Mini-course on algorithmic aspects of stochastic games and related models

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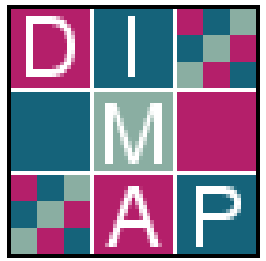
Oct. 31 – Nov. 2, 2011

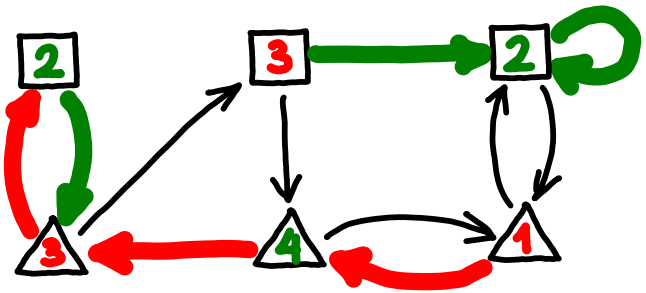
Day 2
Tuesday, November 1

Marcin Jurdziński

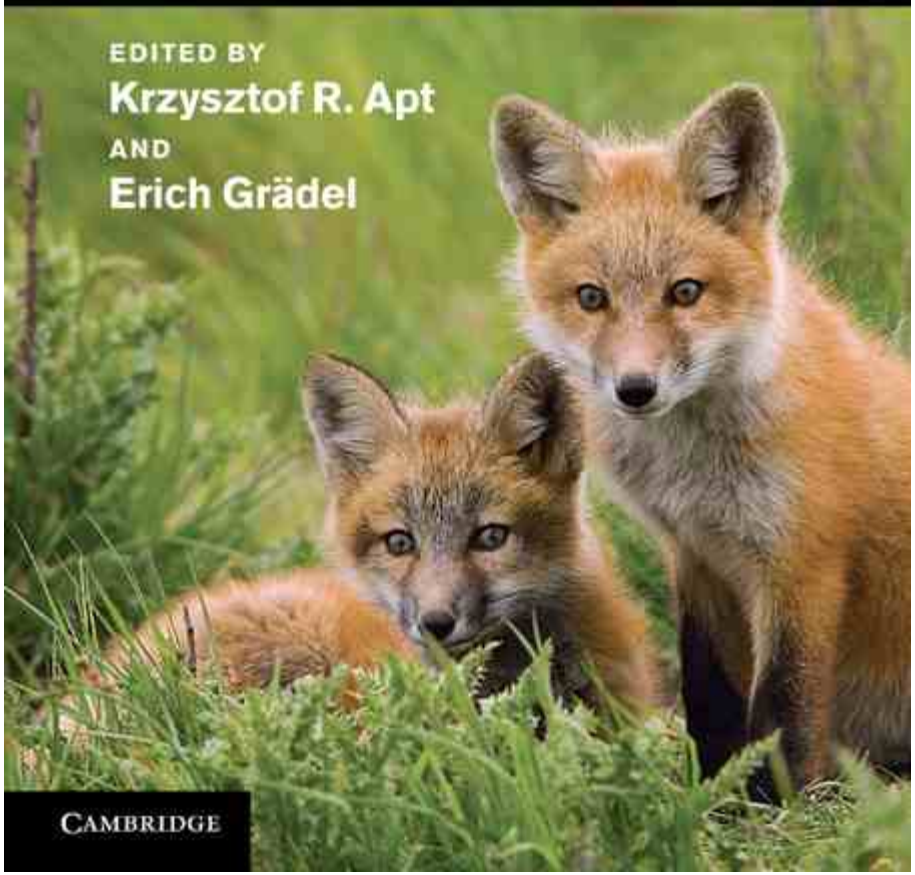
(University of Warwick)



Parity Games



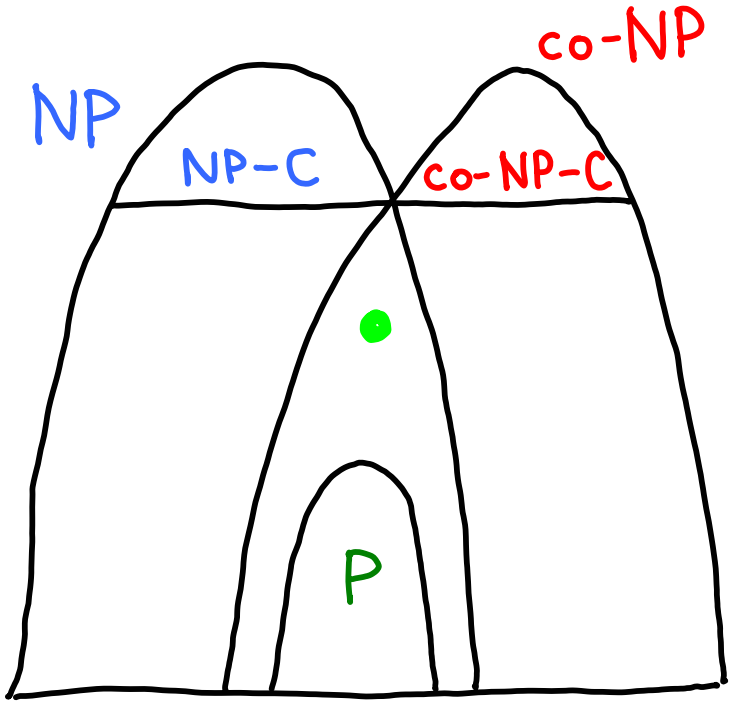
Lectures in Game Theory for Computer Scientists



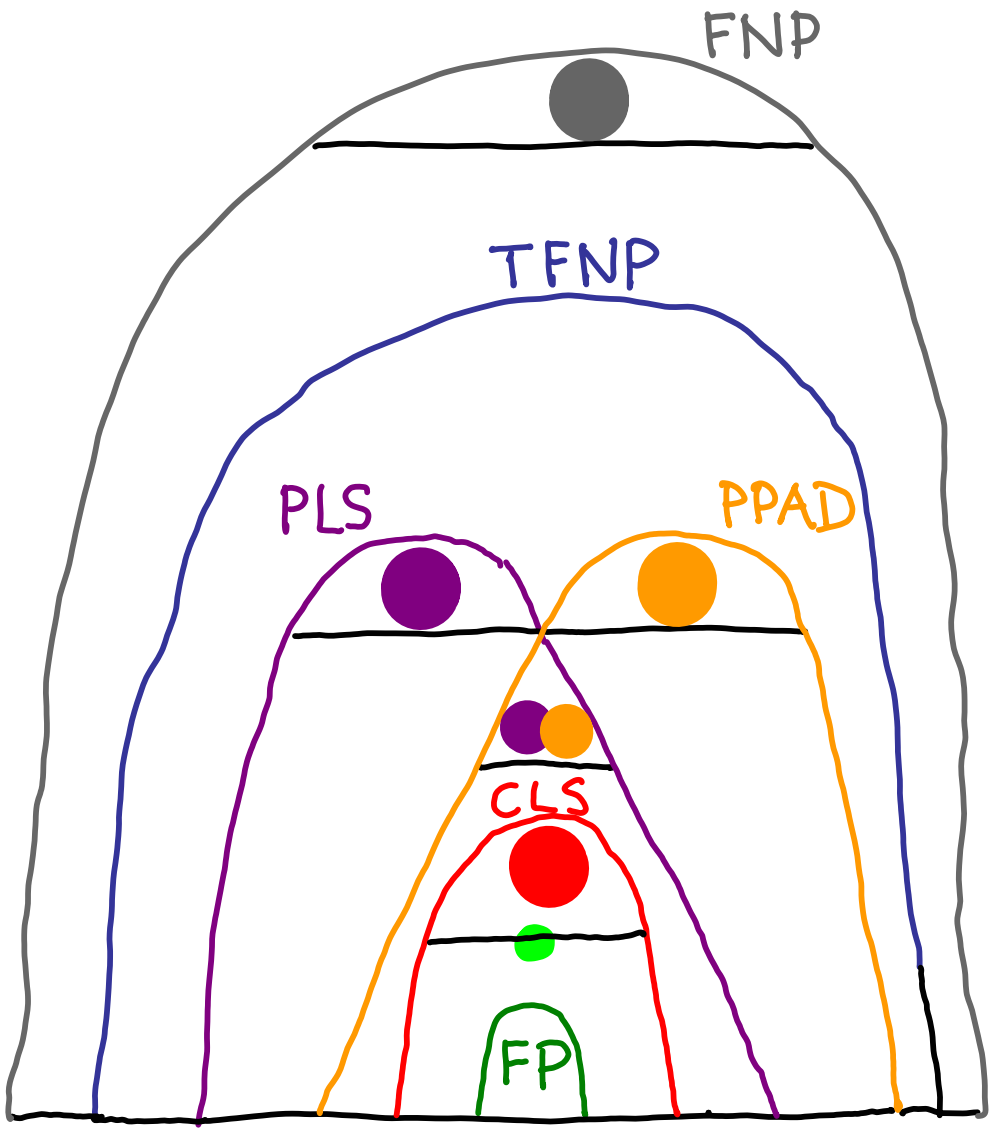
EDITED BY
Krzysztof R. Apt
AND
Erich Grädel

CAMBRIDGE

How difficult is it to find a winning strategy?



COMPUTING OPTIMAL STRATEGIES



PLAN

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. Parity games
6. Better algorithms for parity games
7. Search complexity of computing optimal strategies

Boolean formula evaluation

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$$s \stackrel{?}{\models} \varphi$$

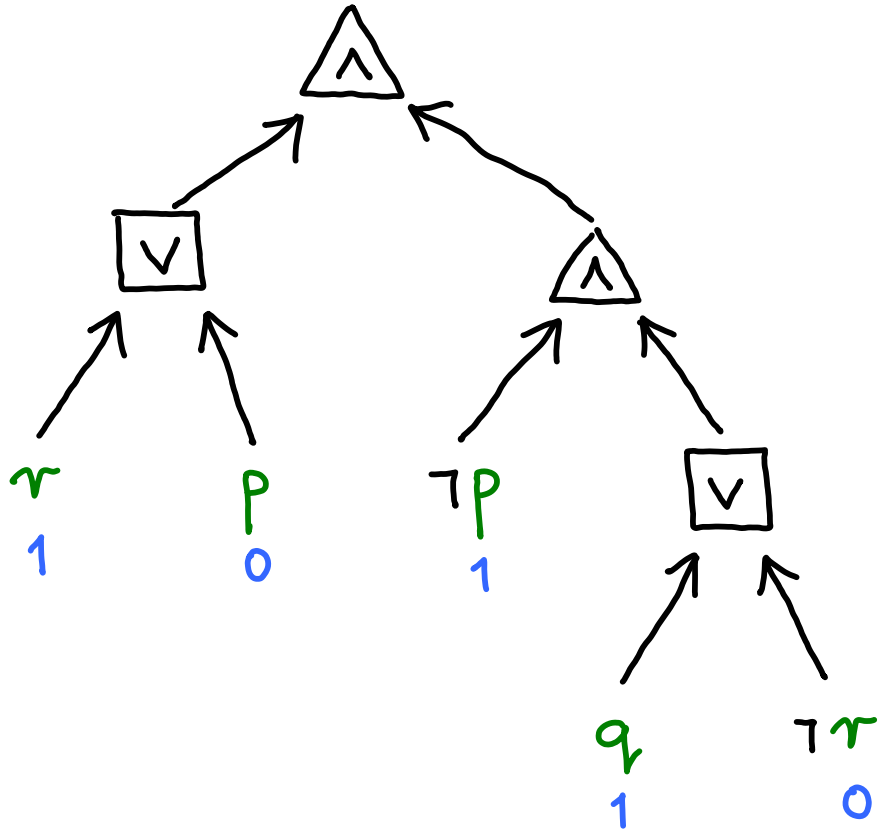
Boolean formula evaluation

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$C_{s,\varphi}$:



Boolean formula evaluation

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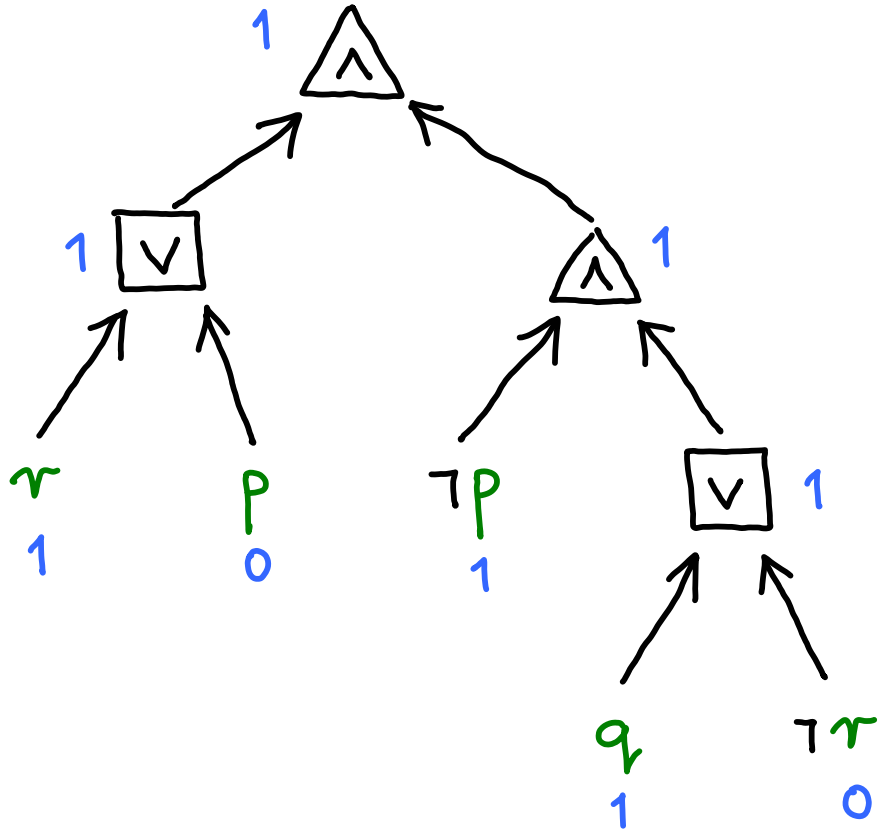
$\mathcal{C}_{s,\varphi}$:



