as long as the Hartree equation does not blow-up, the linear evolution is approximated by the Hartree dynamics.

Theorem 2 [Michelangeli-S., 2009]: As before, let $\varphi \in H^2(\mathbb{R}^3)$, φ_t be the solution of the Hartree equation, $\psi_{N,t} = e^{-iH_N^\alpha t}\varphi^{\otimes N}$.

Assume that $\alpha(N) \geq N^{-\ell}$ for some $\ell \in \mathbb{N}$.

Suppose that there exists $T < \infty$ such that $\|\varphi_t\|_{H^{1/2}} < \infty$ for all $0 \leq t < T$, and

$$\|\varphi_t\|_{H^{1/2}(\mathbb{R}^3)} \to \infty \quad \text{as } t \to T^- \, .$$

Then there exists $N(t)$, defined for $t \in [0, T)$ such that $N(t) \to \infty$ as $t \to T^-$ and

$$\lim_{t \to T^-} \mathrm{Tr}\,(1 - \Delta)^{1/2}\,\gamma^{(1)}_{N(t),t} = \infty$$

In words: if the solution of the nonlinear equation blows up at time $T < \infty$, also the linear evolution collapses as $N \to \infty$.

Remarks:

1) To prove Theorem 2, we need convergence of $\gamma_{N,t}^{(1)}$ to $|\varphi_t\rangle\langle\varphi_t|$ for all $t < T$, in the energy-norm. We show that

$$\mathrm{Tr}\left|(1-\Delta)^{1/4}\left(\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right)(1-\Delta)^{1/4}\right| \leq \frac{K}{\sqrt{N}}$$

for a constant K depending only on $\sup_{|\tau|<t}\|\varphi_\tau\|_{H^{1/2}}$.

2) Proof based on approach introduced by Hepp, in 1973. Use Fock space representation, and consider coherent states as initial data.

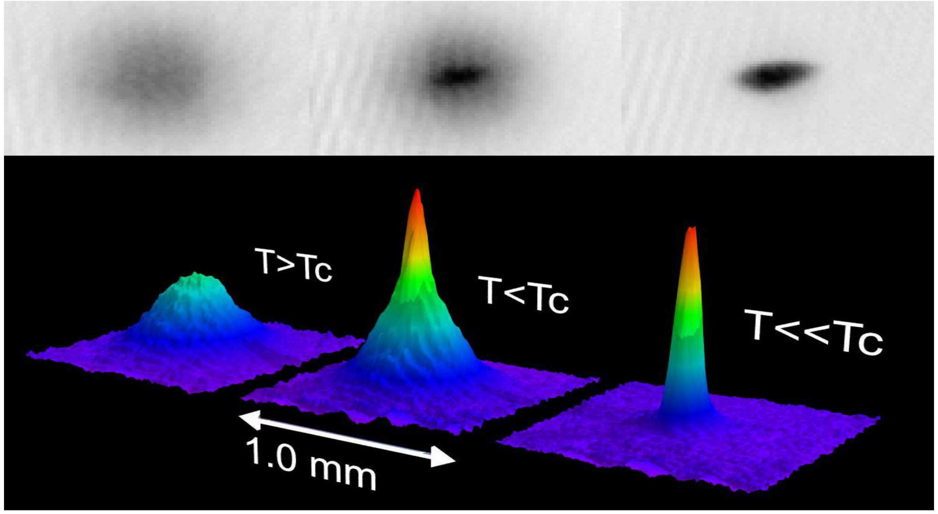
Advantage: one can easily identify the Hartree component of evolution. Need to control the fluctuations. Crucial observation: fluctuation dynamics is "sub-critical".

IV. Dynamics of Bose Einstein Condensates

In the last 15 years, BEC have become accessible to experiments.

In 2001, Cornell-Ketterle-Wieman received Nobel prize in physics for experiments which first proved the existence of BEC for trapped Bose gas.

Goal: give a mathematical description of these experiments.



Trapped Bose Gases: will be described by Hamiltonian

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left(-\Delta_{x_j} + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^{N} V_a(x_i - x_j)$$

where V_a has scattering length $a > 0$.

We will consider regime

$$N \gg 1, \quad a \ll 1, \quad a_0 = Na \simeq 1.$$

Therefore, we write

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left(-\Delta_{x_j} + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^{N} V_N(x_i - x_j)$$

where

$$V_N(x) = N^2 V(Nx)$$

and V has fixed scattering length a_0.

