

Center for Scientific Computing, University of Warwick

Optical lattices and Bose gases

MIR@W-Day and CSC One-Day Workshop

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Anisotropic Generalised BEC with Two Critical Densities

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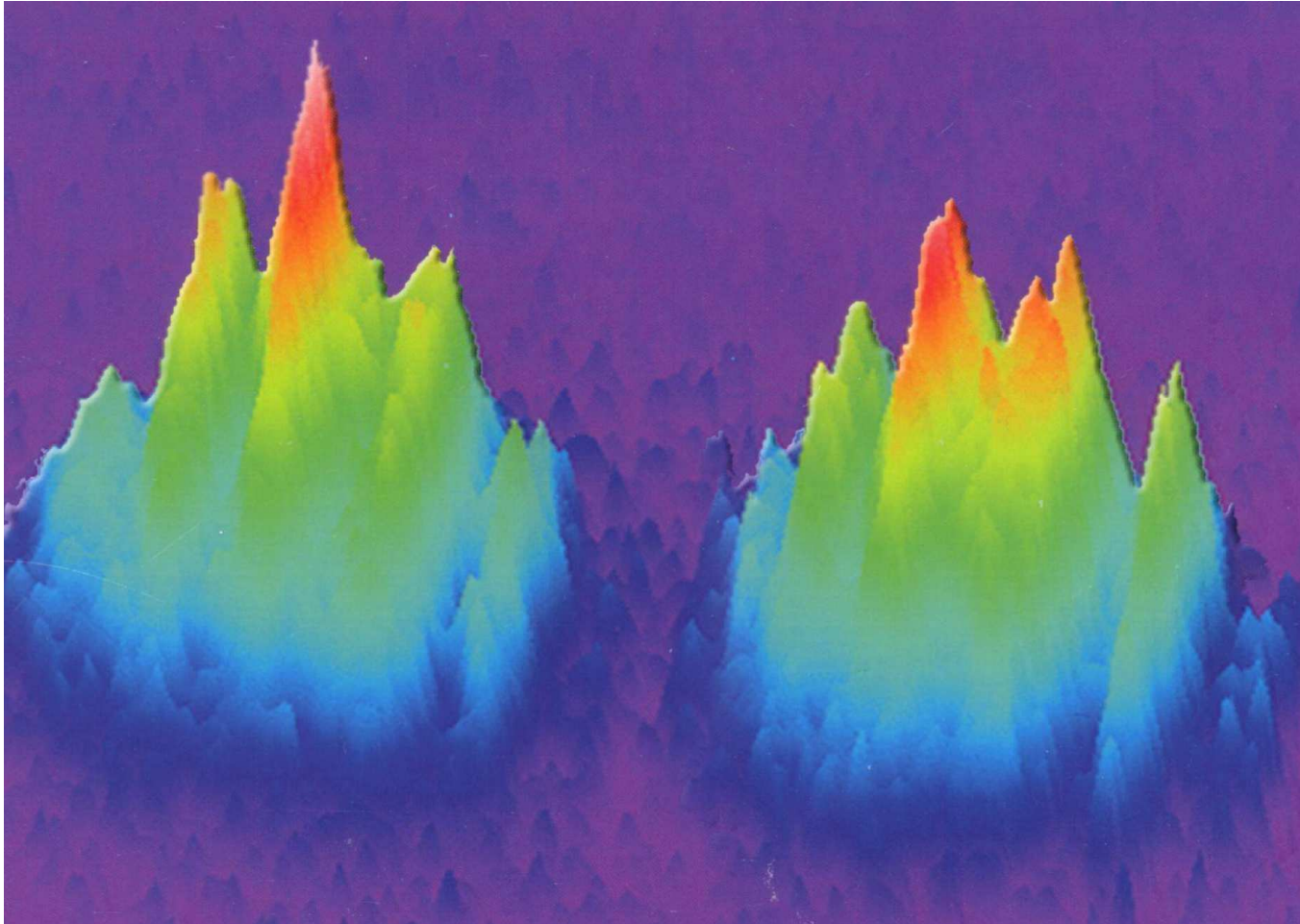
Université Aix-Marseille II

Centre de Physique Théorique, Luminy

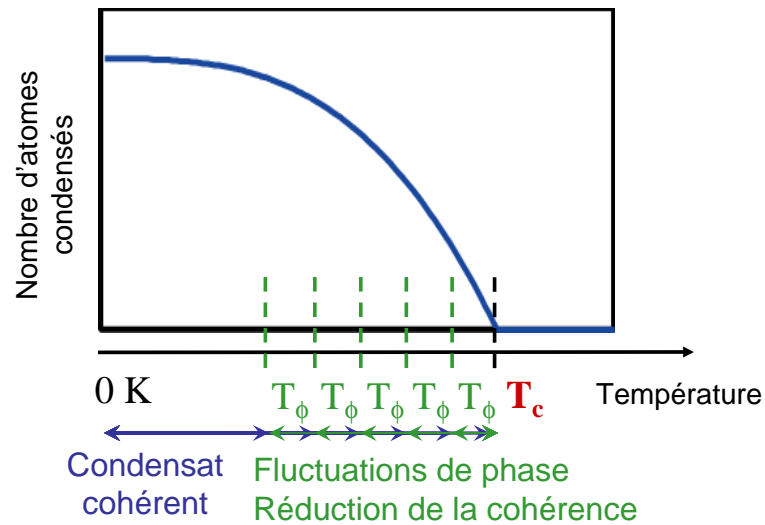
- 0. Experimental Data.
- 1. Perfect Bose-gas.
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- 3. Exponential BEAM and CIGAR Traps.
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M. Beau and V.A.Z arXiv:1002.1242 (February 2010)

0. Experimental Data Optical lattices and Bose gases - Warwick 2010

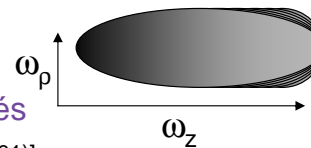


Température de phase T_ϕ (caractérise la cohérence)