Fact

$s \models \varphi$

iff

the value of circuit $\mathcal{C}_{s,\varphi}$ is 1

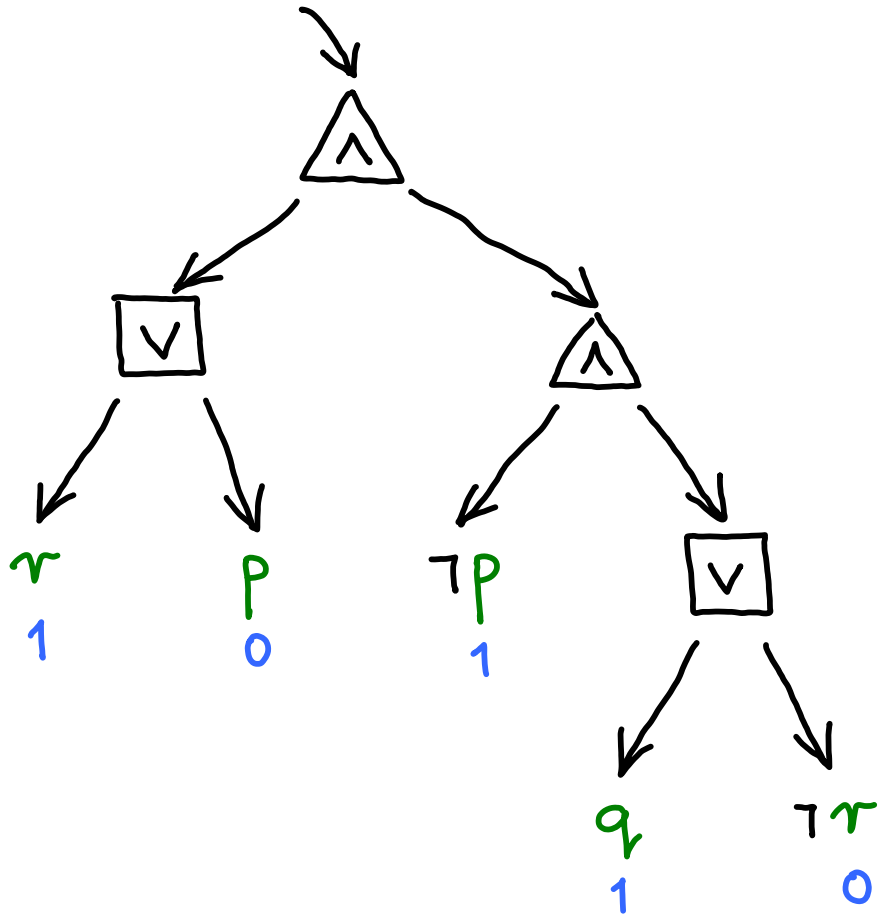
Boolean formula evaluation: Game

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$\mathcal{G}_{s,\varphi}$:



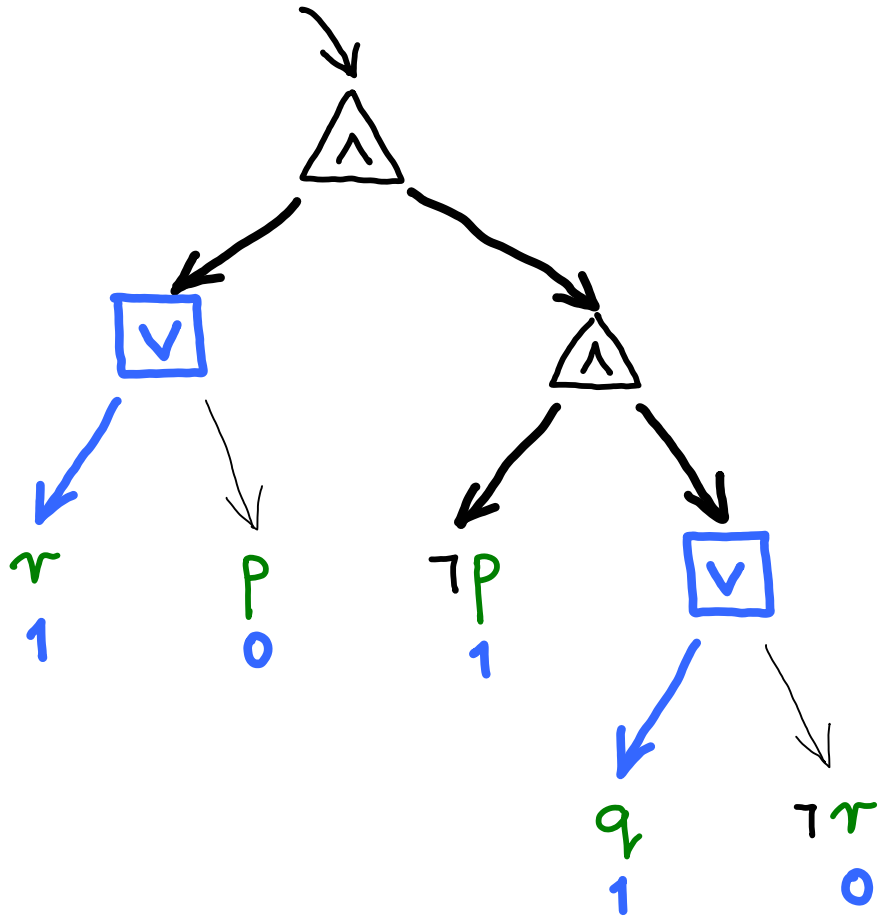
Boolean formula evaluation: Game

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$\mathcal{G}_{s,\varphi}$:



Fact

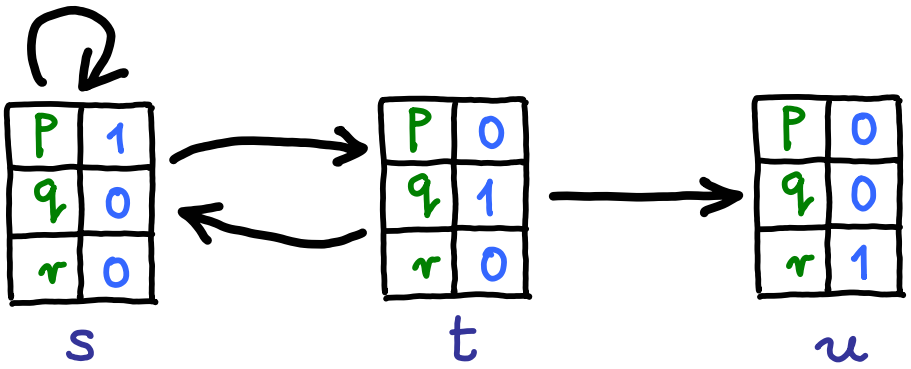
$s \models \varphi$

iff

∨ has a **strategy** to reach 1 in $\mathcal{G}_{s,\varphi}$

Temporal formula evaluation

(Model of a) program :



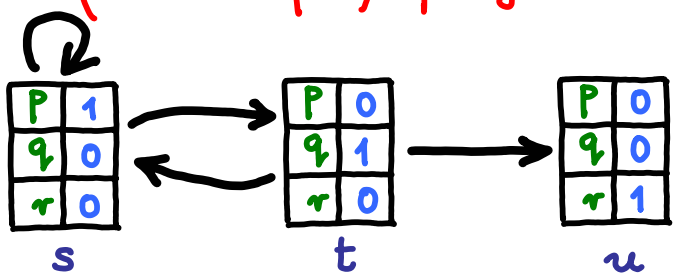
CTL property φ :

$$E((p \vee q) \, U \, r)$$

$$s \stackrel{?}{\models} \varphi$$

Temporal formula evaluation: Game

(Model of a) program:



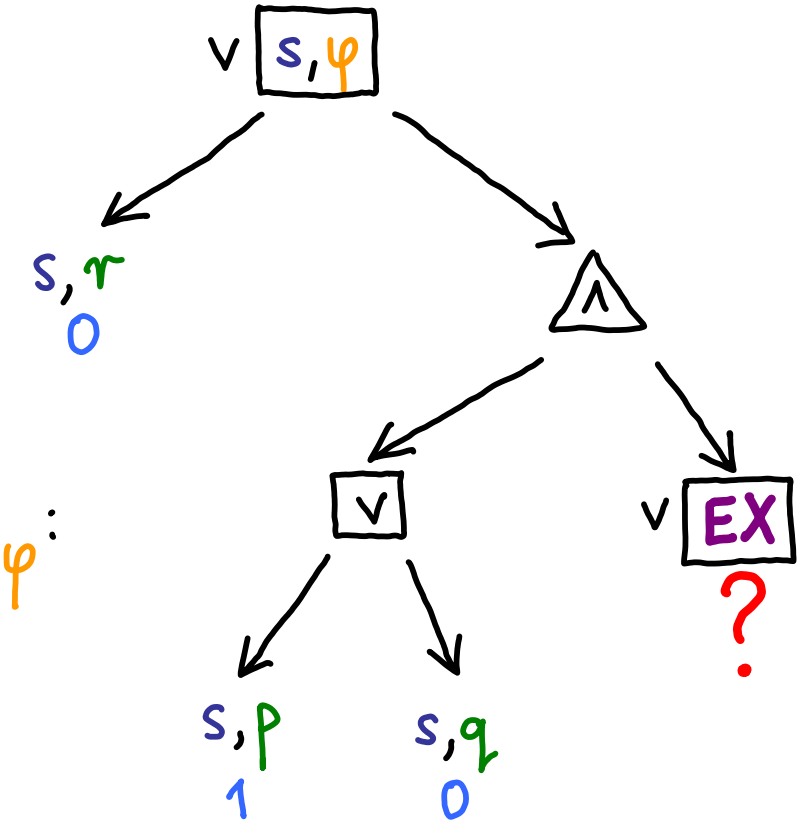
CTL property φ:

φ: $E((p \vee q) U r)$

$\equiv$

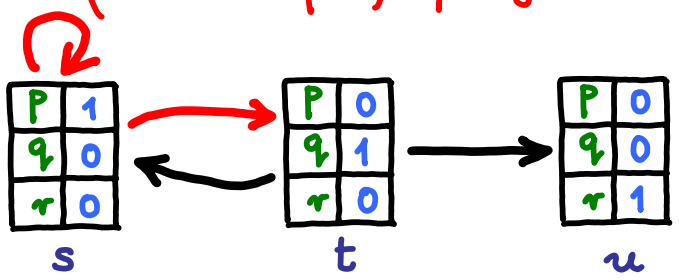
$r \vee ((p \vee q) \wedge EX\varphi)$

$G_{s,\varphi}$:



Temporal formula evaluation: Game

(Model of a) program:



CTL property φ:

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$\Vert\!\Vert$

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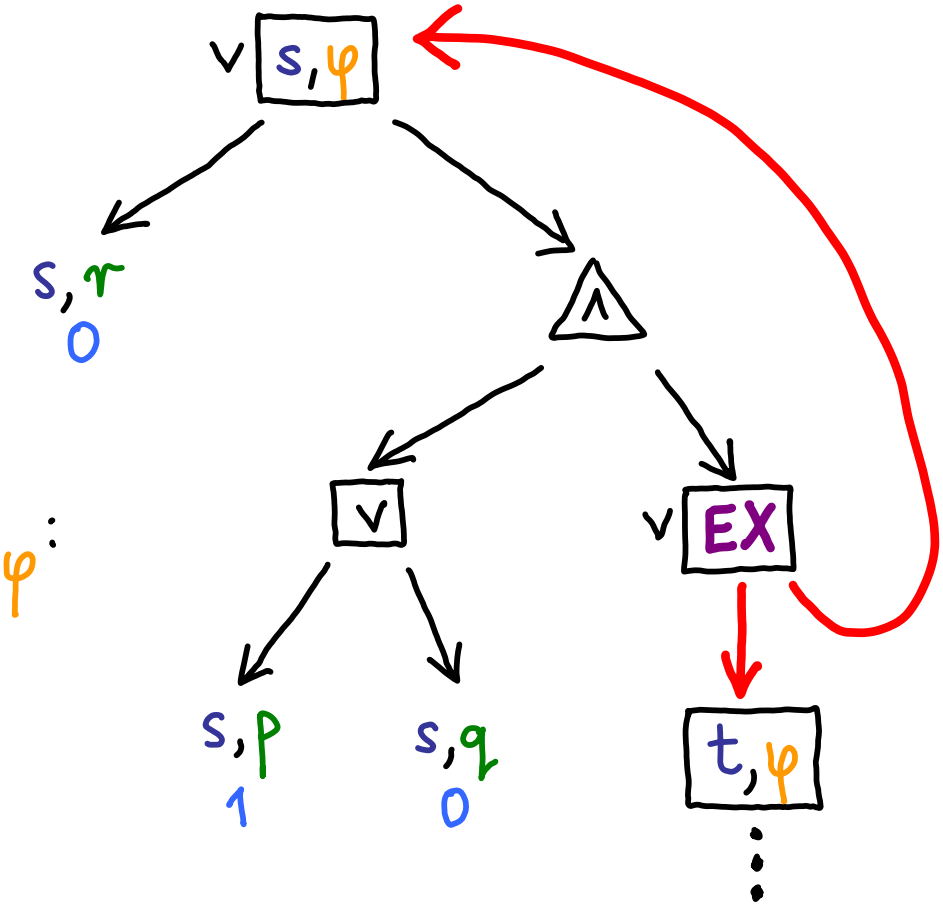
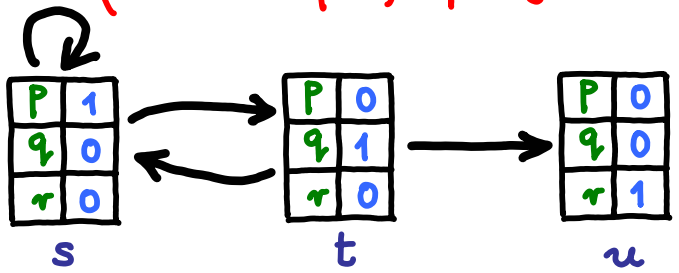
$\mathcal{G}_{s,\varphi}$:



Temporal formula evaluation: Game

(Model of a) program :