Scattering Length: Recall that the scattering length a_0 of V is defined by

$$\left(-\Delta + \frac{1}{2}V(x)\right) f(x) = 0 \qquad \text{with } f(x) \to 1 \text{ for } |x| \to \infty\,.$$

Then

$$f(x) \simeq 1 - \frac{a_0}{|x|}, \qquad \text{for } |x| \text{ large} \qquad \left(\Rightarrow \qquad 8\pi a_0 = \int \mathrm{d}x V(x) f(x)\right)$$

By scaling, $V_N(x) = N^2 V(Nx)$ has scattering length $a = a_0/N$. In fact, if

$$f_N(x) = f(Nx) \simeq 1 - \frac{a_0}{N|x|} = 1 - \frac{a}{|x|}$$

we find

$$\left(-\Delta + \frac{1}{2}V_N(x)\right) f_N(x) = 0, \qquad \text{and } f_N(x) \to 1 \quad \text{as } |x| \to \infty.$$

Properties of the Ground State: in 2000, Lieb-Seiringer-Yngvason, proved that ground state energy of H_N^{trap} is

$$\lim_{N\to\infty} \frac{E_{\text{GS}}(N)}{N} = \inf_{\varphi:\|\varphi\|=1} \mathcal{E}_{\text{GP}}(\varphi)$$

with

$$\mathcal{E}_{\text{GP}}(\varphi) = \int \mathrm{d}x \left(|\nabla\varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4 \right)$$

$$a_0 = \text{scattering length of } V(x)$$

In 2002, Lieb-Seiringer proved that the ground state exhibits complete condensation in the minimizer of $\mathcal{E}_{\text{GP}}$, that is

$$\gamma_N^{(1)} \to |\phi\rangle\langle\phi|, \qquad \phi = \text{minimizer of } \mathcal{E}_{\text{GP}}$$

What happens if traps are switched off? The condensate evolves.

Our goal: show that the Gross-Pitaevskii theory also describe the dynamics of the condensates, after traps are switched off.

Theorem [Erdös - S. - Yau, 2006-2008]: suppose $V \geq 0$, $|V(x)| \leq C\langle x\rangle^{-\sigma}$, for some $\sigma > 5$, and let

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^2 V(N(x_i - x_j))\,.$$

Assume that ψ_N has finite energy per particle $\langle\psi_N, H_N\psi_N\rangle \leq CN$ and that it exhibits complete BEC

$$\gamma_N^{(1)} \to |\varphi\rangle\langle\varphi| \qquad \text{for some } \varphi \in L^2(\mathbb{R}^3)$$

Let $\psi_{N,t} = e^{-iH_N t}\psi_N$. Then, for every fixed $t \in \mathbb{R}$, $k \in \mathbb{N}$,

$$\text{Tr}\,\left|\gamma_{N,t}^{(k)} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \to 0 \qquad \text{as } N \to \infty\,,$$

where φ_t is the solution of the Gross-Pitaevskii equation

$$i\partial_t\varphi_t = -\Delta\varphi_t + 8\pi a_0|\varphi_t|^2\varphi_t$$

with $\varphi_{t=0} = \varphi$.

Remarks:

- The result concerns the stability of condensation with respect to the time-evolution.

- Examples of initial data include (by the results of Lieb-Seiringer-Yngvason) the ground state of H_N^{trap}, the Hamiltonian with traps.

- Since

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^{N} N^3 V(N(x_i - x_j))$$

H_N looks like a mean-field Hamiltonian with potential

$$v_N(x) = N^3 V(Nx).$$

Physically, however, we are not at all in a mean-field situation, because here interactions are very rare and very strong.

Evolution of Marginal Densities: recall the definition of the k-particle marginal density $\gamma^{(k)}_{N,t}$ associated with $\psi_{N,t}$:

$$\gamma^{(k)}_{N,t} = \mathrm{Tr}_{k+1,\dots,N}\, |\psi_{N,t}\rangle\langle\psi_{N,t}|$$

Its kernel is given by

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\; \psi_{N,t}(\mathbf{x}_k,\mathbf{x}_{N-k})\overline{\psi}_{N,t}(\mathbf{x}'_k,\mathbf{x}_{N-k})$$

The family $\{\gamma^{(k)}_{N,t}\}_{k=1}^N$ satisfies the BBGKY Hierarchy

$$\begin{aligned} i\partial_t\gamma^{(k)}_{N,t} = {} & \sum_{j=1}^k \left[-\Delta_{x_j},\gamma^{(k)}_{N,t}\right] + \sum_{1\le i<j\le k}\left[N^2V(N(x_i-x_j)),\gamma^{(k)}_{N,t}\right] \\ & + (N-k)\sum_{j=1}^k \mathrm{Tr}_{k+1}\left[N^2V(N(x_j-x_{k+1})),\gamma^{(k+1)}_{N,t}\right]. \end{aligned}$$

For $k = 1$, we find

$$i\partial_t \gamma^{(1)}_{N,t} = \left[-\Delta, \gamma^{(1)}_{N,t}\right] + (N-1)\mathrm{Tr}_2 \left[N^2 V(N(x_1 - x_2)), \gamma^{(2)}_{N,t}\right].$$

Naively, if $\gamma^{(1)}_{N,t} \to \gamma^{(1)}_{\infty,t}$ and $\gamma^{(2)}_{N,t} \to \gamma^{(2)}_{\infty,t}$, we obtain

$$i\partial_t \gamma^{(1)}_{\infty,t} = \left[-\Delta, \gamma^{(1)}_{\infty,t}\right] + b_0 \mathrm{Tr}_2 \left[\delta(x_1 - x_2), \gamma^{(2)}_{\infty,t}\right]$$

because

$$(N-1)N^2 V(Nx) \simeq N^3 V(Nx) \to b_0\, \delta(x) \quad \text{with} \quad b_0 = \int \mathrm{d}x V(x).$$