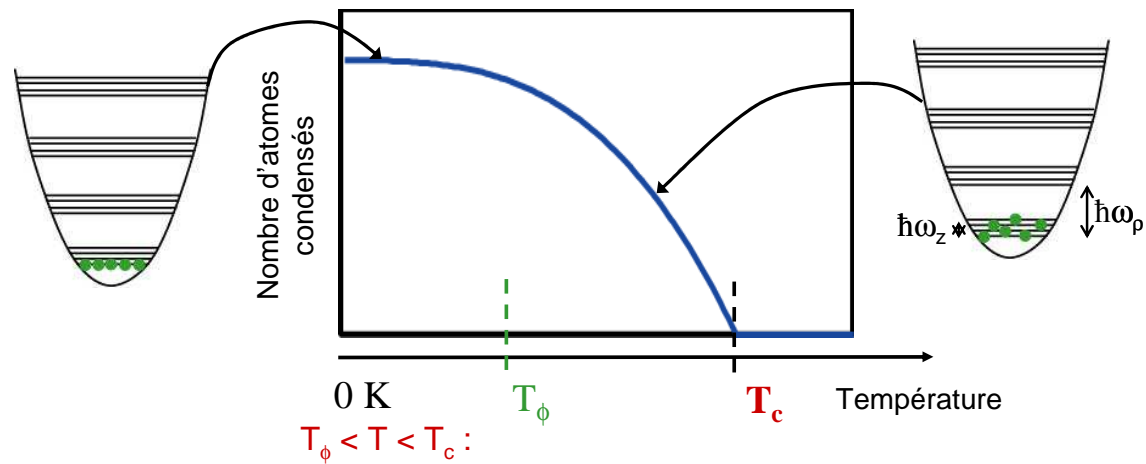


$$T_\phi = \frac{15}{32} \frac{(\hbar\omega_z)^2 N}{\mu}$$

T_ϕ petite :
 - condensat long
 - peu d'atomes condensés
 D. Petrov et al. [PRL 87, 050404 (2001)]



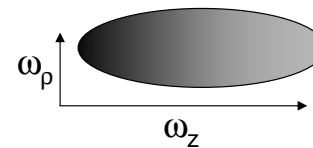
Origine des fluctuations de phase



Distribution aléatoire sur plusieurs niveaux d'énergie très proches
 \Rightarrow Fluctuations de phase suivant l'axe long du condensat

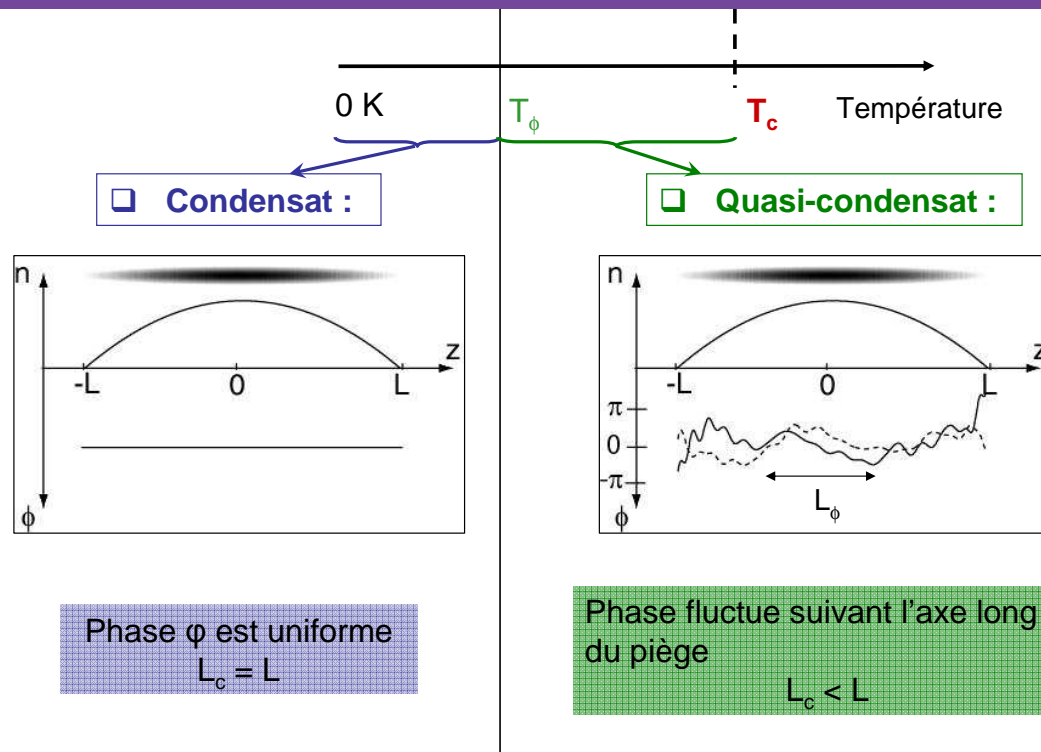
Amplitude des fluctuations de phase :