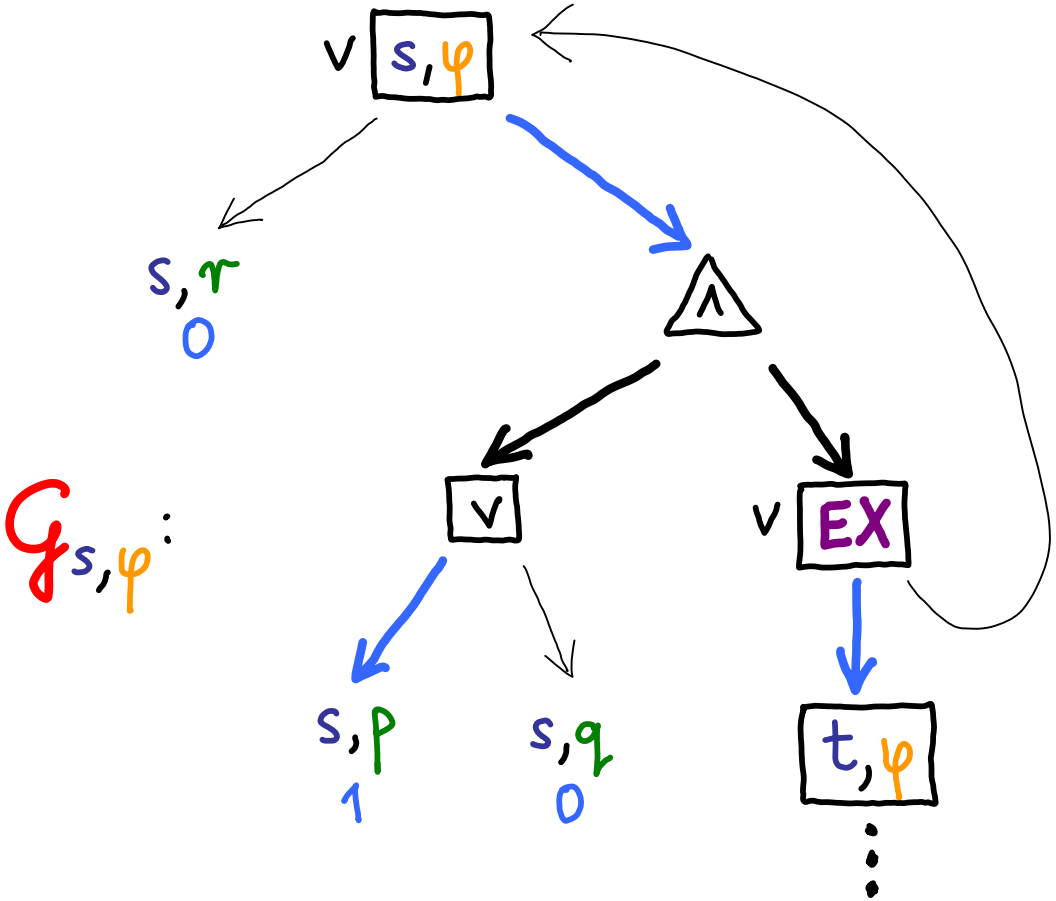


CTL property φ:

φ : $E((p \vee q) U r)$

$\equiv$

$r \vee ((p \vee q) \wedge EX\varphi)$



Thm

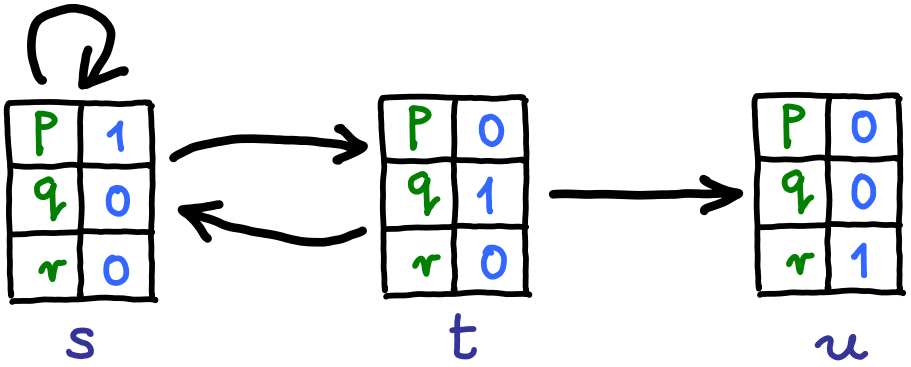
$s \models \varphi$

iff

∨ has a strategy to reach 1 in $G_{s,\varphi}$

Fixpoint formula evaluation

(Model of a) program :



μ-calculus property:

φ : $\boxed{\mu Z . \psi(Z)}$

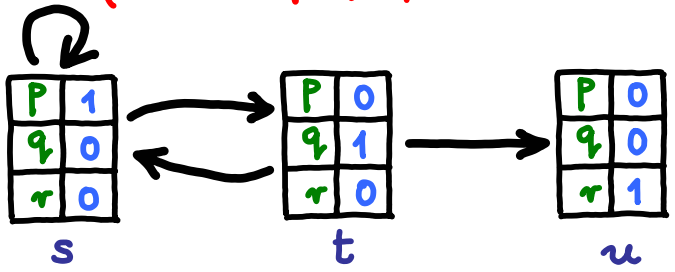
$\equiv$

$\psi(\varphi)$

$$s \stackrel{?}{\models} \varphi$$

Fixpoint formula evaluation

(Model of a) program:

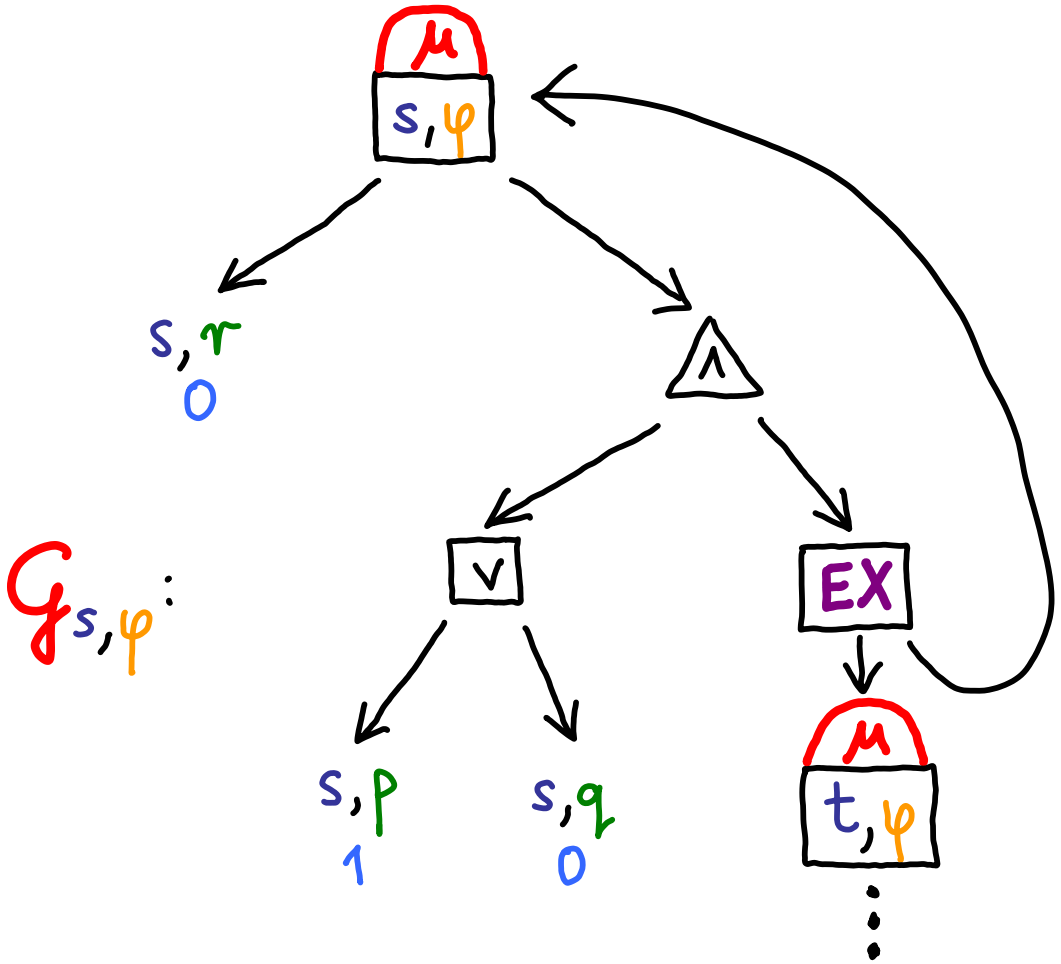


μ-calculus property φ:

$\varphi: \boxed{\mu Z.\, \psi(Z)}$

$\equiv$

$\psi(\varphi)$



Thm

$s \models \varphi$

iff

∨ has a **strategy** in $G_{s,\varphi}$

- to reach 1, or
- outermost infinitely occuring fixpoint subformula is **not** a μ (i.e., least fixpoint)

PLAN

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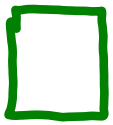
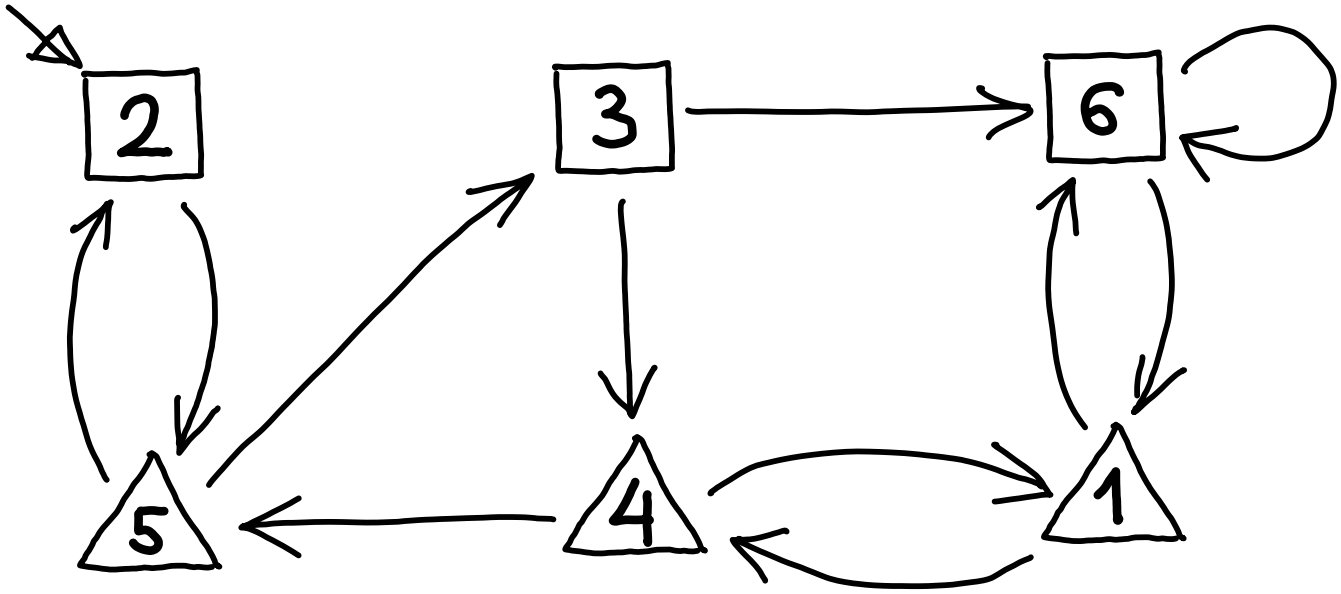
GAMES ON GRAPHS: GAME GRAPH

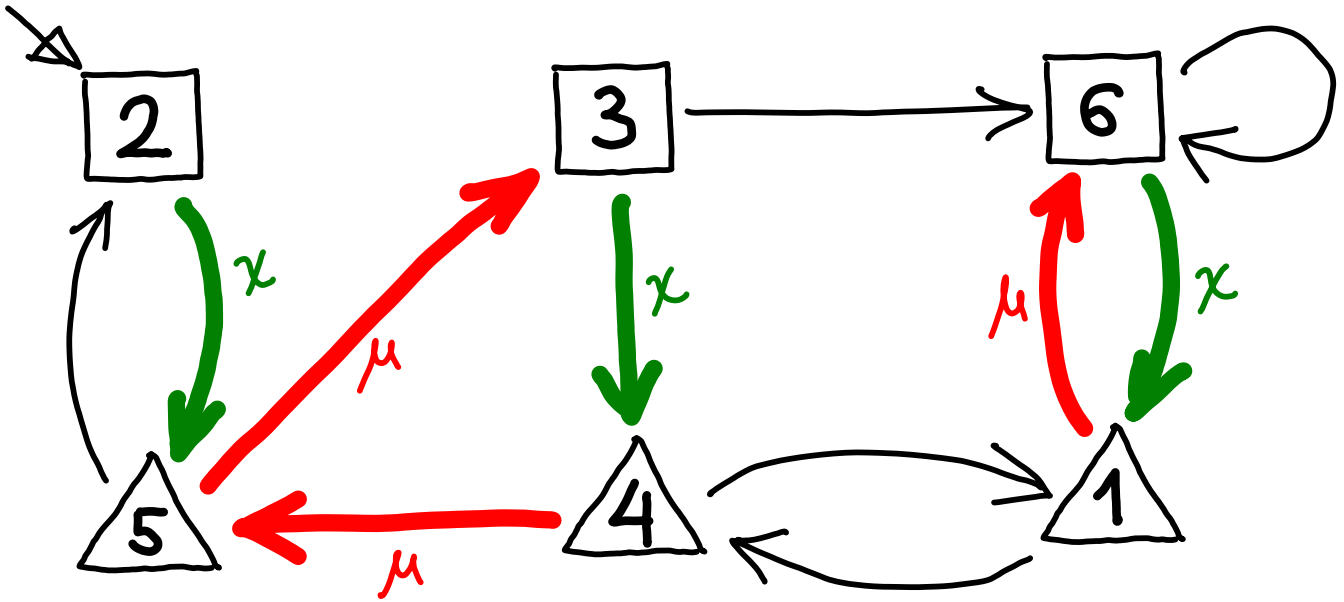
Players:



$n = |V|$ vertices, $m = |E|$ edges



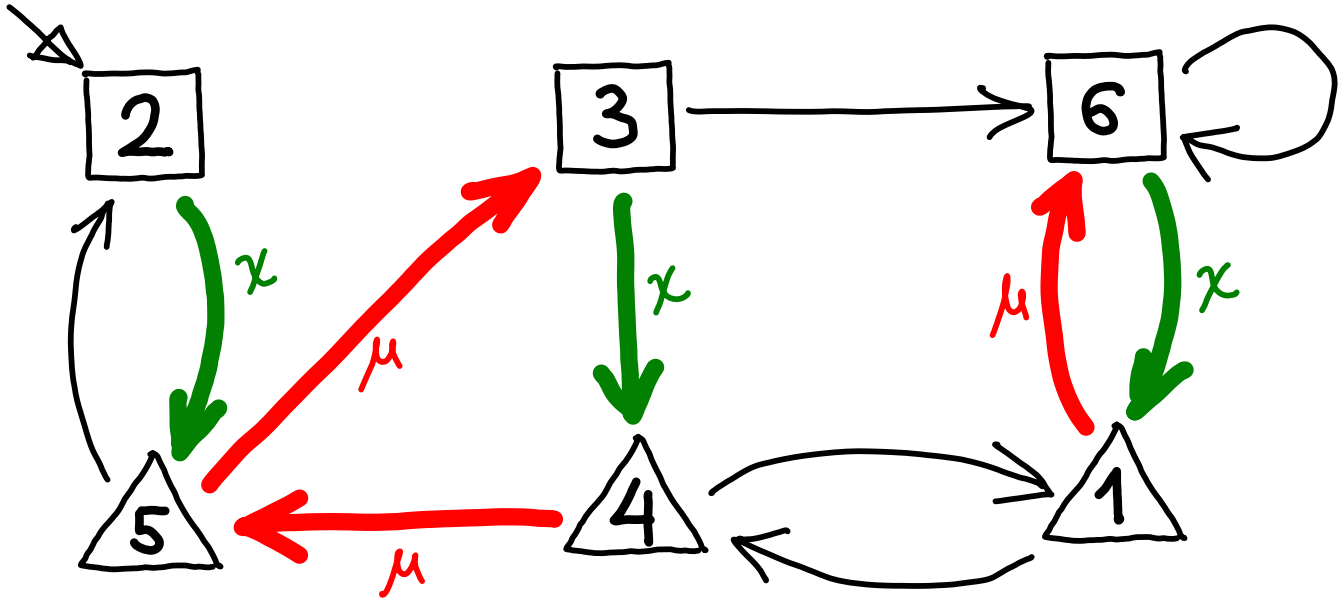
Games on graphs: positional strategies



Positional **strategy** for □: $\chi : V_{\square} \to V$

Positional **strategy** for △: $\mu : V_{\triangle} \to V$

Games on graphs: plays



$$\mathrm{Play}(2,\chi,\mu) = (2,5,3,4,5,(3,4,5)^{\omega}) \in V^{\omega}$$

GAMES ON GRAPHS: OBJECTIVES

Objectives: $W_\Box \subseteq V^\omega$

$W_\Delta = V^\omega \setminus W_\Box$

GAMES ON GRAPHS: OBJECTIVES

Objectives: $W_\square \subseteq V^\omega$

$W_\Delta = V^\omega \setminus W_\square$

Examples of objectives:

- Reachability: $W_\square = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$
- Safety: $W_\square = \{(v_0, v_1, v_2, \ldots) : v_i \in S \text{ for all } i \in \mathbb{N}\}$

where $T, S \subseteq V$

Games on graphs: winning strategies

$\chi \in \Sigma_{\square}$ is a **winning strategy** for $\square$ from $v \in V$,

if $\boxed{Play(v, \chi, \mu) \in W_{\square}, \quad \text{for all } \mu \in \Sigma_{\Delta}}$

$\mu \in \Sigma_{\Delta}$ is a **winning strategy** for Δ from $v \in V$,

if $\boxed{Play(v, \chi, \mu) \in W_{\Delta}, \quad \text{for all } \chi \in \Sigma_{\square}}$

GAMES ON GRAPHS: ALGORITHMIC PROBLEM

Given: game graph $(V = V_\Box \uplus V_\triangle, E)$

starting vertex $v \in V$

objective $W_\Box \subseteq V^\omega$

answer: if $\Box$ has a winning strategy from v

Games on graphs: Determinacy

A game $(V = V_\square \uplus V_\triangle, E; W_\square)$ is **determined**

if for every starting vertex $v \in V$

- $\square$ has a winning strategy from v, **or**
- $\triangle$ has a winning strategy from v

MARTIN'S DETERMINACY THEOREM

THM [MARTIN 1975]

Every game $\left(V = V_{\square} \uplus V_{\triangle}, E ; W_{\square}\right)$,

such that $W_{\square} \subseteq V^{\omega}$ is a Borel set,

is determined

GAMES ON GRAPHS: ALGORITHMIC PROBLEM

Given: game graph $(V = V_\square \uplus V_\triangle, E)$

objective $W_\square \subseteq V^\omega$

compute: $V = W_\square \uplus W_\triangle$ such that

- $\square$ has a winning strategy from $W_\square$, and
- $\triangle$ has a winning strategy from $W_\triangle$

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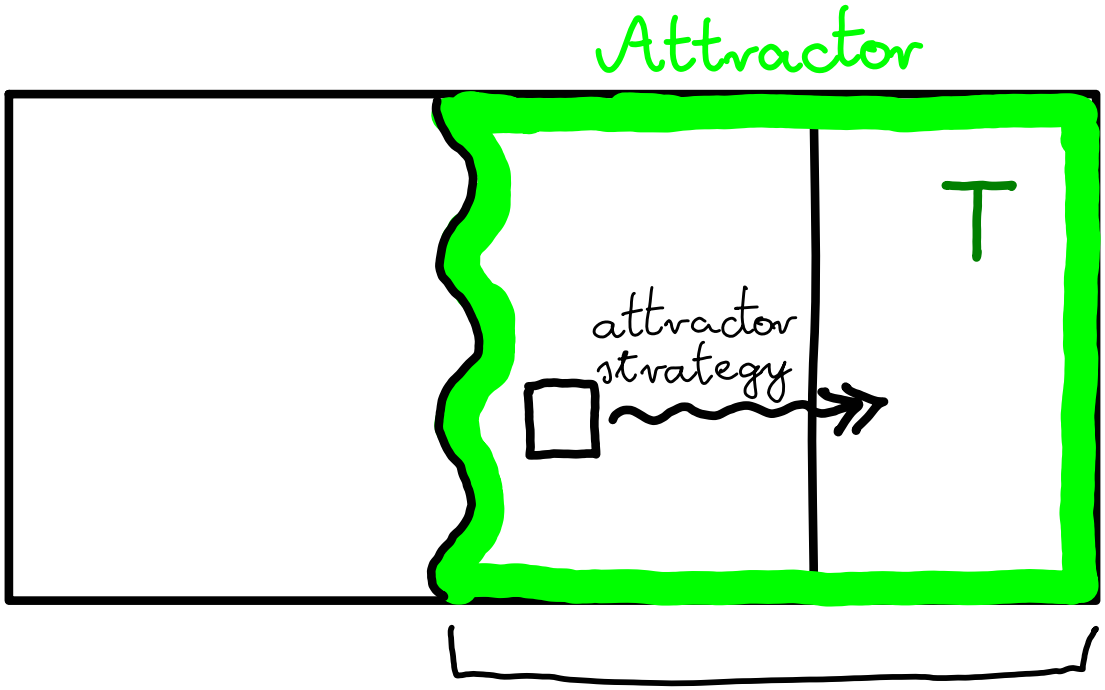
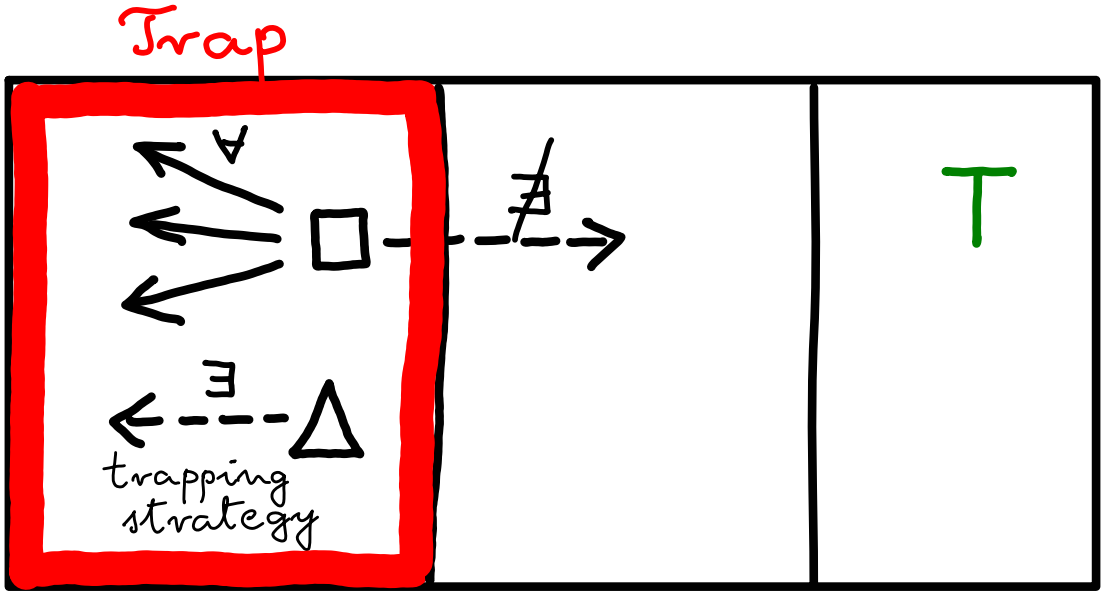
Reachability games



$W_\square = attr_\square(T)$

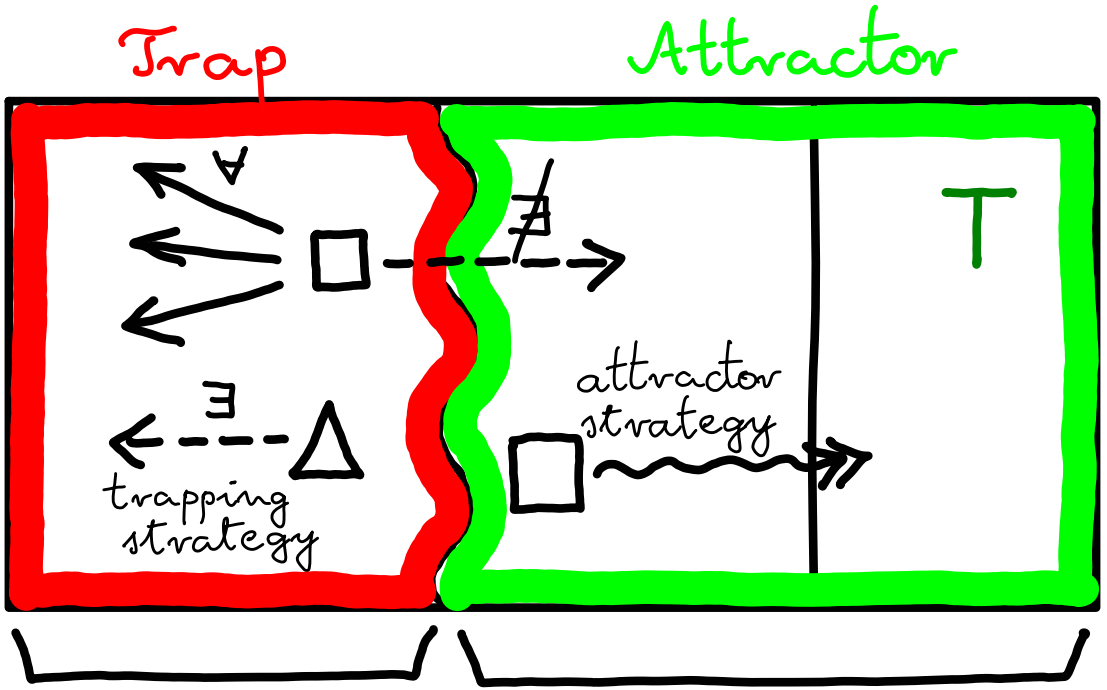
- **Reachability**: $W_\square = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$

Safety games



- **Reachability**: $W_{\square} = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$
- **Safety**: $W_{\Delta} = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for no } i \in \mathbb{N}\}$

Reachability/Safety games



$W_\Delta = V \setminus attr_\square(T)$ $W_\square = attr_\square(T)$

Fact

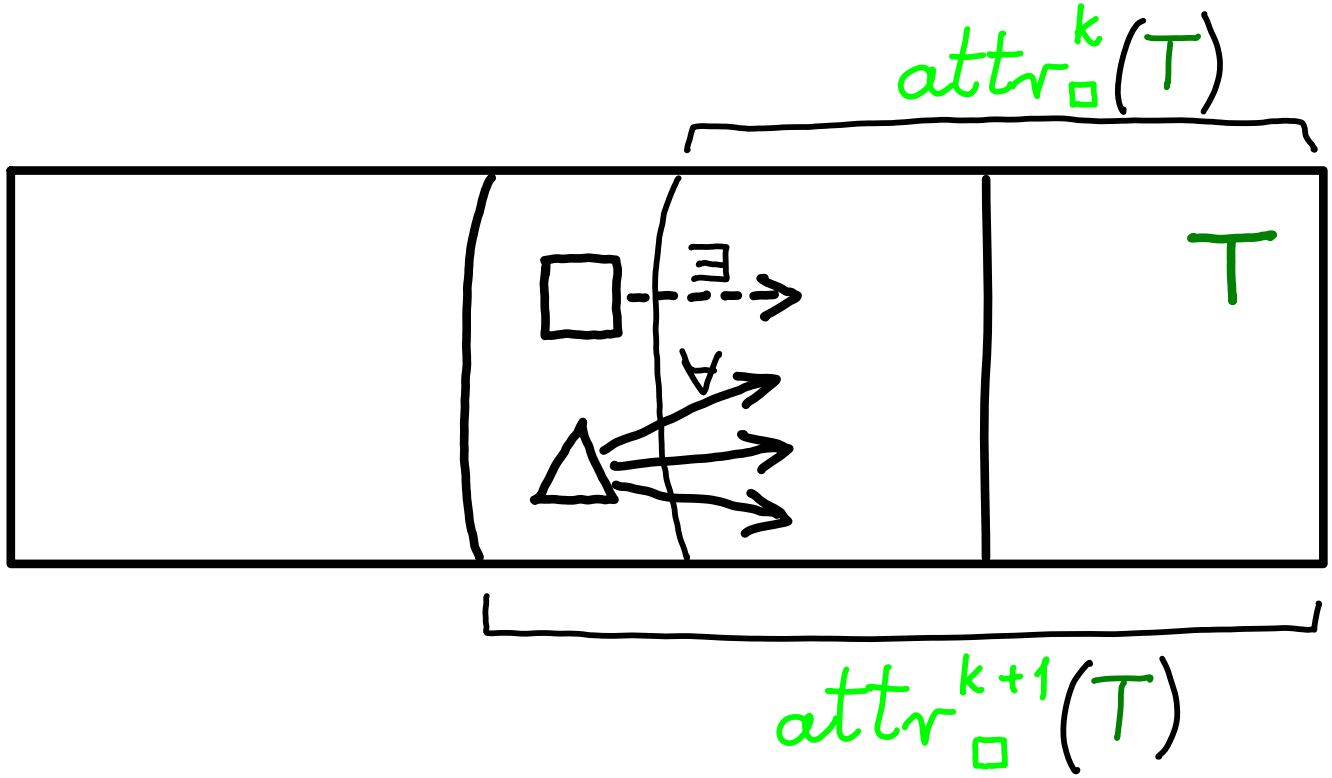
Reachability/safety games are **positionally determined**

Exercise 1

Explain in detail
why the complement of an attractor
is a trap for the other player,
and how to compute
positional attractor and trapping strategies.

Conclude that reachability/safety games
are positionally determined.

Solving Reachability/Safety games: Computing Attractors



Fact $attr_{\square}(T) = \bigcup_{k \geq 0} attr_{\square}^{k}(T)$

can be computed in $O(m)$ time

Exercise 2

Discuss implementation details that make it possible to compute attractors in time $O(m)$.