For condensates,

$$\gamma^{(1)}_{\infty,t} = |\varphi_t\rangle\langle\varphi_t| \qquad \left(\gamma^{(1)}_{\infty,t}(x_1; x_1') = \varphi_t(x_1)\overline{\varphi}_t(x_1')\right)$$

$$\gamma^{(2)}_{\infty,t} = |\varphi_t\rangle\langle\varphi_t|^{\otimes 2} \quad \left(\gamma^{(2)}_{\infty,t}(x_1, x_2; x_1', x_2') = \varphi_t(x_1)\varphi_t(x_2)\overline{\varphi}_t(x_1')\overline{\varphi}_t(x_2')\right)$$

$$\Rightarrow i\partial_t \varphi_t = -\Delta\varphi_t + b_0 |\varphi_t|^2 \varphi_t$$

Right equation but wrong coupling constant.

Emergence of Scattering Length: correlations effect.

$$\gamma^{(1)}_{N,t}(x_1; x'_1) \simeq \varphi_t(x_1)\overline{\varphi}_t(x'_1)$$

$$\gamma^{(2)}_{N,t}(x_1, x_2; x'_1, x'_2) \simeq f_N(x_1 - x_2)\, f_N(x'_1 - x'_2)\varphi_t(x_1)\varphi_t(x_2)\overline{\varphi}_t(x'_1)\overline{\varphi}_t(x'_2)$$

where

$$\left(-\Delta + \frac{N^2}{2}V(Nx)\right) f_N(x) = 0 \qquad \Rightarrow \qquad f_N(x) \simeq 1 - \frac{a_0}{N|x|}$$

From

$$i\partial_t \gamma^{(1)}_{N,t} = \left[-\Delta, \gamma^{(1)}_{N,t}\right] + (N-1)N^2 \mathrm{Tr}\left[V(N(x_1 - x_2)), \gamma^{(2)}_{N,t}\right].$$

we obtain

$$\begin{aligned} i\partial_t \varphi_t(x) &= -\Delta\varphi_t(x) + \left(\lim_{N\to\infty} \int \mathrm{d}x\, N^3 V(Nx) f(Nx)\right) |\varphi_t(x)|^2 \varphi_t(x) \\ &= -\Delta\varphi_t(x) + 8\pi a_0 |\varphi_t(x)|^2 \varphi_t(x) \end{aligned}$$

⇒ Gross-Pitaevskii equation with correct coupling constant.

Strategy to prove theorem:

(1) Show compactness of $\gamma^{(k)}_{N,t}$ w.r.t. appropriate weak topology.

(2) Show existence of short scale correlation structure in $\gamma^{(k)}_{N,t}$.

(1)+(2) $\Rightarrow$ $\gamma^{(k)}_{N,t}$ has limit points $\gamma^{(k)}_{\infty,t}$ with

$$\gamma^{(k)}_{N,t} \simeq f_N(x_i - x_j)\gamma^{(k)}_{\infty,t} \quad \text{when } |x_i - x_j| \simeq N^{-1}.$$

(3) Prove that every limit point $\gamma^{(k)}_{\infty,t}$ satisfies the infinite hierarchy

$$i\partial_t \gamma^{(k)}_{\infty,t} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma^{(k)}_{\infty,t}\right] + 8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}_{\infty,t}\right]$$

and observe $\gamma^{(k)}_{\infty,t} = |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$ is a solution iff φ_t solves GP eq.

(4) Theorem follows if we show uniqueness for infinite hierarchy.

Existence of correlation structure: follows from a-priori bounds of the form

$$\int \mathrm{d}\mathbf{x} \left| \nabla_{x_i} \nabla_{x_j} \frac{\psi_{N,t}(\mathbf{x})}{f_N(x_i - x_j)} \right|^2 \leq C$$

uniformly in $N \in \mathbb{N}$ and in $t \in \mathbb{R}$.

Remarks:

- It is crucial that we divide by f_N. In fact

$$\int \mathrm{d}\mathbf{x} \, \left| \nabla_{x_1} \nabla_{x_2} \psi_{N,t}(\mathbf{x}) \right|^2 \simeq N$$

- A-priori estimates follow from energy conservation. More precisely, we show that

$$\langle \psi_N, \, H_N^2 \, \psi_N \rangle \geq \, C N^2 \int \mathrm{d}\mathbf{x} \, \left| \nabla_{x_i} \nabla_{x_j} \frac{\psi_N(\mathbf{x})}{f_N(x_i - x_j)} \right|^2$$

Remark: theorem also applies to factorized initial data.

This suggests that correlations are formed dynamically within very short times (Erdős-Michelangeli-S., 2008).

The formation of correlations lowers the local energy; the excess energy is scattered away through incoherent waves, which do not influence the macroscopic dynamics.

In particular, this implies that, in general, we do not have convergence in the energy-norm.

Open Problems:

Control on rate of convergence?