$$\frac{T}{T_\phi}$$



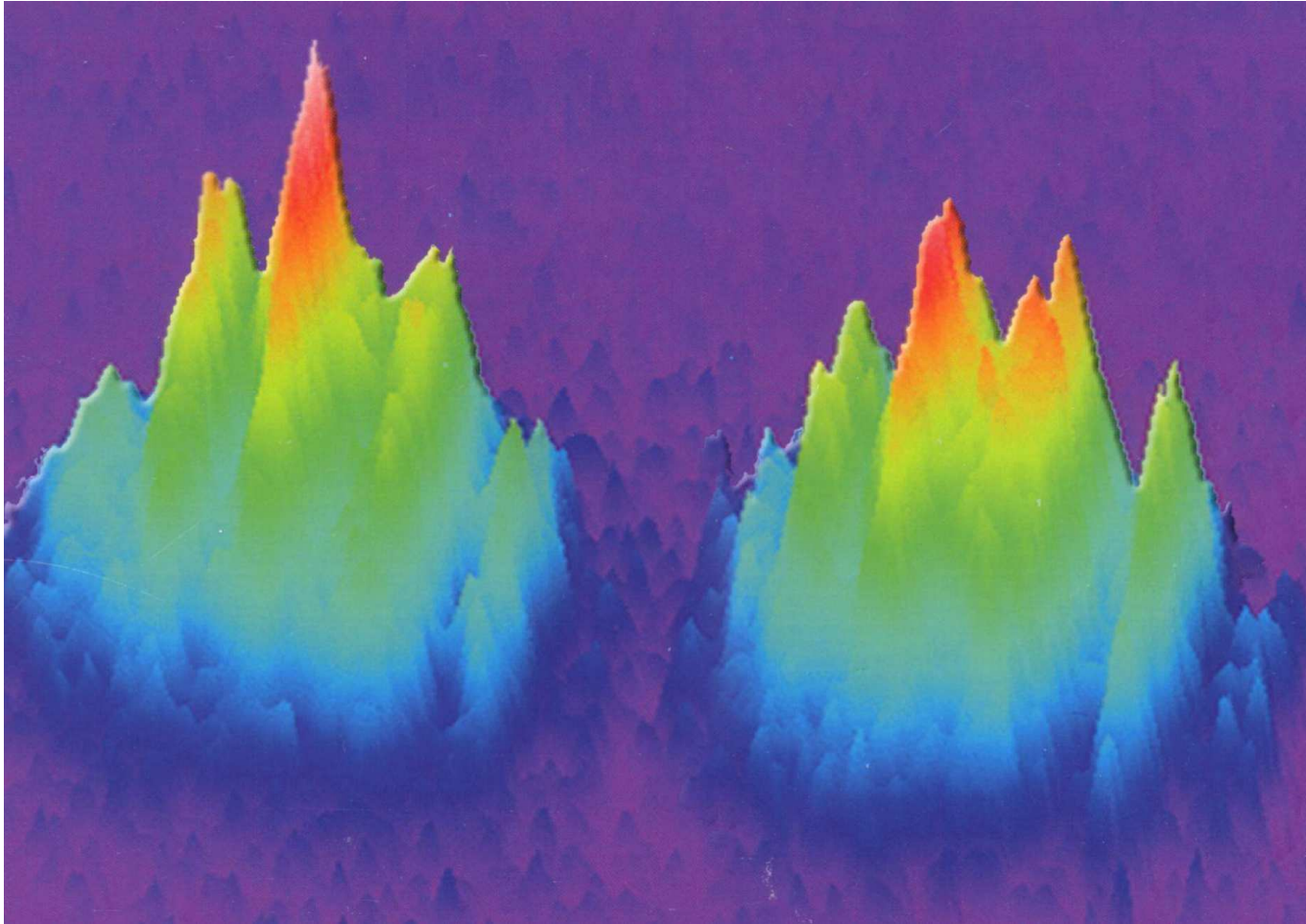
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Densité et phase du quasi-condensat



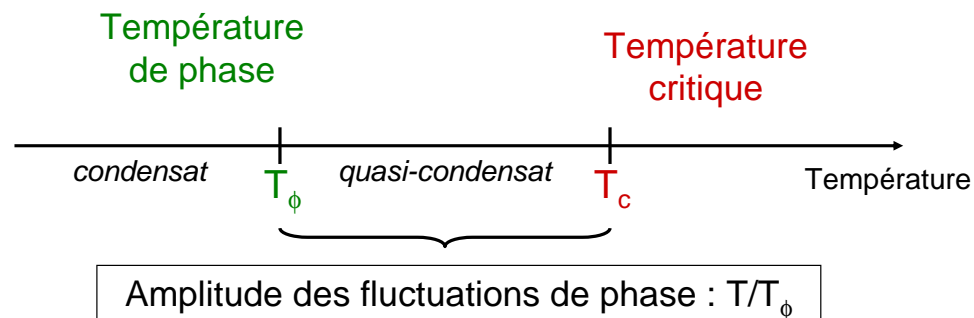
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Optical lattices and Bose gases - Warwick 2010



Résumé

- Deux températures pour la caractérisation de la condensation :



1. Perfect Bose-gas

- For $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^3$ and $T_\Lambda^{(N=1)} = (-\hbar^2 \Delta / (2m))_D$ the spectrum:

$$\left\{ \varepsilon_s = \frac{\hbar^2}{2m} \sum_{j=1}^3 (\pi s_j / L_j)^2 \right\}_{s_j \in \mathbb{N}}$$

- Eigenfunctions: $\{\phi_{s,\Lambda}(x) = \prod_{j=1}^3 \sqrt{2/L_j} \sin(\pi s_j x_j / L_j)\}_{s_j \in \mathbb{N}}$,
 $s := (s_1, s_2, s_3) \in \mathbb{N}^3$
- In (T, V, μ) , $V = L_1 L_2 L_3$ the Gibbs mean occupation number of $\phi_{s,\Lambda}$ is $N_s(\beta, \mu) = (e^{\beta(\varepsilon_s - \mu)} - 1)^{-1}$, $\mu < \inf_s \varepsilon_s$.
- Particle density $\rho_\Lambda(\beta, \mu) = \sum_{s \in \mathbb{N}^3} N_s(\beta, \mu) / V =: N_\Lambda(\beta, \mu) / V$
- The **first critical density**: $\rho_c(\beta) := \sup_{\mu \leq 0} \lim_\Lambda \rho_\Lambda(\beta, \mu) = \zeta(3/2) / \lambda_\beta^3$, $\lambda_\beta := \hbar \sqrt{2\pi\beta/m}$, de Broglie thermal length.

2. Exponential SLAB and the Second Critical Point

2.1 Let $\Lambda = Le^{\alpha L} \times Le^{\alpha L} \times L$. For **any fixed** s_1, s_2 and $\mu \leq 0$

$$\lim_{L \rightarrow \infty} \sum_{s \neq (s_1, s_2, 1)} \frac{N_s(\beta, \mu)}{V_L} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3 k}{e^{\beta(\hbar^2 k^2 / 2m - \mu)} - 1}.$$

2.2 Let $\mu_L(\beta, \rho) := \varepsilon_{(1,1,1)} - \Delta_L(\beta, \rho)$, where $\Delta_L(\beta, \rho) \geq 0$ is a **unique solution** of the equation:

$$\rho = \sum_{s=(s_1, s_2, 1)} \frac{N_s(\beta, \mu)}{V_L} + \sum_{s \neq (s_1, s_2, 1)} \frac{N_s(\beta, \mu)}{V_L}. \quad (1)$$

2.3 Since: $\lim_{L \rightarrow \infty} \sum_{s \neq (s_1, s_2, 1)} N_s(\beta, \mu = 0) / V_L = \rho_c(\beta)$, for $\rho > \rho_c(\beta)$ the limit of the **first sum** is

$$\begin{aligned} \lim_{L \rightarrow \infty} \sum_{s=(s_1, s_2, 1)} \frac{N_s(\beta, \mu)}{V_L} &= \\ \lim_{L \rightarrow \infty} \frac{1}{L} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{d^2 k}{e^{\beta(\hbar^2 k^2 / 2m + \Delta_L(\beta, \rho))} - 1} &= \\ \lim_{L \rightarrow \infty} -\frac{1}{\lambda_\beta^2 L} \ln[\beta \Delta_L(\beta, \rho)] &= \rho - \rho_c(\beta). \end{aligned}$$