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GAMES ON GRAPHS: (co-)BÜCHI OBJECTIVES

- **Büchi** (repeated reachability)

  An infinite play is winning if

  vertices in $T \subseteq V$ occur **infinitely often**

- **co-Büchi** (eventual safety)

  An infinite play is winning if

  vertices in $T \subseteq V$ occur **finitely often**

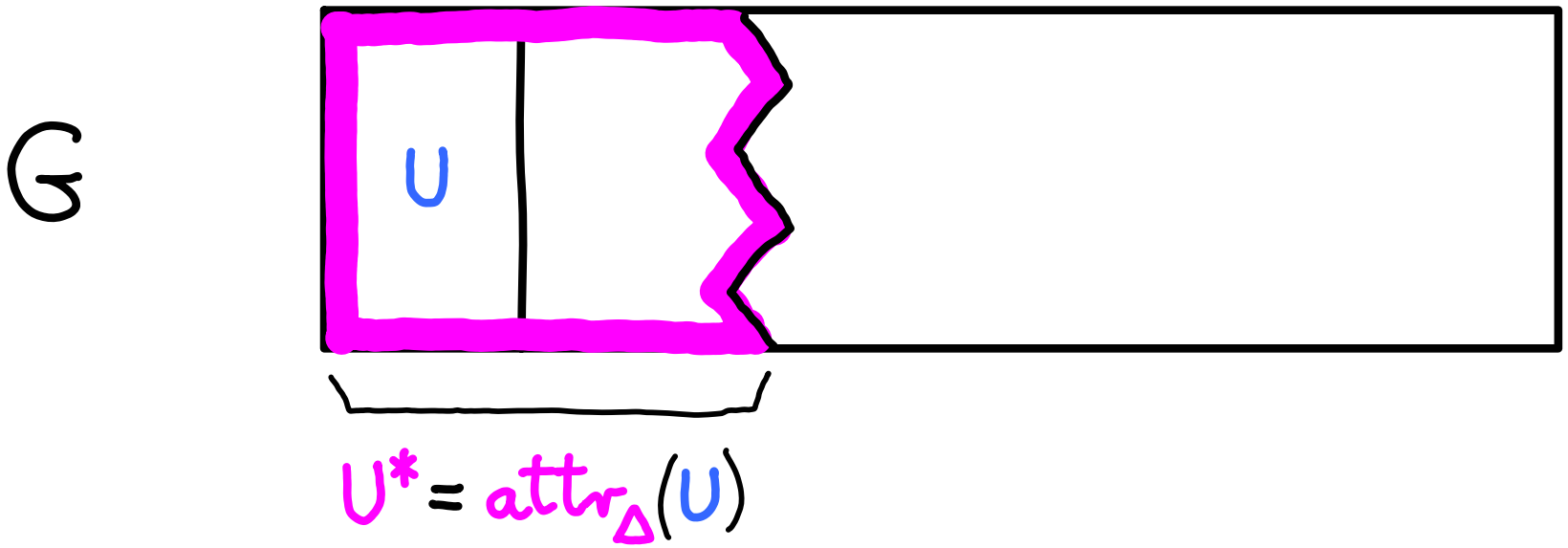
Oracle Lemma

Lemma If $U \subseteq win_{\Delta}(G)$

then $win_{\Delta}(G) = U^{*} \cup win_{\Delta}(G'')$

$win_{\square}(G) = win_{\square}(G'')$

Oracle Lemma



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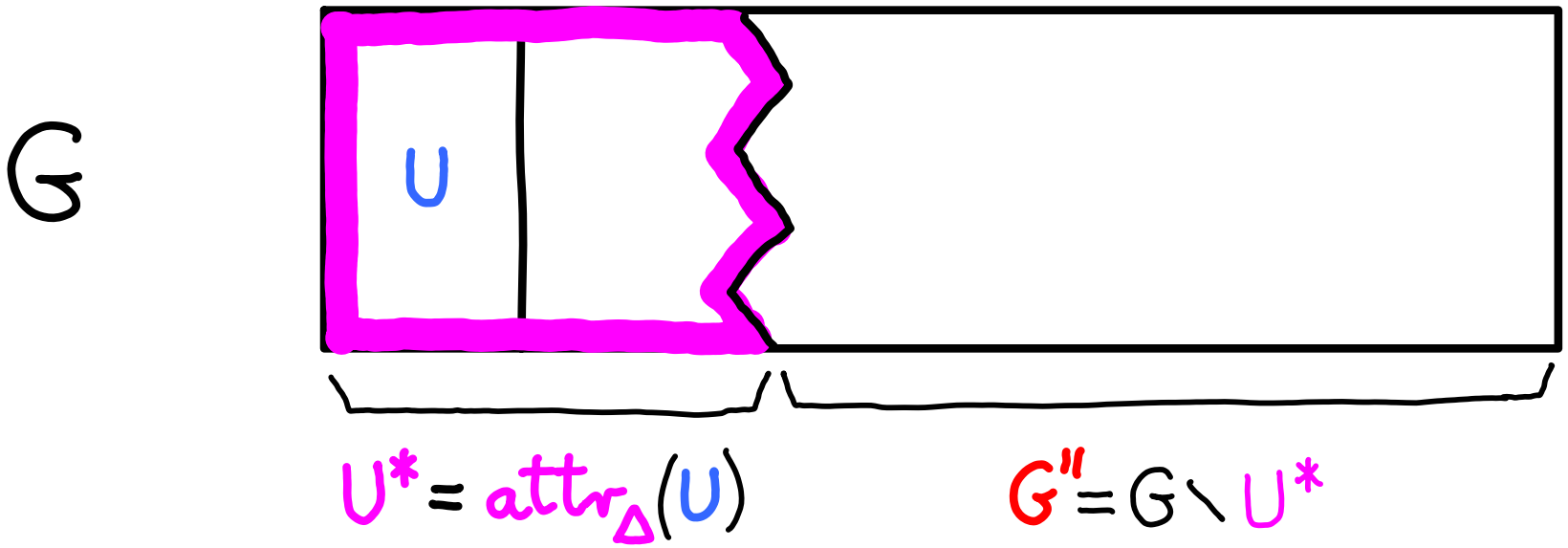
Oracle Lemma



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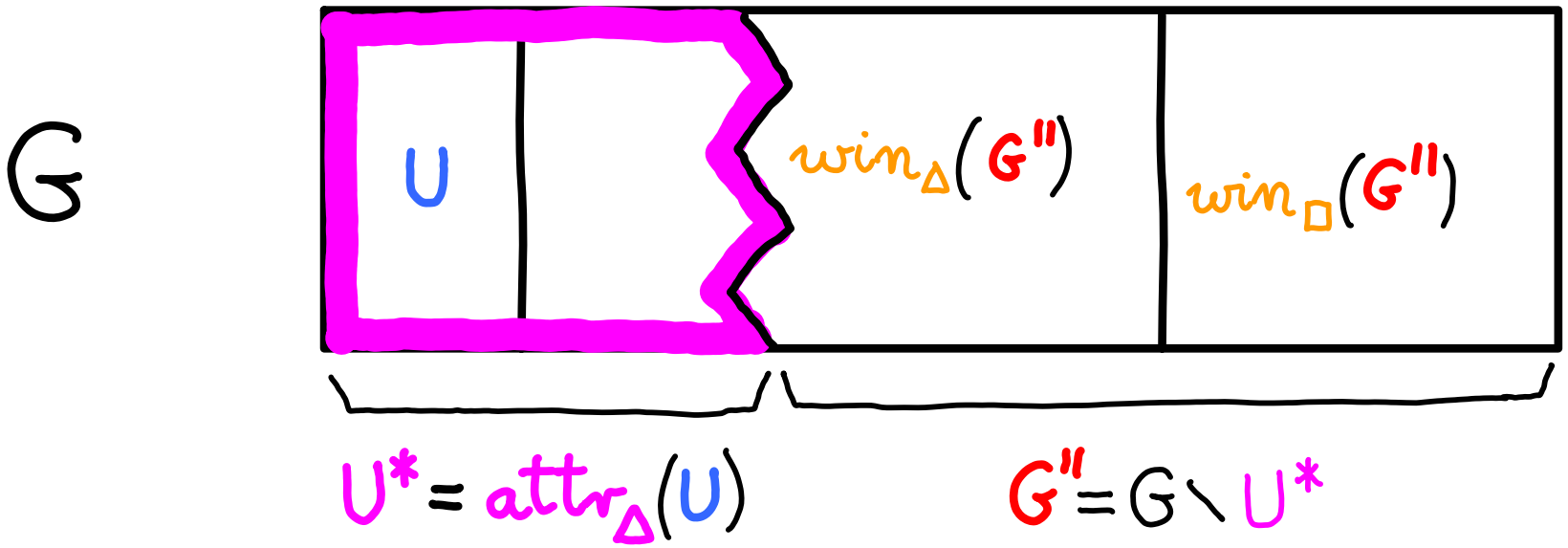
Oracle Lemma



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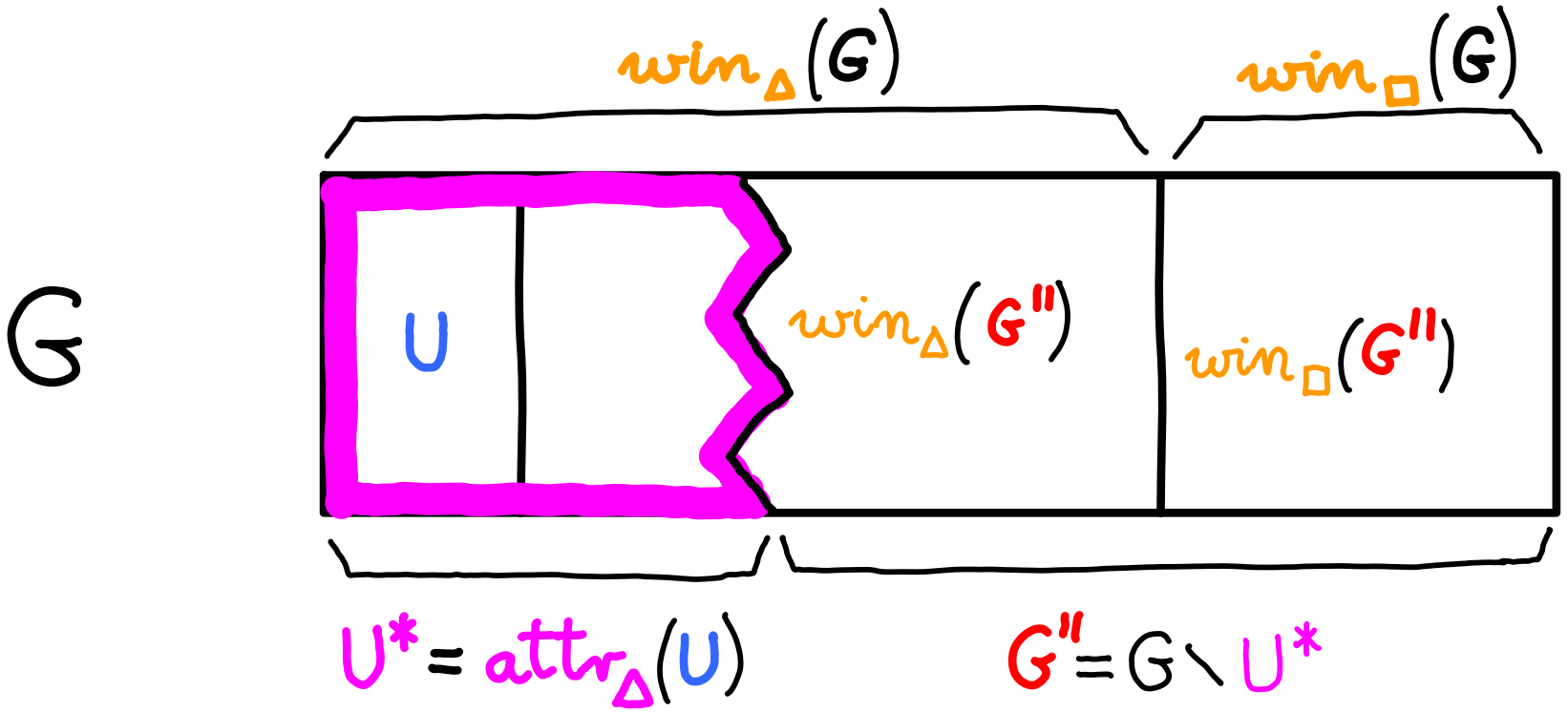
Oracle Lemma



Lemma If $U \subseteq win_\Delta(G)$

then $win_\Delta(G) = U^* \cup win_\Delta(G'')$

$win_\square(G) = win_\square(G'')$

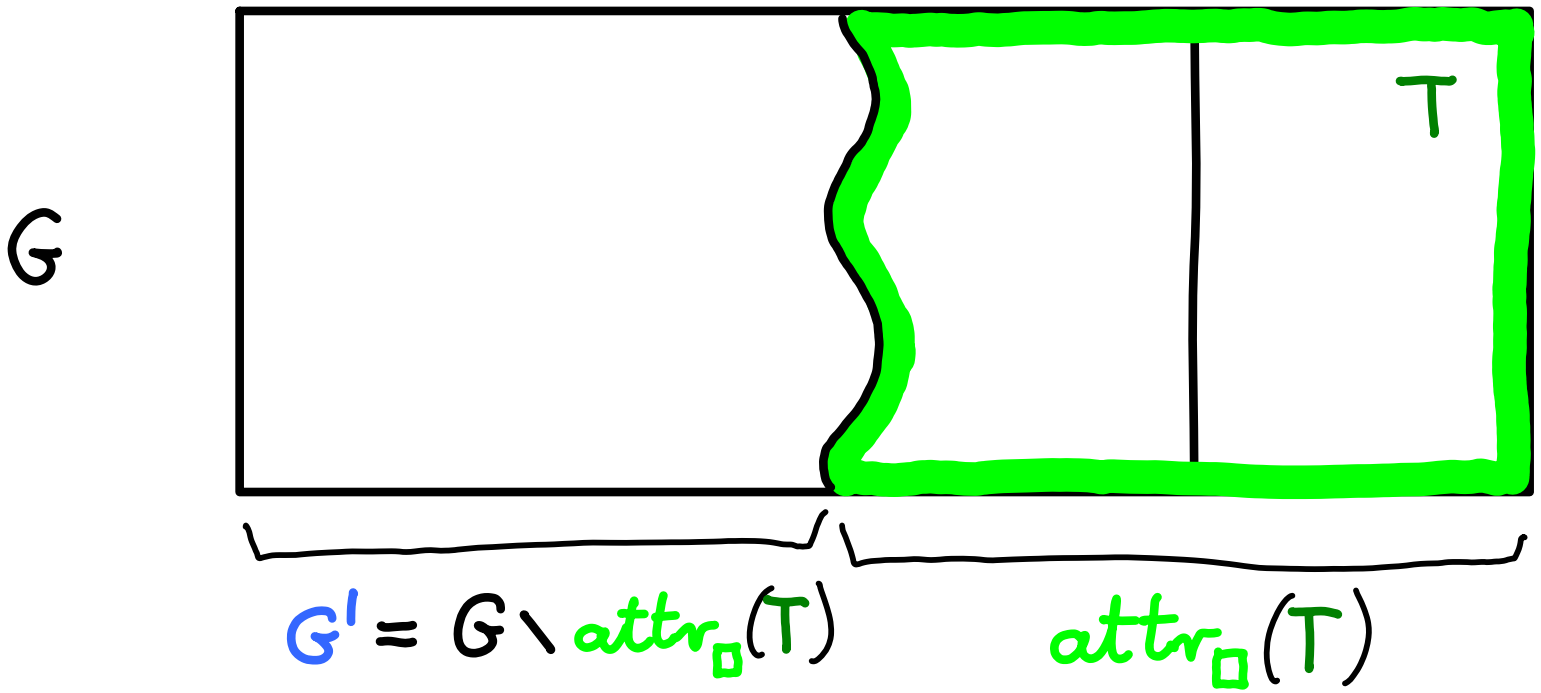
Exercise 3

Prove the oracle lemma for Büchi and co-Büchi games.