What happens if $a_0 < 0$?

Uniqueness for infinite hierarchy: we prove uniqueness in the class of densities with

$$\text{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\,\gamma_t^{(k)} \leq C^k$$

with a constant C independent of $k \geq 1$ and t.

Need to prove that any limit point $\{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ of the marginals $\{\gamma_{N,t}^{(k)}\}_{k=1}^N$ satisfies these a-priori bounds.

Problem: the estimates

$$\text{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\,\gamma_{N,t}^{(k)} \leq C^k$$

cannot be true uniformly in N, because of short scale structure.

Choose a length scale ℓ with $N\ell^2 \gg 1$ and $N\ell^3 \ll 1$. For $j = 1, \dots, N$ define

$$\theta_j(\mathbf{x}) \simeq \begin{cases} 1 & \text{if } |x_i - x_j| \gg \ell \quad \forall i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Proposition (higher order energy estimates):

$$\langle \psi_N, (H_N + N)^k \psi_N \rangle \geq C^k N^k \int d\mathbf{x}\, \theta_1(\mathbf{x}) \dots \theta_{k-1}(\mathbf{x}) \, |\nabla_{x_1} \dots \nabla_{x_k} \psi_N(\mathbf{x})|^2$$

$$\Rightarrow \int d\mathbf{x}\, \theta_1(\mathbf{x}) \dots \theta_{k-1}(\mathbf{x}) \, |\nabla_{x_1} \dots \nabla_{x_k} \psi_{N,t}(\mathbf{x})|^2 \leq C^k$$

The cutoff $\theta_j(\mathbf{x})$ is effective only when x_j falls into a volume of order $N\ell^3$ in $\mathbb{R}^3$.

Since $N\ell^3 \to 0$ as $N \to \infty$, the cutoff can be removed in the limit $N \to \infty$, and we obtain the a-priori bounds

$$\text{Tr}\,(1 - \Delta_{x_1}) \dots (1 - \Delta_{x_k}) \gamma^{(k)}_{\infty,t} \leq C^k.$$

Theorem: given a family $\{\gamma^{(k)}\}_{k\geq 1}$ with

$$\mathrm{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\gamma^{(k)} \leq C^k$$

there exists at most one solution $\{\gamma_t^{(k)}\}_{k\geq 1}$ of

$$i\partial_t \gamma_t^{(k)} = \sum_{j=1}^k \left[-\Delta_{x_j}, \gamma_t^{(k)}\right] + 8\pi a_0 \sum_{j=1}^k \mathrm{Tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma_t^{(k+1)}\right]$$

such that

$$\mathrm{Tr}(1-\Delta_1)\dots(1-\Delta_k)\gamma_t^{(k)} \leq C^k \quad \text{for all} \quad t \in \mathbb{R}\,.$$

Main difficulty: in 3 dimensions,

$$\delta(x) \not\leq C(1-\Delta) \qquad\qquad (\delta(x) \leq C(1-\Delta)^\alpha \quad \text{only if } \alpha > 3/2)$$

A-priori bounds are not enough to show uniqueness; instead we need to make use of the smoothing effects of free evolution.

To this end, we developed diagrammatic expansion in terms of Feynman graphs.

Hierarchy in Integral Form: rewrite infinite hierarchy

$$i\partial_t \gamma_t^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_t^{(k)} \right] + 8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_t^{(k+1)} \right]$$

as

$$\gamma_t^{(k)} = \mathcal{U}^{(k)}(t)\gamma_0^{(k)} + \int_0^t \mathrm{d}s\, \mathcal{U}^{(k)}(t-s) B^{(k)} \gamma_s^{(k+1)}, \qquad k \geq 1$$

with

$$\mathcal{U}^{(k)}(t)\gamma^{(k)} = \exp\left(it \sum_{j=1}^{k} \Delta_{x_j} \right) \gamma^{(k)} \exp\left(-it \sum_{j=1}^{k} \Delta_{x_j} \right)$$

$$B^{(k)}\gamma^{(k+1)} = -i8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \Big[\delta(x_j - x_{k+1}), \gamma^{(k+1)} \Big]$$

Duhamel Series: expand arbitrary solution $\gamma_t^{(k)}$ as

$$\gamma_t^{(k)} = \mathcal{U}^{(k)}(t)\gamma_0^{(k)} + \sum_{m=1}^{n-1} \xi_{m,t}^{(k)} + \eta_{n,t}^{(k)}$$

with

$$\xi_{m,t}^{(k)} = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{m-1}} \mathrm{d}s_m \, \mathcal{U}^{(k)}(t-s_1)\, B^{(k)} \mathcal{U}^{(k+1)}(s_1-s_2)\, B^{(k+1)} \dots$$
$$\dots \mathcal{U}^{(k+m-1)}(s_{m-1}-s_m) B^{(k+m-1)} \mathcal{U}^{(k+m)}(s_m) \gamma_0^{(k+m)}$$

$$\eta_{n,t}^{(k)} = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{n-1}} \mathrm{d}s_n \, \mathcal{U}^{(k)}(t-s_1) B^{(k)} \mathcal{U}^{(k+1)}(s_1-s_2) B^{(k+1)} \dots$$
$$\dots \mathcal{U}^{(k+n-1)}(s_{n-1}-s_n) B^{(k+n-1)} \gamma_{s_n}^{(k+n)}$$

with

$$B^{(k)}\gamma^{(k+1)} = -i8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}\right]$$