This implies the asymptotics:

$$\Delta_L(\beta, \rho) = \frac{1}{\beta} e^{-\lambda_\beta^2 (\rho - \rho_c(\beta)) L} + \dots$$

2.4 Remark 2.1. Since $L_{j=1,2} = Le^{\alpha L}$ and

$$\varepsilon_{(s_1, s_2, 1)} - \mu_L(\beta, \rho) = \frac{\hbar^2}{2m} \sum_{j=1}^2 [(\pi s_j / L_j)^2 - 1] + \Delta_L(\beta, \rho)$$

the representation of the **first sum** by the **integral** is valid **only** when $\lambda_\beta^2(\rho - \rho_c(\beta)) \leq 2\alpha$, i.e.

$$L^{-2}e^{-2\alpha L} < \beta^{-1} e^{-\lambda_\beta^2(\rho - \rho_c(\beta))L} + \dots .$$

2.5 Definition 2.2. The **second** critical density:

$$\rho_m(\beta) := \rho_c(\beta) + 2\alpha / \lambda_\beta^2 > \rho_c(\beta) .$$

2.6 Remark 2.3. For $\rho > \rho_m(\beta)$ the convergence $\Delta_L(\beta, \rho) \rightarrow 0$ **must** be **faster** than $e^{-2\alpha L}$.

2.7 To keep the difference $\rho - \rho_m(\beta) > 0$ one **must** return back to the **finite volume sum representation** to take into account the **input of the ground state** occupation density.

Theorem 2.4. The asymptotics of $\Delta_L(\beta, \rho > \rho_m(\beta))$ is

$$\Delta_L(\beta, \rho) = [\beta(\rho - \rho_m(\beta))V_L]^{-1} + \dots < L^{-2}e^{-2\alpha L} .$$

2.8 Since $V_L = L^3e^{2\alpha L}$, the **first sum without** the ground-state:

$$\lim_{L \rightarrow \infty} \sum_{s=(s_1 > 1, s_2 > 1, 1)} \frac{N_s(\beta, \mu)}{V_L} = \lim_{L \rightarrow \infty} \frac{1}{\lambda_\beta^2 L} \ln[\beta \Delta_L(\beta, \rho)]^{-1} =$$

$$2\alpha / \lambda_\beta^2 = \rho_m(\beta) - \rho_c(\beta).$$

2.9 The ground-state term gives the **macroscopic** occupation:

$$\rho - \rho_m(\beta) = \lim_{L \rightarrow \infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_{(1,1,1)} - \mu_L(\beta, \rho))} - 1} .$$

2.10 Corollary 2.5 Since for $\rho_c(\beta) < \rho < \rho_m(\beta)$

$$\varepsilon_s - \mu_L(\beta, \rho) = \Delta_L(\beta, \rho) + \varepsilon_s - \varepsilon_{(1,1,1)} = \mathcal{O}(\beta^{-1} e^{-\lambda_{\beta}^2(\rho - \rho_c(\beta))L}) ,$$

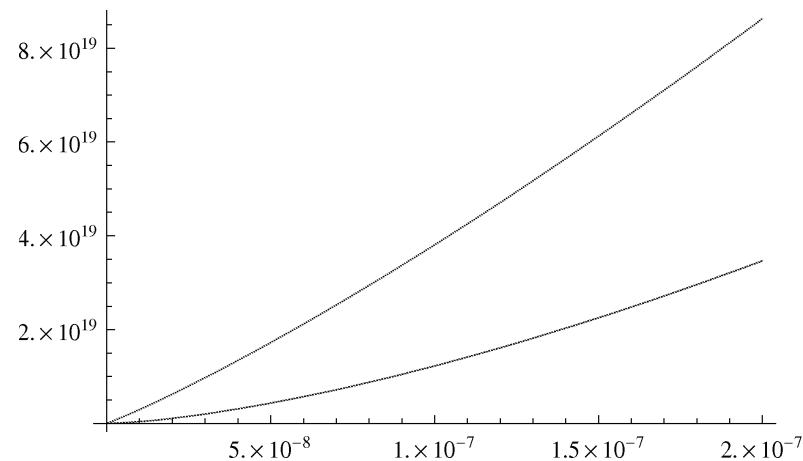
one gets the **type III** van den Berg-Lewis-Pulé generalised condensation (vdBLP-GC): when **none** of the single-particle states are *macroscopically* occupied:

$$\rho_s(\beta, \rho) := \lim_{L \rightarrow \infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta, \rho))} - 1} = 0 .$$

The asymptotics $\Delta_L(\beta, \rho > \rho_m(\beta)) = [\beta(\rho - \rho_m(\beta))V_L]^{-1}$ implies

$$\lim_{L \rightarrow \infty} \rho_{s \neq (1,1,1)}(\beta, \rho) := \lim_{L \rightarrow \infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta, \rho))} - 1} = 0 ,$$

and $\lim_{L \rightarrow \infty} \rho_{(1,1,1)}(\beta, \rho) = \rho - \rho_m(\beta) > 0$, the **type I** vdBLP-GC.



2.11 For $\rho > \rho_m(\beta)$ there is a *coexistence* of the *saturated type III* vdBLP-GC, with the **constant** density $\rho_m(\beta) - \rho_c(\beta)$, and the standard BEC (the **type I** vdBLP-GC) in the the **ground state** with the density $\rho - \rho_m(\beta)$.

3. Exponential BEAM and CIGAR Traps

3.1 Remark 3.1 It is curious to note that neither **Casimir shaped boxes** $\Lambda = L^{\alpha_1} \times L^{\alpha_2} \times L^{\alpha_3}$, nor the **van den Berg boxes** $\Lambda = Le^{\alpha L} \times L \times L$, with **one-dimensional anisotropy** do **not** produce the *second* critical density $\rho_m(\beta) \neq \rho_c(\beta)$.