Are the winning strategies constructed positional?

If not, what assumptions do you need to make them so?

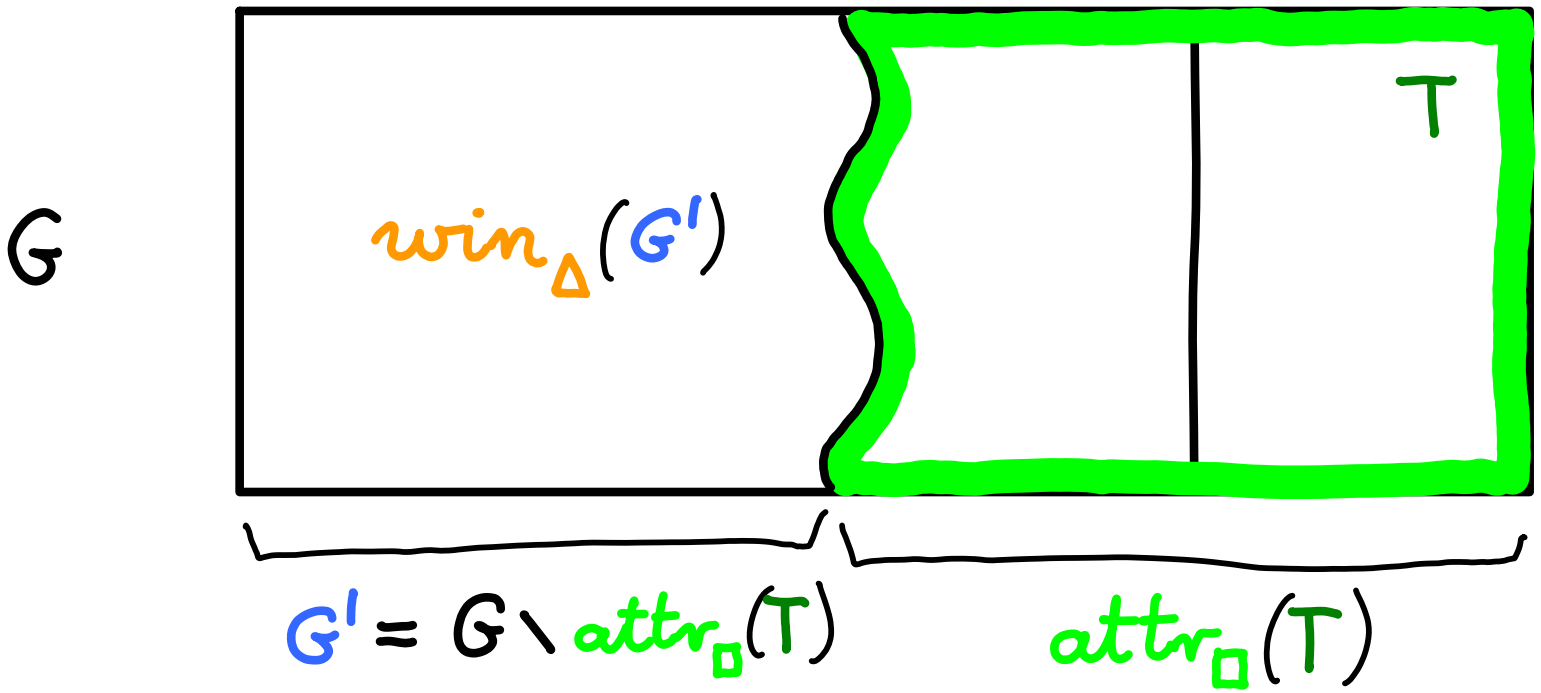
T-ATTRACTOR LEMMA



LEMMA

1. $win_{\triangle}(G') = V \setminus attr_{\square}(T) \subseteq win_{\triangle}(G)$
2. If $G' = \emptyset$ then $win_{\square}(G) = V$

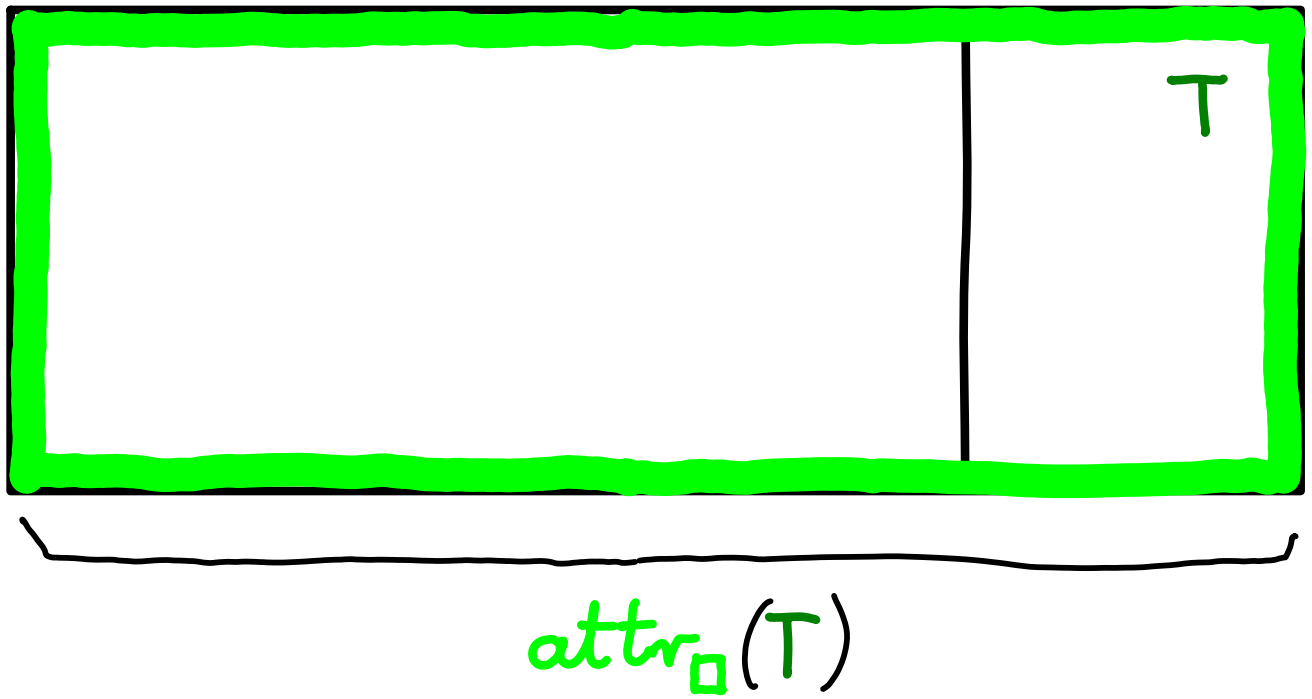
T-attractor Lemma



Lemma

1. $win_\triangle(G') = V \setminus attr_\square(T) \subseteq win_\triangle(G)$
2. If $G' = \emptyset$ then $win_\square(G) = V$

T-ATTRACTOR LEMMA

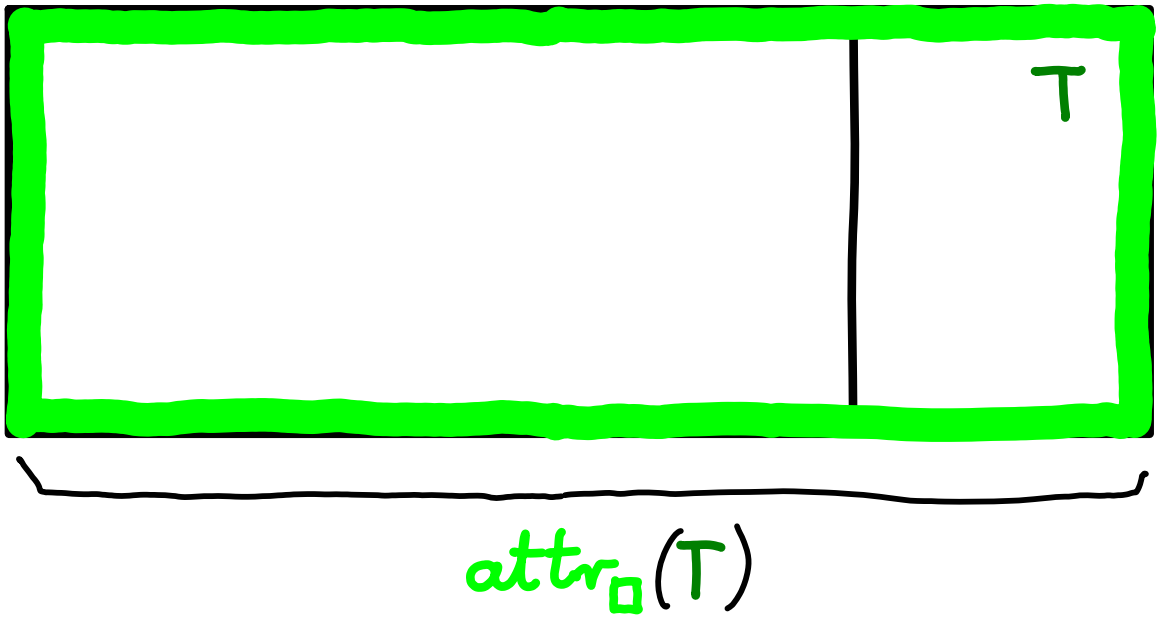


LEMMA

1. $win_{\triangle}(G') = V \setminus attr_{\square}(T) \subseteq win_{\triangle}(G)$

2. If $G' = \emptyset$ then $win_{\square}(G) = V$

A DIVIDE-AND-CONQUER ALGORITHM



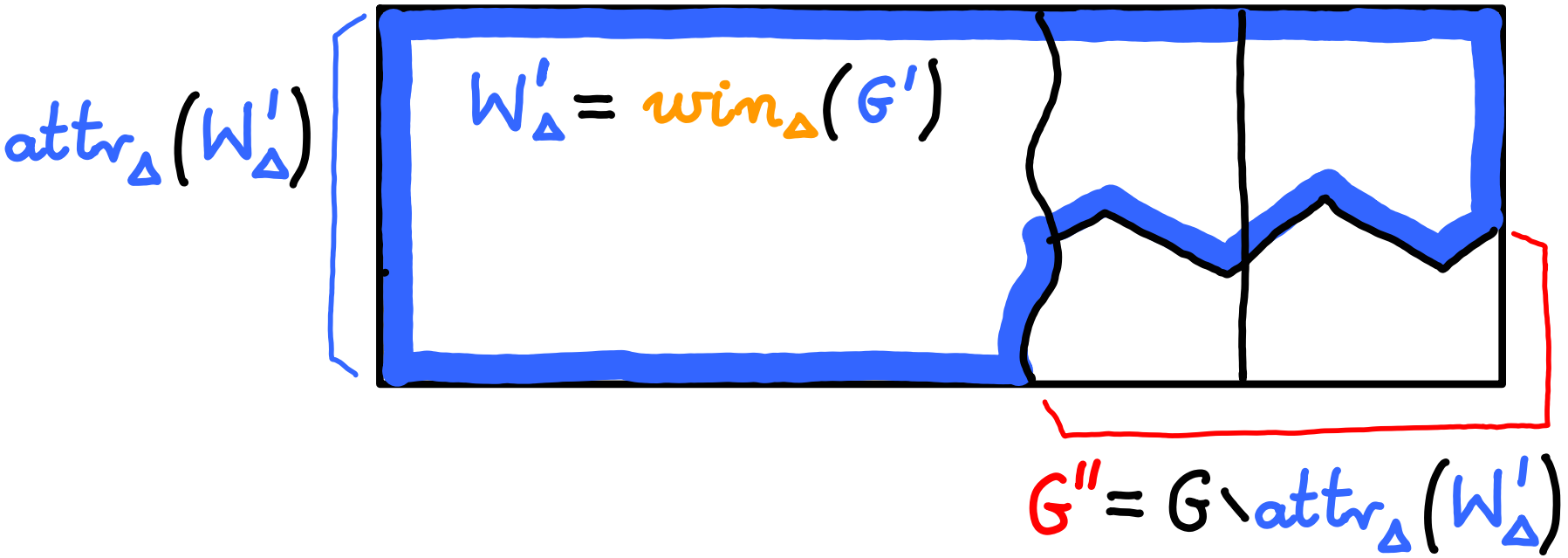
WIN(G): if $G' = \emptyset$ then $(W_{\square}, W_{\triangle}) := (V, \emptyset)$

else $(W''_{\square}, W''_{\triangle}) := \text{WIN}(G'')$

$(W_{\square}, W_{\triangle}) := \left(W''_{\square},\ attr_{\triangle}(W'_{\triangle}) \cup W''_{\triangle}\right)$

return $(W_{\square}, W_{\triangle})$

A DIVIDE-AND-CONQUER ALGORITHM



WIN(G):

if $G' = \emptyset$ then $(W_\square, W_\Delta) := (V, \emptyset)$

else $(W''_\square, W''_\Delta) := \text{WIN}(G'')$

$(W_\square, W_\Delta) := \left(W''_\square,\ attr_\Delta(W'_\Delta) \cup W''_\Delta\right)$

return $(W_\square, W_\Delta)$

POSITIONAL DETERMINACY OF (co-) BÜCHI GAMES

LEMMA

Büchi / co-Büchi games are **positionally determined**

(both on finite and **infinite graphs**)

Exercise 4

Prove the positional determinacy of Büchi and co-Büchi games.