For example:

$$\xi^{(k)}_{m,t} = (-8\pi i a_0)^m \sum_{j_1=1}^{k} \sum_{j_2=1}^{k+1} \cdots \sum_{j_m=1}^{k+m-1} \int_0^t \mathrm{d}s_1 \ldots \int_0^{s_{m-1}} \mathrm{d}s_m$$

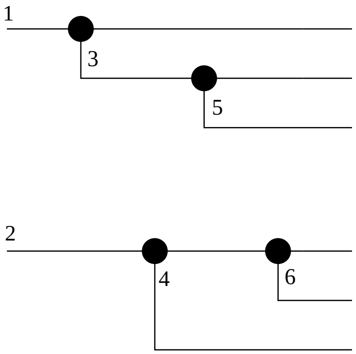
$$\times\, \mathcal{U}^{(k)}(t-s_1)\, \mathrm{Tr}_{k+1}\Big[\delta(x_{j_1}-x_{k+1}),$$

$$\times\, \mathcal{U}^{(k+1)}(s_1-s_2)\mathrm{Tr}_{k+2}\Big[\delta(x_{j_2}-x_{k+2}),\ \ldots$$

$$\ldots$$

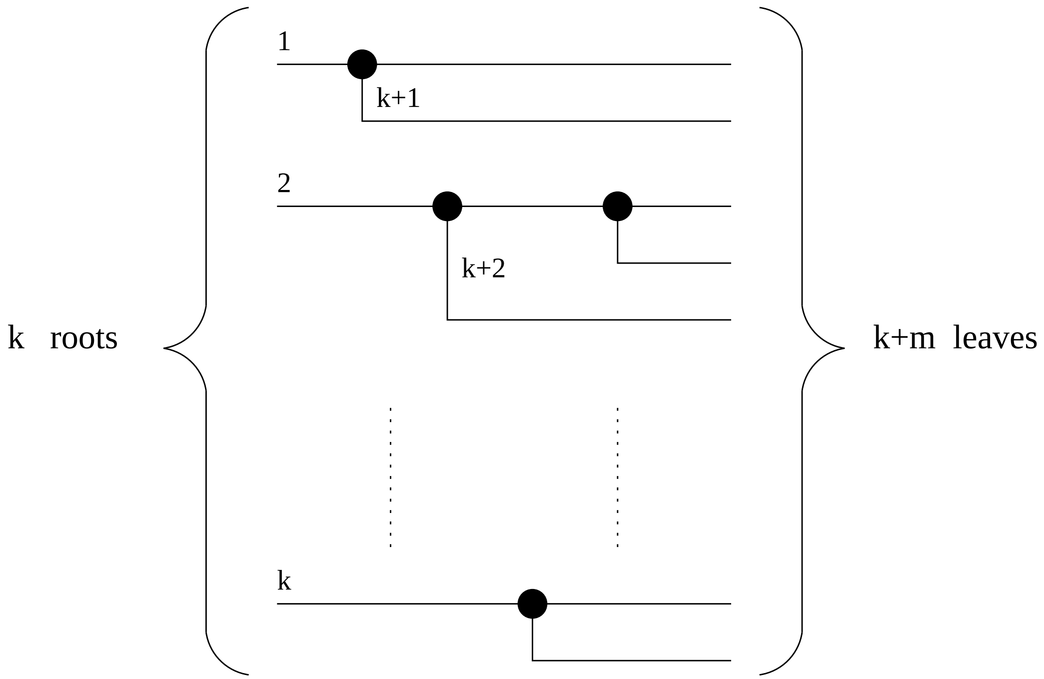
$$\times\, \mathcal{U}^{(k+m-1)}(s_{m-1}-s_m)\, \mathrm{Tr}_{k+m}\Big[\delta(x_{j_m}-x_{k+m}), \mathcal{U}^{(k+m)}(s_m)\gamma_0^{(k+m)}\Big]\ldots\Big]$$

Classical Graphs: the graphs should describe the collision history of the different terms. For example, for $k=2$, $m=4$,