3.2 Remark 3.2 (BEAM) For beams with **two** critical densities we consider the Hamiltonian: $T_{\Lambda}^{(N=1)} = -\hbar^2 \Delta / (2m) + m\omega_1^2 x_1^2 / 2$, with **harmonic trap** in direction x_1 and Dirichlet boundary conditions in directions x_2, x_3 . Then the spectrum:

$$\left\{ \epsilon_s := \hbar\omega_1(s_1 + 1/2) + \frac{\hbar^2}{2m} \sum_{j=2}^3 (\pi s_j / L_j)^2 \right\}_{s \in \mathbb{N}}$$

$s = (s_1, s_2, s_3) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2$, the **ground-state** energy: $\epsilon_{(0,1,1)}$.

3.3 For $\mu_L(\beta, \varrho) := \epsilon_{(0,1,1)} - \Delta_L(\beta, \varrho)$, the $\Delta_L(\beta, \varrho) \geq 0$ is a unique solution of the equation:

$$\varrho := \sum_{s=(s_1,1,1)} \omega_1 \frac{N_s(\beta, \mu)}{L_2 L_3} + \sum_{s \neq (s_1,1,1)} \omega_1 \frac{N_s(\beta, \mu)}{L_2 L_3},$$

$N_s(\beta, \mu) = (e^{\beta(\epsilon_s - \mu)} - 1)^{-1}$, $\omega_1 := \hbar/(mL_1^2)$ and $L_2 = L_3 = L$.

3.4 Similar to SLAB, for any $s_1 \geq 0$ and $\mu \leq 0$

$$\varrho(\beta, \mu) := \lim_{L_1, L \rightarrow \infty} \sum_{s \neq (s_1,1,1)} \omega_1 \frac{N_s(\beta, \mu)}{L_2 L_3} = \frac{1}{(2\pi)^2} \int_0^\infty dp \int_{\mathbb{R}^2} \frac{d^2 k}{e^{\beta(\hbar p + \hbar^2 k^2 / 2m - \mu)} - 1}.$$

The **first critical density** is *finite*: $\varrho_c(\beta) := \sup_{\mu \leq 0} \varrho(\beta, \mu) = \varrho(\beta, \mu = 0) < \infty$.

3.5 For $\varrho > \varrho_c(\beta)$ the limit $L \rightarrow \infty$ of the **first sum** in **3.3**

$$\begin{aligned} \lim_{L_1, L \rightarrow \infty} \sum_{s=(s_1, 1, 1)} \omega_1 \frac{N_s(\beta, \mu_L)}{L_2 L_3} &= \\ \lim_{L \rightarrow \infty} \frac{1}{L^2} \int_0^\infty \frac{dp}{e^{\beta(\hbar p + \Delta_L(\beta, \varrho))} - 1} &= \\ \lim_{L \rightarrow \infty} \frac{1}{\hbar \beta L^2} \ln[\beta \Delta_L(\beta, \varrho)]^{-1} &= \varrho - \varrho_c(\beta). \end{aligned}$$

This gives the **asymptotics** : $\Delta_L(\beta, \varrho) = \beta^{-1} e^{-\hbar \beta(\varrho - \varrho_c(\beta))} L^2 + \dots$

3.6 Let $L_1 := L e^{\gamma L^2}$, $\gamma > 0$. Then, similar to SLAB, the representation of the limit in **3.5** by the integral is valid for $\hbar \beta(\varrho - \varrho_c(\beta)) \leq 2\gamma$ and we reach to necessity of the **second critical density** $\varrho_m(\beta) := \varrho_c(\beta) + 2\gamma/(\hbar \beta)$.

3.7 The rest of scenario is **identical** to the case of the SLAB.

3.8 Remark 3.3 (CIGAR) A "cigar"-type geometry is ensured by the **anisotropic harmonic trap**:

$$T_{\Lambda}^{(N=1)} = -\hbar^2 \Delta / (2m) + \sum_{1 \leq j \leq 3} m \omega_j^2 x_j^2 / 2 .$$

with $\omega_1 = \hbar / (mL_1^2)$, $\omega_2 = \omega_3 = \hbar / (mL^2)$. Here $L_1, L_2 = L_3 = L$ are the *characteristic* sizes of the trap in three directions and the **spectrum** $\eta_s = \sum_{1 \leq j \leq 3} \hbar \omega_j (s_j + 1/2)$.

3.9 For $\mu_L(\beta, n) := \eta_{(0,0,0)} - \Delta_L(\beta, n)$ and factor $\kappa > 0$:

$$\lim_{L_1, L \rightarrow \infty} \sum_{s=(s_1, 0, 0)} \kappa^3 \omega_1 \omega_2 \omega_3 N_s(\beta, \mu_L) =$$

$$\lim_{L \rightarrow \infty} \frac{\kappa^3 \hbar}{\beta (mL^2)^2} \ln[\beta \Delta_L(\beta, n)]^{-1} = n - n_c(\beta).$$

3.10 Again the **first critical density** $n_c(\beta) := n(\beta, \mu = 0)$ is **finite**:

$$n(\beta, \mu) := \lim_{L_1, L \rightarrow \infty} \sum_{s \neq (s_1, 0, 0)} \kappa^3 \omega_1 \omega_2 \omega_3 N_s(\beta, \mu) = \int_{\mathbb{R}^3_+} \frac{\kappa^3 d\omega_1 d\omega_2 d\omega_3}{e^{\beta[(\omega_1 + \omega_2 + \omega_3) - \mu]} - 1},$$

and asymptotics:

$$\Delta_L(\beta, n > n_c(\beta)) = \beta^{-1} e^{-\beta(n - n_c(\beta))m^2 L^4 / (\hbar \kappa^3)} + \dots$$

3.11 If $L_1 := L e^{\hat{\gamma} L^4}$, $\hat{\gamma} > 0$, then the **second critical density**:

$$n_m(\beta) := n_c(\beta) + (\hat{\gamma} \hbar \kappa^3) / (\beta m^2).$$

is defined by the standard argument of the **energy level spacing**.