Running time of the divide-and-conquer algorithm

$T(n)$

Win(G):

if $G' = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$

else $(W''_\square, W''_\triangle) := \textbf{Win}(G'')$

$(W_\square, W_\triangle) := \left(W''_\square, attr_\triangle(W'_\triangle) \cup W''_\triangle\right)$

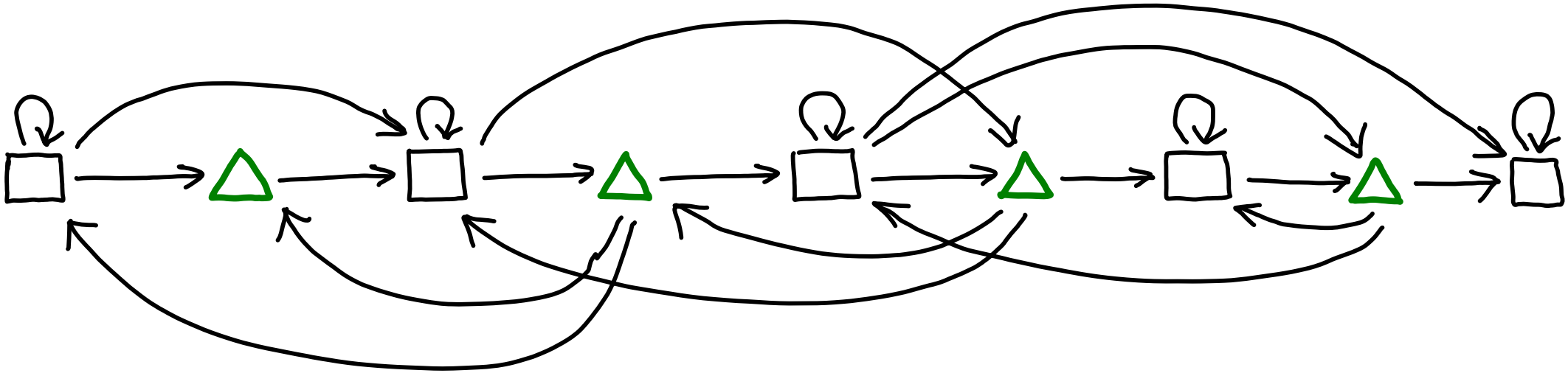
return $(W_\square, W_\triangle)$

$T(n-1)$

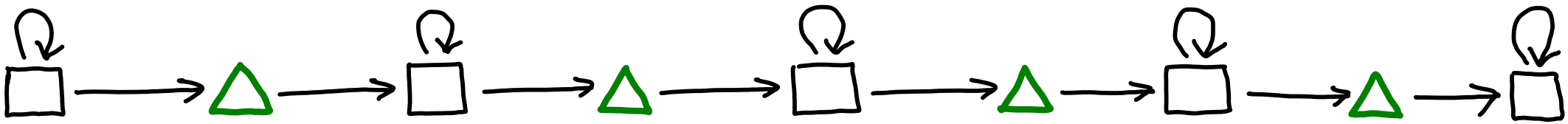
Recurrence: $T(n) \leq T(n-1) + O(m)$

Solution: $T(n) = O(n \cdot m)$

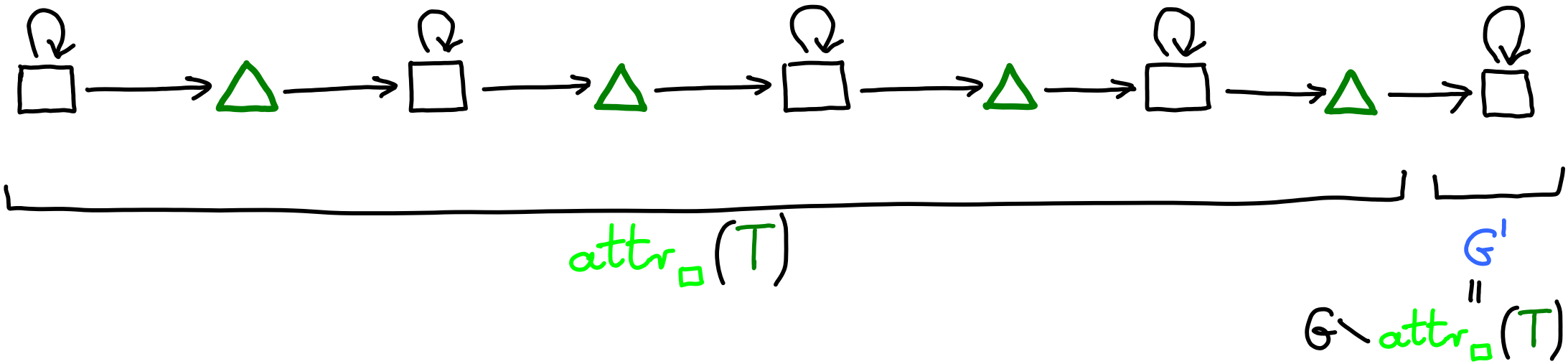
A Büchi game



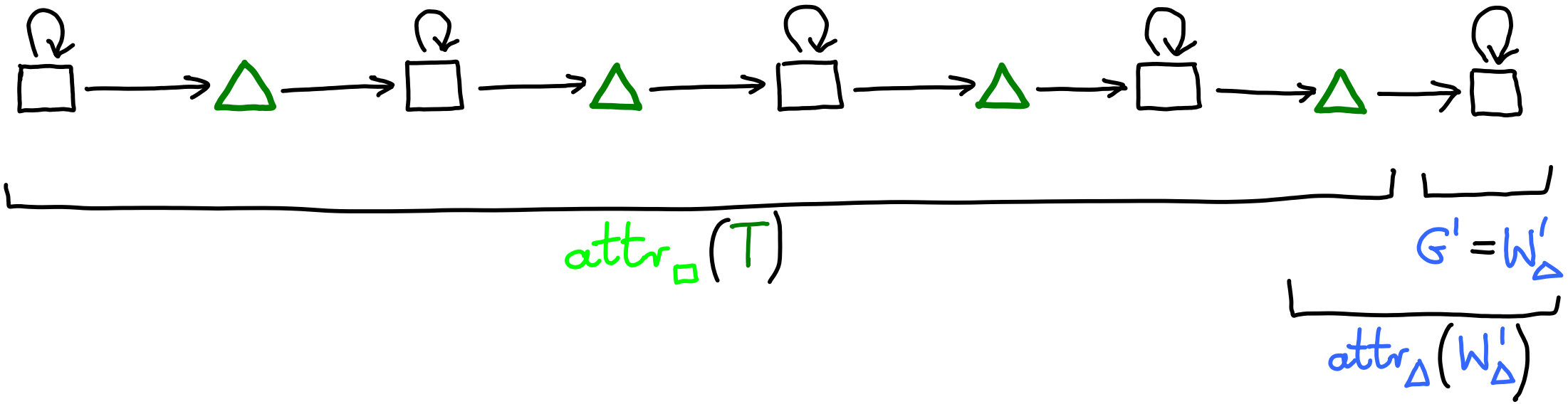
A Büchi game



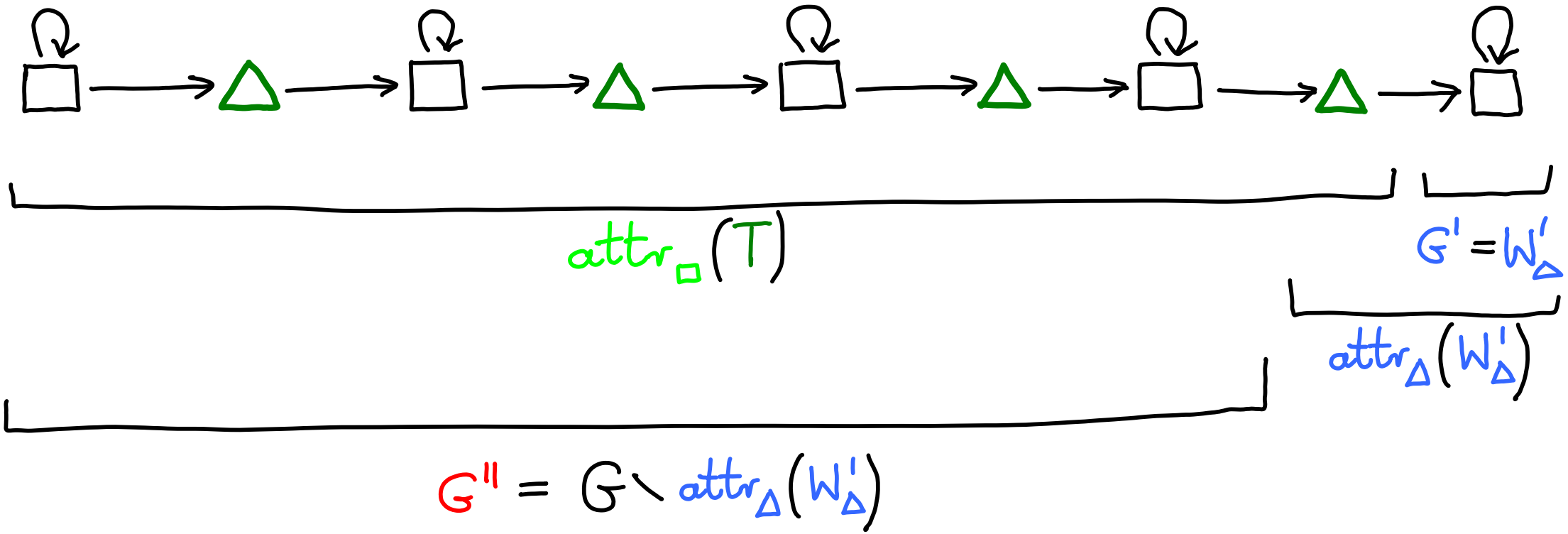
A Büchi game



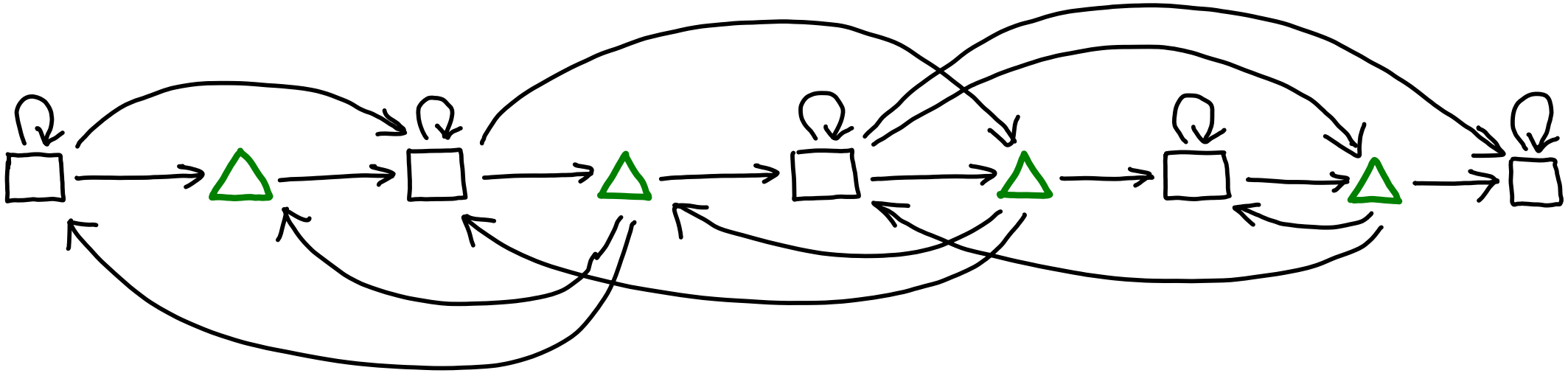
A Büchi game


$\mathrm{attr}_{\square}(T)$
$G' = W'_{\triangle}$
$\mathrm{attr}_{\triangle}(W'_{\triangle})$

A Büchi game


$\mathrm{attr}_{\square}(T)$
$G' = W'_{\triangle}$
$\mathrm{attr}_{\triangle}(W'_{\triangle})$
$G'' = G \setminus \mathrm{attr}_{\triangle}(W'_{\triangle})$

A Büchi game



$attr_{\square}(T)$

$G' = W'_{\triangle}$

$attr_{\triangle}(W'_{\triangle})$

$G'' = G \setminus attr_{\triangle}(W'_{\triangle})$

FACT The divide and conquer algorithm may take $\Theta(mn)$ time

AN $O(n^2)$ ALGORITHM FOR BÜCHI GAMES

[CHATTERJEE, HENZINGER 2012]

- Consider game graphs $G_1, G_2, G_3, \ldots, G_{\log n} = G$ each G_i with $\leq 2^i \cdot n$ edges, respectively
- Keep removing traps (complements of attractors)

  each time proceeding through graphs $G_1, G_2, G_3, \ldots, G_{\log n}$ until one of them gives a non-empty trap

---

KEY PROPERTY If a trap found in G_i first then its size is $\geq 2^i$

- Ammortized analysis charges $(2+2^2+\cdots+2^i)n \leq 2^{i+1} \cdot n$ work to the 2^i vertices removed

PLAN

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. **Parity games**
6. Better algorithms for parity games
7. Search complexity of computing optimal strategies

GAMES ON GRAPHS: PARITY OBJECTIVES

$$\mathrm{Inf}(a_0, a_1, a_2, \ldots) = \{ a : a_i = a \text{ for } \textbf{infinitely many } i \in \mathbb{N} \}$$