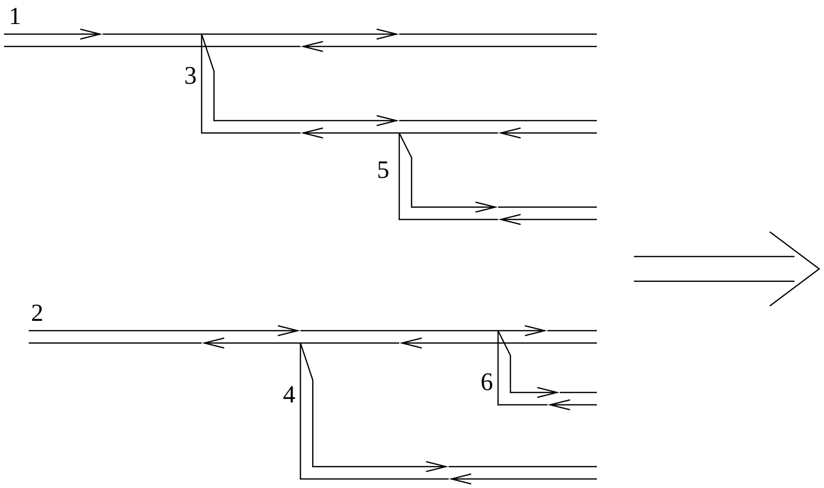
$$\Longrightarrow \quad \delta(x_1-x_3)\ \delta(x_2-x_4)\ \delta(x_3-x_5)\ \delta(x_2-x_6)$$

More generally, contributions to $\xi_{m,t}^{(k)}$ can be represented by ordered forests of k disjoint trees with m vertices



$$\text{Number of ordered graphs } = \frac{(k+m)!}{k!}$$

Doubled Graphs: because of the commutators, for every collision we have a binary choice. To represent all contributions we double the classical graphs. For example ($k = 2$, $m = 4$)

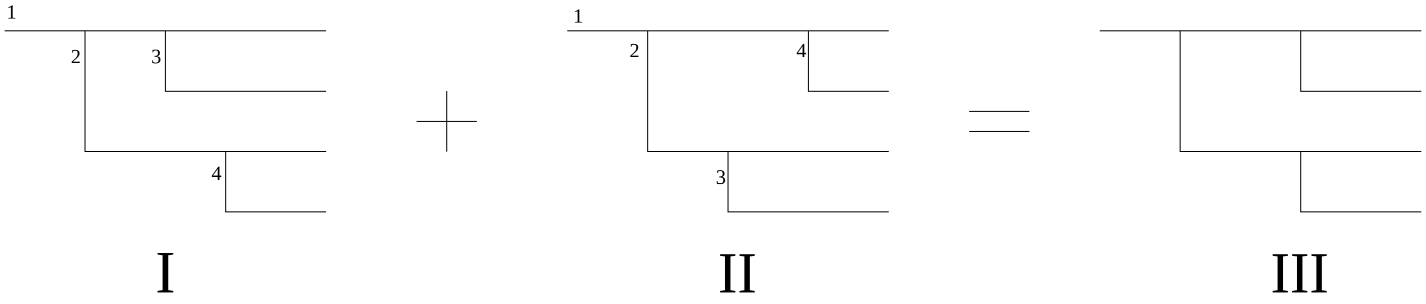


$$\delta\,(x_1 - x_3)\ \delta\,(x_2 - x_6)\ \gamma^{(6)}\ \delta\,(x_3 - x_5)\ \delta\,(x_2 - x_4)$$

The vertices are still completely ordered, and

$$\text{number of doubled graphs} = 2^m \, \frac{(k+m)!}{k!}$$

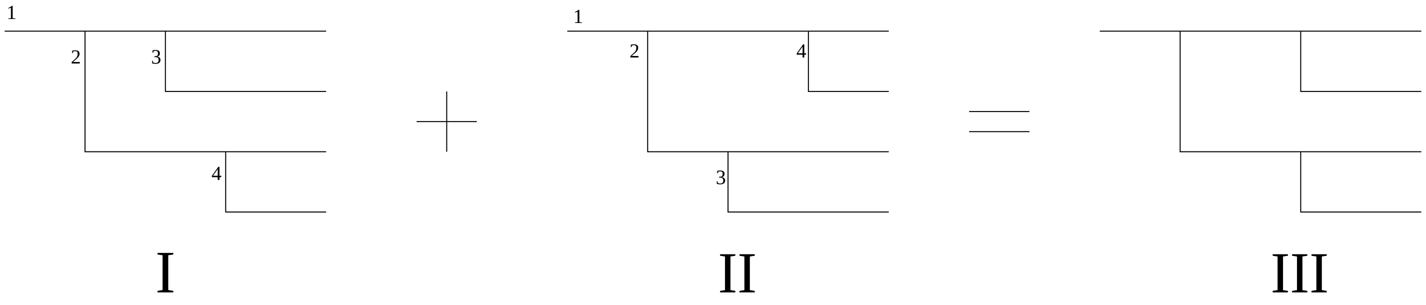
Removing the Order: next we combine the contributions of topologically equivalent ordered graphs.