3.12 Bose-condensation (CIGAR) For $n_c(\beta) < n < n_m(\beta)$ we obtain the *type III vdBLP-GC*, when *none* of the single-particle states are *macroscopically* occupied:

$$n_s(\beta, \rho) := \lim_{L \rightarrow \infty} \frac{\kappa^3 \omega_1 \omega_2 \omega_3}{e^{\beta(\eta_s - \mu_L(\beta, n))} - 1} = 0 .$$

Although for $n_m(\beta) < n$ there is a coexistence of the *type III vdBLP-GC*, with the **saturated constant** density $n_m(\beta) - n_c(\beta)$, and the standard BEC (*type I vdBLP-GC*) in the ground-state:

$$n - n_m(\beta) = \lim_{L \rightarrow \infty} \frac{\kappa^3 \omega_1 \omega_2 \omega_3}{e^{\beta(\eta_{(0,0,0)} - \mu_L(\beta, n))} - 1} > 0 .$$

4. Temperature Dependence of the Bose-Condensate

4.1 The *first* critical temperatures: $T_c(\rho)$, $\tilde{T}_c(\rho)$ or $\hat{T}_c(\rho)$ are well-known. For a given density ρ they verify the identities:

$$\rho = \rho_c(\beta_c(\rho)) , \quad \varrho = \varrho_c(\tilde{\beta}_c(\varrho)) , \quad n = n_c(\hat{\beta}_c(n)) ,$$

respectively for slabs, squared beams or "cigars".

4.2 Since $\rho_c(\beta) =: T^{3/2} I_{sl}$, $\varrho_c(\beta) =: T^2 I_{bl}$, $n_c(\beta) =: T^3 I_{cg}$, the expressions for the **second critical densities** one gets relations between the *first* and the *second* critical temperatures:

$$\begin{aligned} T_m^{3/2}(\rho) + \tau^{1/2} T_m(\rho) &= T_c^{3/2}(\rho) \quad (\text{slab}) , \\ \tilde{T}_m^2(\varrho) + \tilde{\tau} \tilde{T}_m(\varrho) &= \tilde{T}_c^2(\varrho) \quad (\text{beam}) , \\ \hat{T}_m^3(n) + \hat{\tau}^2 \hat{T}_m(n) &= \hat{T}_c^3(n) \quad (\text{cigar}) . \end{aligned}$$

$\tau = [\alpha m k_B / (\pi \hbar^2 I_{sl})]^2$, $\tilde{\tau} = 2\gamma k_B / (\hbar I_{bl})$, $\hat{\tau} = [(\hat{\gamma} \hbar \kappa^3 k_B) / (m^2 I_{cg})]^{1/2}$ are "**effective**" temperatures related to the corresponding geometrical shapes.

4.3 Since the **total** condensate density is $\rho - \rho_c(\beta) := \rho_0(\beta) = \rho_{0c}(\beta) + \rho_{0m}(\beta)$, where $\rho_{0m}(\beta) := (\rho - \rho_m(\beta)) \theta(\rho - \rho_m(\beta))$, the **second** critical temperature **modifies** the usual law for the condensate fractions temperature dependence.

4.4 For the **type III vdBLP-GC**, $\rho_{0c}(\beta)$, in the SLAB geometry:

$$\frac{\rho_{0c}(\beta)}{\rho} = \begin{cases} 1 - (T/T_c)^{3/2}, & T_m \leq T \leq T_c, \\ \sqrt{\tau} T/T_c^{3/2}, & T \leq T_m. \end{cases}$$

For the BEC (**type I vdBLP-GC**) in the ground state $\rho_{0m}(\beta)$:

$$\frac{\rho_{0m}(\beta)}{\rho} = \begin{cases} 0, & T_m \leq T \leq T_c, \\ 1 - (T/T_c)^{3/2}(1 + \sqrt{\tau/T}), & T \leq T_m, \end{cases}$$

The **total** condensate density $\rho_0(\beta) := \rho_{0c}(\beta) + \rho_{0m}(\beta)$ results from **coexistence** of both of them: this gives the **standard PBG** expression $\rho_0(\beta)/\rho = 1 - (T/T_c)^{3/2}$.

4.5 For the "cigars" geometry the *type III vdBLP-GC* $n_{0c}(\beta)$:

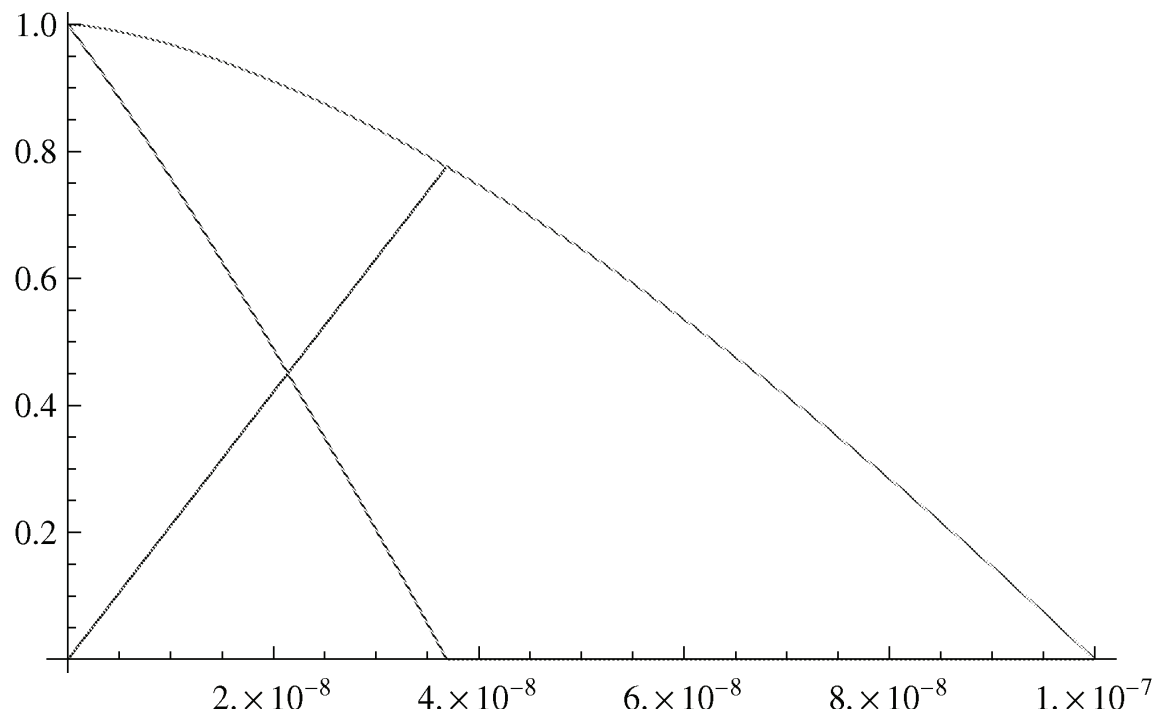
$$\frac{n_{0c}(\beta)}{n} = \begin{cases} 1 - (T/\hat{T}_c)^3, & \hat{T}_m \leq T \leq \hat{T}_c, \\ \hat{\tau}^2 T/\hat{T}_c^3, & T \leq \hat{T}_m. \end{cases}$$