$$\mathrm{I}) = \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_3 \, \mathcal{U}^{(1)}(t-s_1) \mathrm{Tr}_{2,3,4} \, \delta(x_1 - x_2) \mathcal{U}^{(2)}(s_1 - s_2)$$
$$\times \, \delta(x_1 - x_3) \mathcal{U}^{(3)}(s_2 - s_3) \delta(x_2 - x_4) \mathcal{U}^{(4)}(s_3) \gamma_0^{(4)}$$

$$\mathrm{II}) = \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_3 \, \mathcal{U}^{(1)}(t-s_1) \mathrm{Tr}_{2,3,4} \, \delta(x_1 - x_2) \mathcal{U}^{(2)}(s_1 - s_2)$$
$$\times \, \delta(x_2 - x_3) \mathcal{U}^{(3)}(s_2 - s_3) \delta(x_1 - x_4) \mathcal{U}^{(4)}(s_3) \gamma_0^{(4)}$$
$$= \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_3 \int_0^{s_3} \mathrm{d}s_2 \, \mathcal{U}^{(1)}(t-s_1) \mathrm{Tr}_{2,3,4} \, \delta(x_1 - x_2) \mathcal{U}^{(2)}(s_1 - s_2)$$
$$\times \, \delta(x_1 - x_3) \mathcal{U}^{(3)}(s_2 - s_3) \delta(x_2 - x_4) \mathcal{U}^{(4)}(s_3) \gamma_0^{(4)}$$

Removing the Order: next we combine the contributions of topologically equivalent ordered graphs.



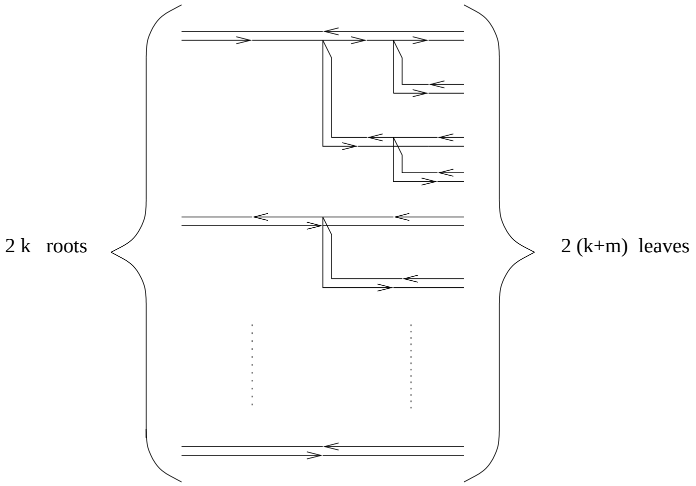
$$\text{III)} := \text{I)} + \text{II)}$$

$$= \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_1} \mathrm{d}s_3\, \mathcal{U}^{(1)}(t-s_1)\mathrm{Tr}_{2,3,4}\, \delta(x_1-x_2)\mathcal{U}^{(2)}(s_1-s_2)$$

$$\times\, \delta(x_1-x_3)\mathcal{U}^{(3)}(s_2-s_3)\delta(x_2-x_4)\mathcal{U}^{(4)}(s_3)\gamma_0^{(4)}$$

Feynman Graphs: different contributions to $\xi_{m,t}^{(k)}$ will be represented by graphs in

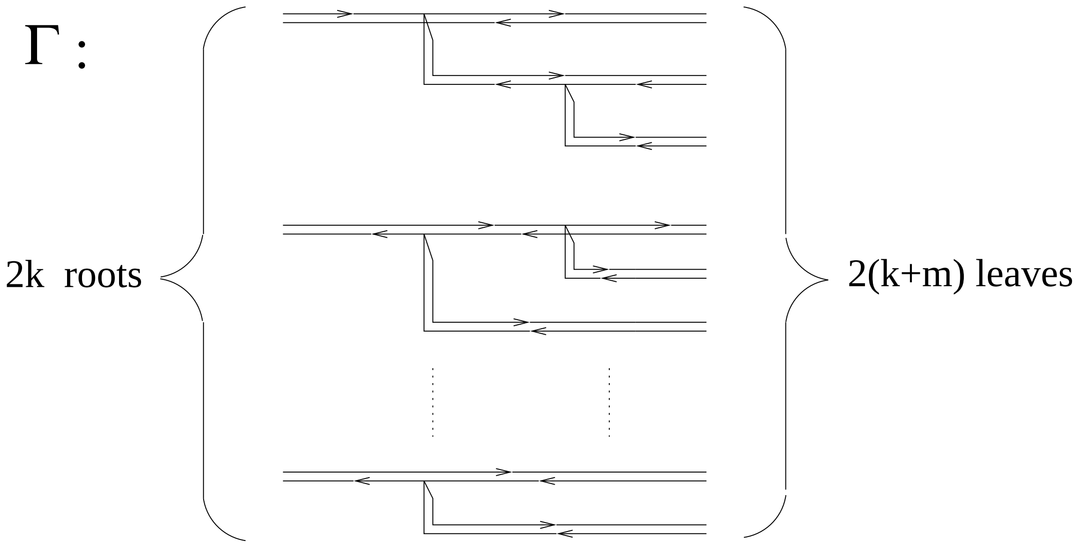
$$\mathcal{F}_{m,k} = \text{set of forests with } 2k \text{ disjoint paired trees}$$
$$\text{with } m \text{ partially ordered vertices}$$



Number of graphs in $\mathcal{F}_{m,k} \leq C^{m+k}$.

Diagrammatic Expansion of $\xi_{m,t}^{(k)}$: we expand

$$\mathrm{Tr} J^{(k)} \xi_{m,t}^{(k)} = \sum_{\Gamma \in \mathcal{F}_{m,k}} \mathrm{Tr}\, J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)}$$



E (Γ) = set of edges of Γ

V (Γ)= set of vertices of Γ

R (Γ)= set of roots of Γ

L (Γ)= set of leaves of Γ

$$\mathrm{Tr}\; J^{(k)}\, K_{\Gamma,t}\, \gamma_0^{(k+m)} =$$

$$= \int \prod_{e\in E(\Gamma)} \frac{\mathrm{d}\alpha_e \mathrm{d}p_e}{\alpha_e - p_e^2 + i\tau_e\eta_e} \prod_{v\in V(\Gamma)} \delta\left(\sum_{e\in v} \pm\alpha_e\right) \delta\left(\sum_{e\in v} \pm p_e\right)$$

$$\times\, J^{(k)} \left(\{(p_e, p_e')\}_{e\in R(\Gamma)}\right)\; \gamma_0^{(k+m)} \left(\{(p_e, p_e')\}_{e\in L(\Gamma)}\right)$$

$$\times\, \exp(-it \sum_{e\in R(\Gamma)} \tau_e(\alpha_e + i\tau_e\eta_e)), \qquad \tau_e = \pm 1$$

Control of the Integral: use $\langle x\rangle = (1+x^2)^{1/2}$.

$$\left|\mathrm{Tr}\, J^{(k)}\, K_{\Gamma,t}\,\gamma_0^{(k+m)}\right| \leq \; C^m t^{m/4}$$

$$\times \int \prod_{e\in E(\Gamma)} \frac{\mathrm{d}\alpha_e \mathrm{d}p_e}{\langle \alpha_e - p_e^2\rangle} \prod_{v\in V(\Gamma)} \delta\left(\sum_{e\in v} \pm\alpha_e\right)\delta\left(\sum_{e\in v}\pm p_e\right)$$

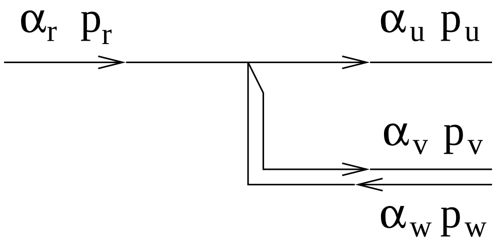
$$\times \left| J^{(k)}\left(\{(p_e,p'_e)\}_{e\in R(\Gamma)}\right)\right| \left|\gamma_0^{(k+m)}\left(\{(p_e,p'_e)\}_{e\in L(\Gamma)}\right)\right|$$

Singularity at $x=0 \Rightarrow$ large momentum problem!!

From a-priori estimates $\Rightarrow$ decay in the momenta of leaves.

Perform integration over all α and p, starting from the leaves and moving towards the roots. At each vertex, we propagate the decay from the son-edges to the father-edge.

Typical Example:



Integrate first the α-variables of the son-edges

$$\int d\alpha_u d\alpha_v d\alpha_w \frac{\delta(\alpha_r = \alpha_u + \alpha_v - \alpha_w)}{\langle \alpha_u - p_u^2 \rangle \langle \alpha_v - p_v^2 \rangle \langle \alpha_w - p_w^2 \rangle} \leq \frac{\text{const}}{\langle \alpha_r - p_u^2 - p_v^2 + p_w^2 \rangle^{1-\varepsilon}}$$

Then integrate over the momenta of the son-edges

$$\int \frac{dp_u dp_v dp_w}{|p_u|^{2+\lambda} |p_v|^{2+\lambda} |p_w|^{2+\lambda}} \frac{\delta(p_r = p_u + p_v - p_w)}{\langle \alpha_r - p_u^2 - p_v^2 + p_w^2 \rangle^{1-\varepsilon}} \leq \frac{\text{const}}{|p_r|^{2+\lambda}}$$

After integrating out all vertices

$$\Rightarrow \left| \text{Tr}\, J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)} \right| \leq C^m t^{m/4} \qquad \forall \Gamma \in \mathcal{F}_{m,k}$$

Convergence of the Expansion: since $|\mathcal{F}_{m,k}| \leq C^m$, we find

$$\left|\text{Tr } J^{(k)} \xi_{m,t}^{(k)}\right| \leq \sum_{\Gamma \in \mathcal{F}_{m,k}} \left|\text{Tr } J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)}\right| \leq C^m t^{m/4}.$$

Analogously, we prove that $\left|\text{Tr } J^{(k)} \eta_{n,t}^{(k)}\right| \leq C^n t^{n/4}$.

⇒ if $\gamma_{1,t}^{(k)}$, $\gamma_{2,t}^{(k)}$ are two solutions with same initial data

$$\left|\text{Tr } J^{(k)} \left(\gamma_{1,t}^{(k)} - \gamma_{2,t}^{(k)}\right)\right| \leq C^n t^{n/4}$$

Since $n \in \mathbb{N}$ is arbitrary ⇒ uniqueness for short time.

A-priori estimates are uniform in time ⇒ uniqueness for all times.