The *ground state* conventional BEC is

$$\frac{n_{0m}(\beta)}{n} = \begin{cases} 0, & \hat{T}_m \leq T \leq \hat{T}_c, \\ 1 - (T/\hat{T}_c)^3(1 + \hat{\tau}^2/T^2), & T \leq \hat{T}_m, \end{cases}$$

and again for the two *coexisting* condensates one gets a standard expression:

$$n - n_c(\beta) := n_0(\beta) = n_{0c}(\beta) + n_{0m}(\beta) = (1 - (T/T_c)^{3/2})n.$$



5. Anisotropy and Localisation

5.1 Global Scaled Particle Density :

$$\xi_L(u) := \sum_s \frac{|\phi_{s,\Lambda}(L_1 u_1, L_2 u_2, L_3 u_3)|^2}{e^{\beta(\varepsilon_s - \mu)} - 1},$$

with the scaled distances $\{u_j = x_j/L_j \in [0, 1]\}_{j=1,2,3}$.

5.2 For a given ρ in the slab geometry

$$\xi_{\rho,L}^{slab}(u) := \sum_s \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j} [\sin(\pi s_j u_j)]^2.$$

Since $2[\sin(\pi s_j u_j)]^2 = 1 - \cos\{(2\pi s_j/L_j)u_j L_j\}$ and $\lim_{L \rightarrow \infty} \mu_L(\beta, \rho < \rho_c(\beta)) < 0$, by the Riemann-Lebesgue lemma we obtain that $\lim_{L \rightarrow \infty} \xi_{\rho,\Lambda}^{slab}(u) = \rho$ for any $u \in (0, 1)^3$.

5.3 If $\rho > \rho_c(\beta)$, then for any $u \in (0, 1)^3$:

$$\begin{aligned}
 & \lim_{L \rightarrow \infty} \sum_{s=(s_1, s_2, 1)} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta, \rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j} [\sin(\pi s_j u_j)]^2 \\
 &= \lim_{L \rightarrow \infty} \frac{2[\sin(\pi u_3)]^2}{(2\pi)^2 L} \int_{\mathbb{R}^2} \frac{\prod_{j=1}^2 (1 - \cos(2k_j u_j L_j)) d^2 k}{e^{\beta(\hbar^2 k^2 / 2m + \Delta_L(\beta, \rho))} - 1} \\
 &= (\rho - \rho_c(\beta)) 2[\sin(\pi u_3)]^2, \\
 & \lim_{L \rightarrow \infty} \sum_{s \neq (s_1, s_2, 1)} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta, \rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j} [\sin(\pi s_j u_j)]^2 \\
 &= \rho_c(\beta) \\
 & \Rightarrow \xi_\rho^{slab}(u) = (\rho - \rho_c(\beta)) 2[\sin(\pi u_3)]^2 + \rho_c(\beta),
 \end{aligned}$$

which manifests a *space anisotropy* of the *type III vdBLP-GC* for $\rho_c(\beta) < \rho < \rho_m(\beta)$ in *direction* u_3 .

5.4 For $\rho > \rho_m(\beta)$ one has to use representations and asymptotics from **2**. Then

$$\xi_\rho^{slab}(u) = (\rho - \rho_m(\beta)) \prod_{j=1}^3 2[\sin(\pi u_j)]^2 + (\rho_m(\beta) - \rho_c(\beta)) 2[\sin(\pi u_3)]^2 + \rho_c(\beta) .$$

So, the anisotropy of the **space particle distribution** is still only in **direction** u_3 due to the **type III vdBLP-GC** ("quasi-condensate") $(\rho_m(\beta) - \rho_c(\beta))$. The input of the standard **type I vdBLP-GC** (one mode BEC) $(\rho - \rho_m(\beta))$ is **isotropic**.

6. Coherence Length and Anisotropy

6.1 ODLRO kernel:

$$K(x, y) := \lim_{L \rightarrow \infty} K_\Lambda(x, y) = \lim_{L \rightarrow \infty} \sum_s \frac{\overline{\phi}_{s,\Lambda}(x) \phi_{s,\Lambda}(y)}{e^{\beta(\varepsilon_s - \mu_L(\beta, \rho))} - 1} .$$

Let us **center** the box Λ at the origin of coordinates: $x_j = \tilde{x}_j + L_j/2$ and $y_j = \tilde{y}_j + L_j/2$. Then the **ODLRO** kernel gets the form:

$$K_\Lambda(\tilde{x}, \tilde{y}) = \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta, \rho)} R_l^{(2)} R_l^{(1)} .$$

6.2 Here after the shift of coordinates and using additive form of the spectrum we put

$$\begin{aligned}
 R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) &= \\
 &\sum_{s=(s_1, s_2)} e^{-l\beta\varepsilon_{s_1, s_2}} \bar{\phi}_{s_1, s_2, \Lambda}(\tilde{x}_1, \tilde{x}_2) \phi_{s_1, s_2, \Lambda}(\tilde{y}_1, \tilde{y}_2) \\
 R_s^{(1)}(\tilde{x}_3, \tilde{y}_3) &= \sum_{s=(s_3)} e^{-l\beta\varepsilon_{s_3}} \sqrt{\frac{2}{L_3}} \sin\left(\frac{\pi s_3}{L_3}\left(\tilde{x}_3 + \frac{L_3}{2}\right)\right) \\
 &\times \sqrt{\frac{2}{L_3}} \sin\left(\frac{\pi s_3}{L_3}\left(\tilde{y}_3 + \frac{L_3}{2}\right)\right) .
 \end{aligned}$$

6.3 By the [Weyl theorem](#) one gets for the first two directions:

$$\lim_{L \rightarrow \infty} R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) = \frac{1}{l\lambda_\beta^2} e^{-\pi \|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2 / l\lambda_\beta^2} .$$

6.4 For **exponentially anisotropic** box and for $\rho_c(\beta) < \rho < \rho_m(\beta)$ we must split the sum over $s = (s_1, s_2, s_3)$ in **6.1** into **two parts**: sum over $s = (s_1, s_2, 1)$ and the rest. For the first sum by **6.3** we obtain:

$$\begin{aligned} & \lim_{L \rightarrow \infty} \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s=(s_1,s_2,1)} e^{-l\beta\varepsilon_{s_1,s_2,1}} \times \\ & \times \bar{\phi}_{s_1,s_2,1\Lambda}(\tilde{x}) \phi_{s_1,s_2,1\Lambda}(\tilde{y}) = \\ & \lim_{L \rightarrow \infty} \sum_{l=1}^{\infty} e^{-l\beta\Delta_L(\beta,\rho)} \frac{1}{l\lambda_\beta^2} e^{-\pi\|\tilde{x}^{(2)}-\tilde{y}^{(2)}\|^2/l\lambda_\beta^2} \times \\ & \times \frac{2}{L} \sin\left(\frac{\pi}{L}\left(\tilde{x}_3 + \frac{L}{2}\right)\right) \sin\left(\frac{\pi}{L}\left(\tilde{y}_3 + \frac{L}{2}\right)\right) . \end{aligned}$$

6.5 For the second part we apply the [Weyl theorem](#) for 3 component function:

$$\lim_{L \rightarrow \infty} \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s \neq (s_1, s_2, 1)} e^{-l\beta\varepsilon_s} \times \\ \times \bar{\phi}_{s,\Lambda}(\tilde{x}) \phi_{s,\Lambda}(\tilde{y}) = \sum_{l=1}^{\infty} \frac{1}{l\lambda_\beta^3} e^{-\pi\|\tilde{x}-\tilde{y}\|^2/l\lambda_\beta^2}.$$

6.6 Since $\Delta_L(\beta, \rho_c(\beta) < \rho < \rho_m(\beta)) \rightarrow 0, L \rightarrow \infty$, the change $l \rightarrow l \Delta_L(\beta, \rho)$ in **6.4** gives the integral Darboux-Riemann sum, where $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2$ is **scaled** as $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2 \Delta_L(\beta, \rho)$.

6.7 Definition 6.1 The **coherence length** L_{ch} in direction perpendicular to x_3 is $L_{ch}(\beta, \rho)/L := \Delta_L^{-1/2}(\beta, \rho)$. A similar argument is valid for $\rho > \rho_m(\beta)$ with obvious modifications due to BEC for $s = (1, 1, 1)$ and adapted asymptotics for $\Delta_L(\beta, \rho)$.

6.7 To compare $L_{ch}(\beta, \rho)$ with the scale $L_{1,2} = Le^{\alpha L}$, we define the **critical exponent** $\gamma(T, \rho)$ such that

$$\lim_{L \rightarrow \infty} (L_{ch}(\beta, \rho)/L)(L_1/L)^{-\gamma(T, \rho)} = 1$$

Then

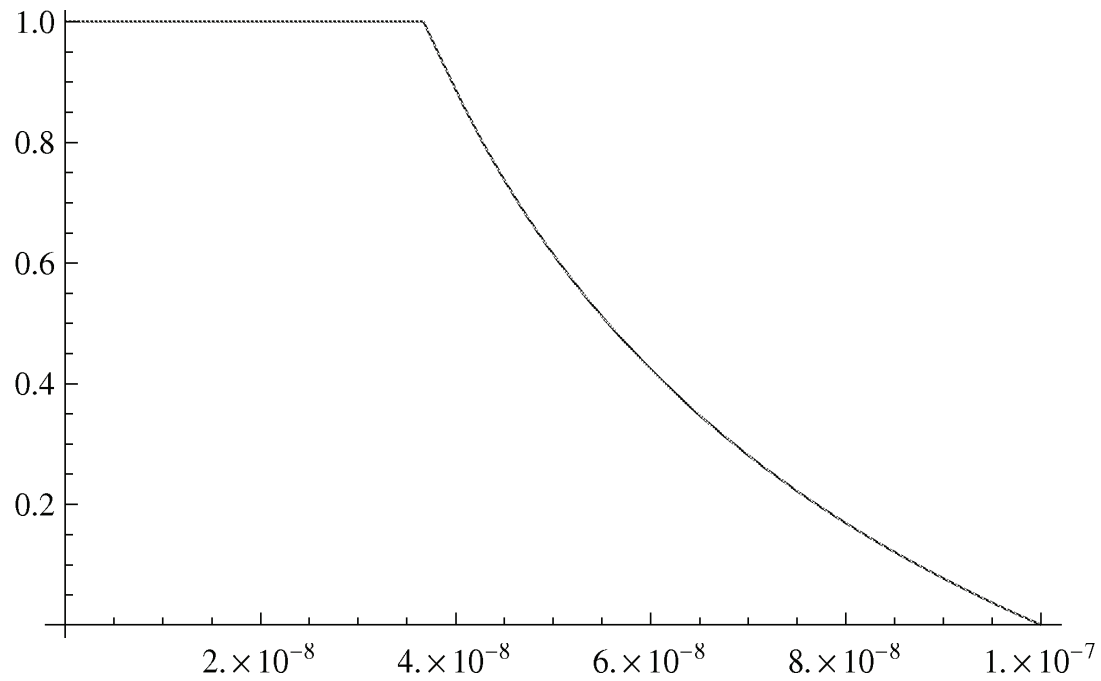
$$\begin{aligned} \gamma(T, \rho) &= \lambda_\beta^2 (\rho - \rho_c(\beta))/2\alpha, \quad \rho_c(\beta) < \rho < \rho_m(\beta) \\ &= \lambda_\beta^2 (\rho_m(\beta) - \rho_c(\beta))/2\alpha, \quad \rho_m(\beta) \leq \rho. \end{aligned}$$

For a fixed density, taking into account temperature dependence of condensates we find the temperature dependence of the exponent $\gamma(T) := \gamma(T, \rho)$, see Fig:

$$\begin{aligned} \gamma(T) &= \sqrt{T/\tau} \{(T_c/T)^{3/2} - 1\}, \quad T_m < T < T_c, \\ &= 1, \quad T \leq T_m. \end{aligned} \tag{2}$$

6.8 Notice that in the both cases the ODLRO kernel is anisotropic due to the type III condensation in the states $s = (s_1, s_2, 1)$, whereas the other states give a symmetric part of correlations, which includes a constant term $\rho_c(\beta)$.

6.9 Numerically, for $L_1 = L_2 = 100\mu m$, $L_3 = 1\mu m$ and $T_m < T = 0.75T_c$ the coherence length of the condensate is equal to $2.8\mu m \ll 100\mu m$. This decreasing of the coherence length for $T_c < T < T_m$ is experimentally observed (2003).



Thank you for your attention !