Parity objective:

$$W_\square = \{ (v_0, v_1, v_2, \ldots) : \max(\mathrm{Inf}(p(v_0), p(v_1), p(v_2), \ldots) \text{ is } \textbf{even} \}$$

where $p : V \to \mathbb{N}$ is the priority function

Parity Games

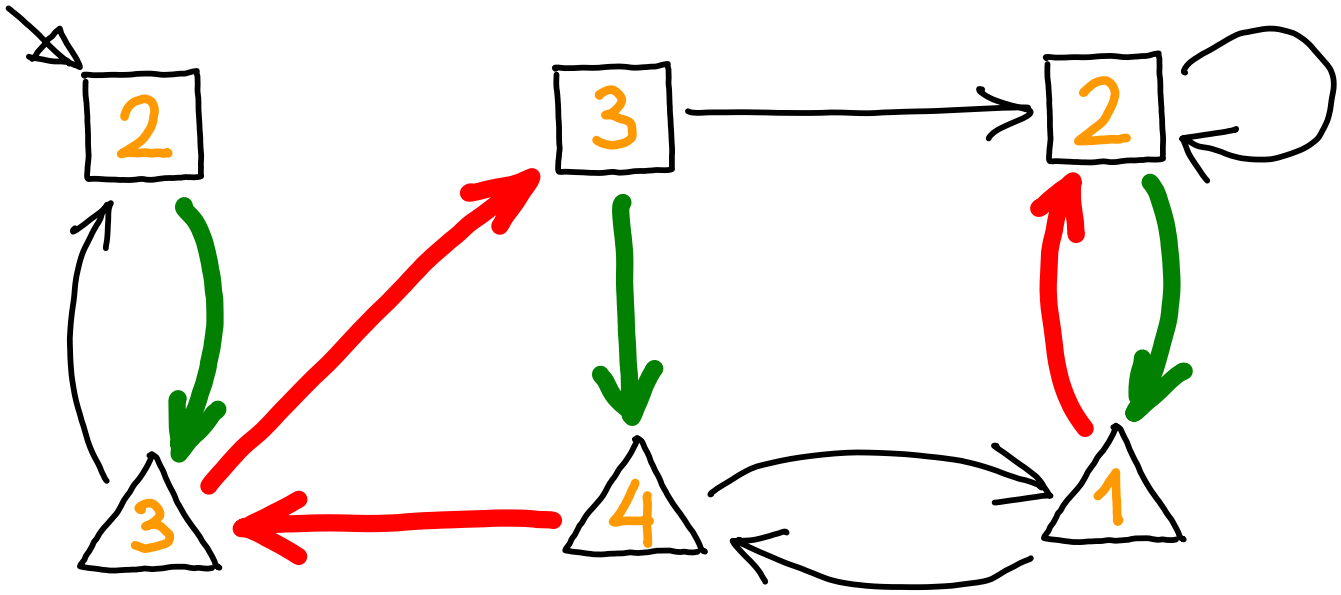
Players:

n vertices, m edges, d priorities

$n = 6$, $m = 11$, $d = 4$

Even = □ = 0

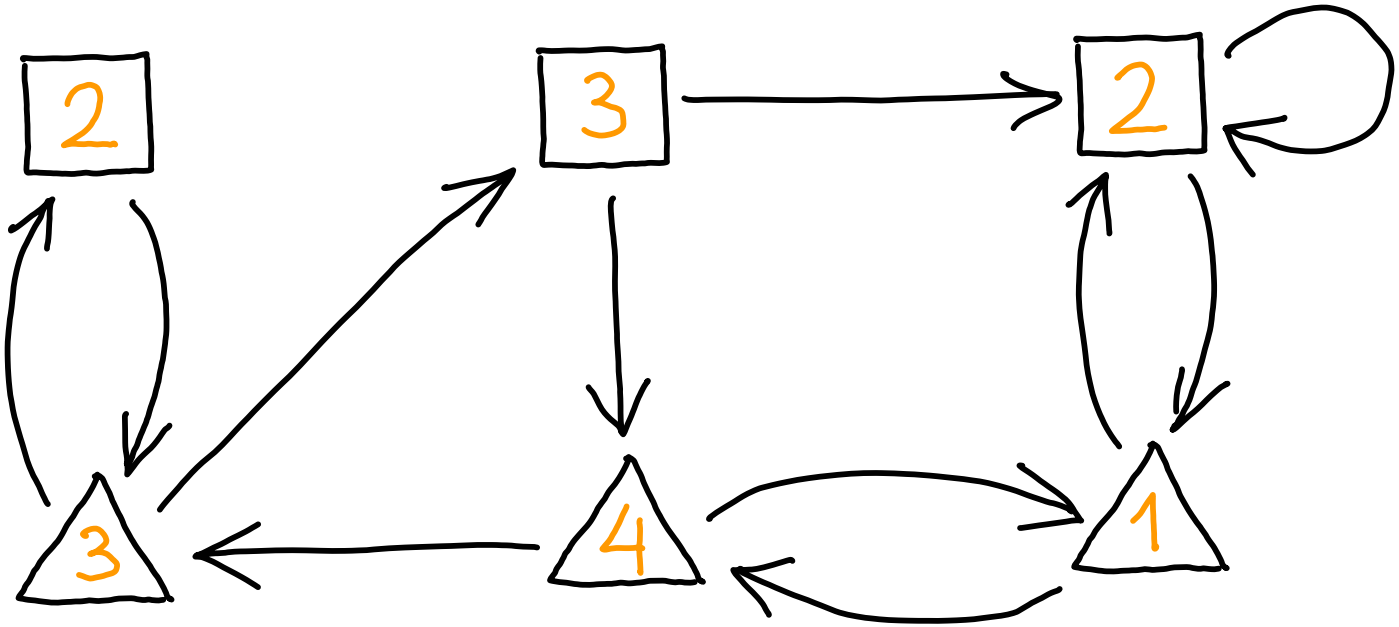
Odd = △ = 1



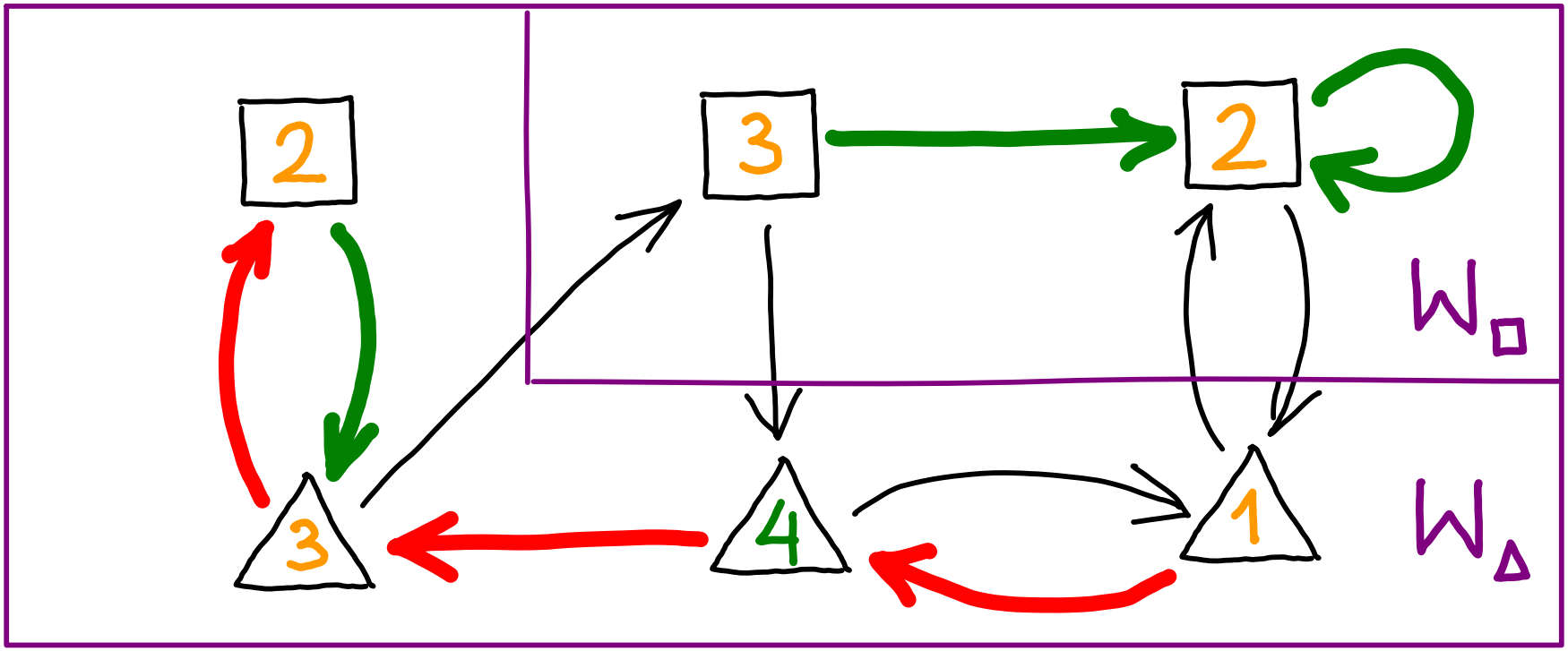
Winner of an infinite play:

parity of the highest priority occurring infinitely often

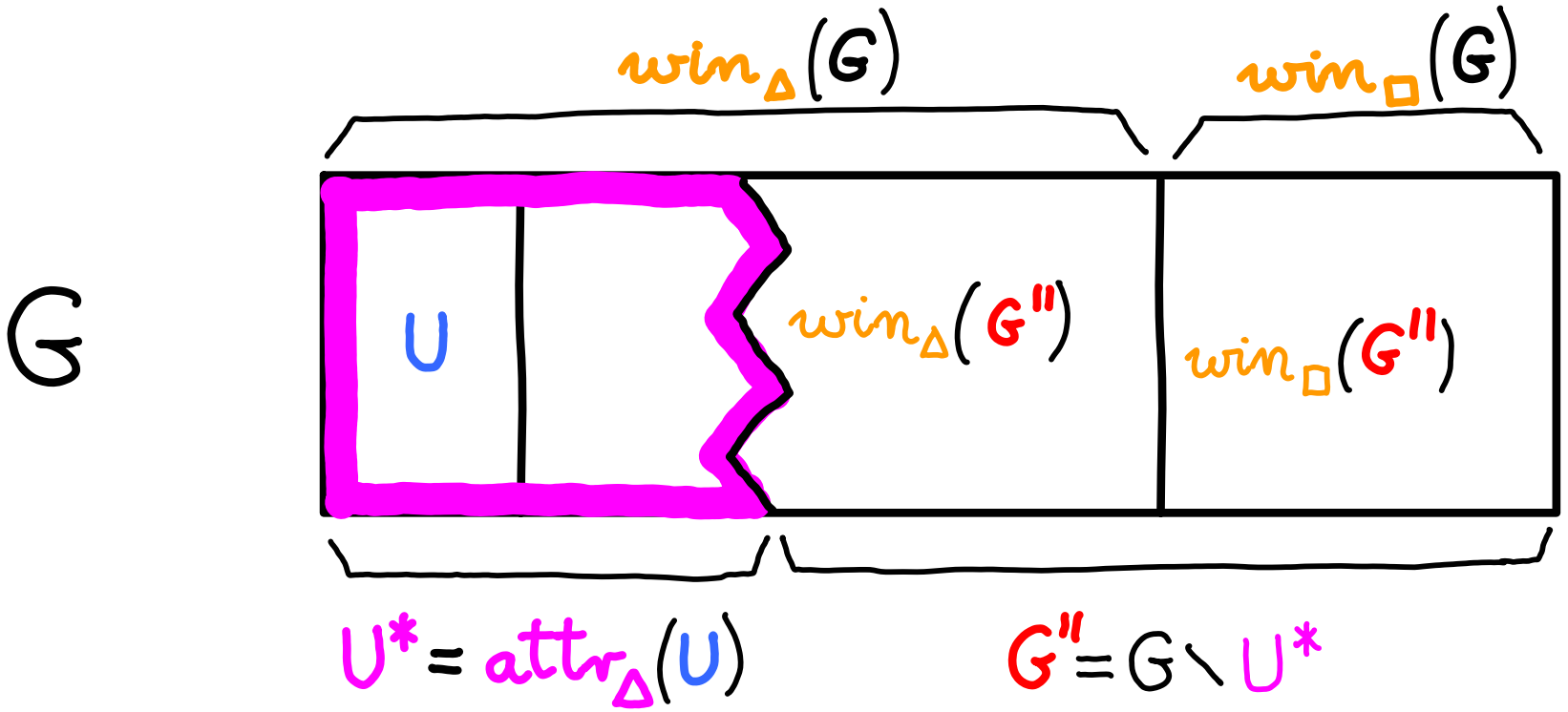
Solving parity games



Solving parity games



Oracle Lemma



Lemma If $U \subseteq win_\Delta(G)$

then $win_\Delta(G) = U^* \cup win_\Delta(G'')$

$win_\square(G) = win_\square(G'')$

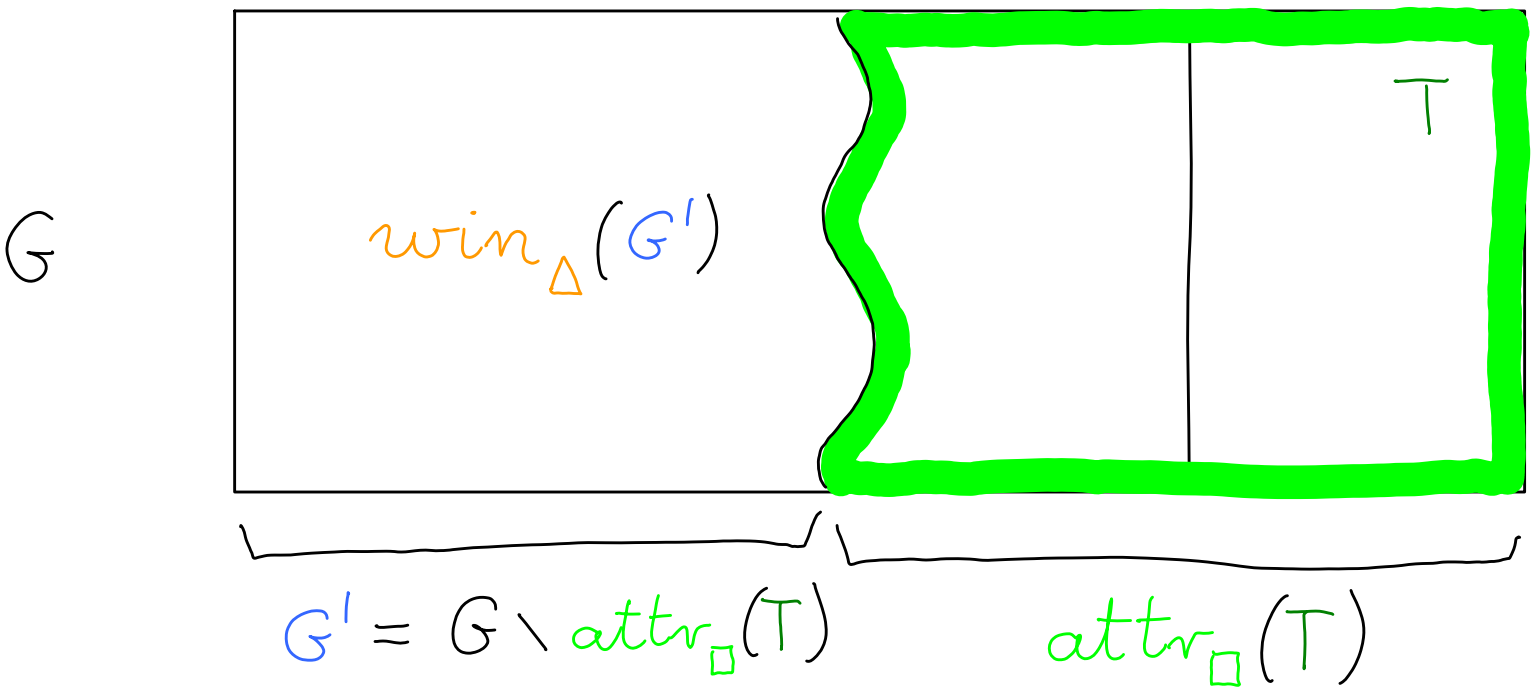
Exercise 5

- Does your proof of the oracle lemma for Büchi/co-Büchi games work also for parity games?

- Are the winning strategies constructed positional?

- If not, what assumptions do you need to make them so?

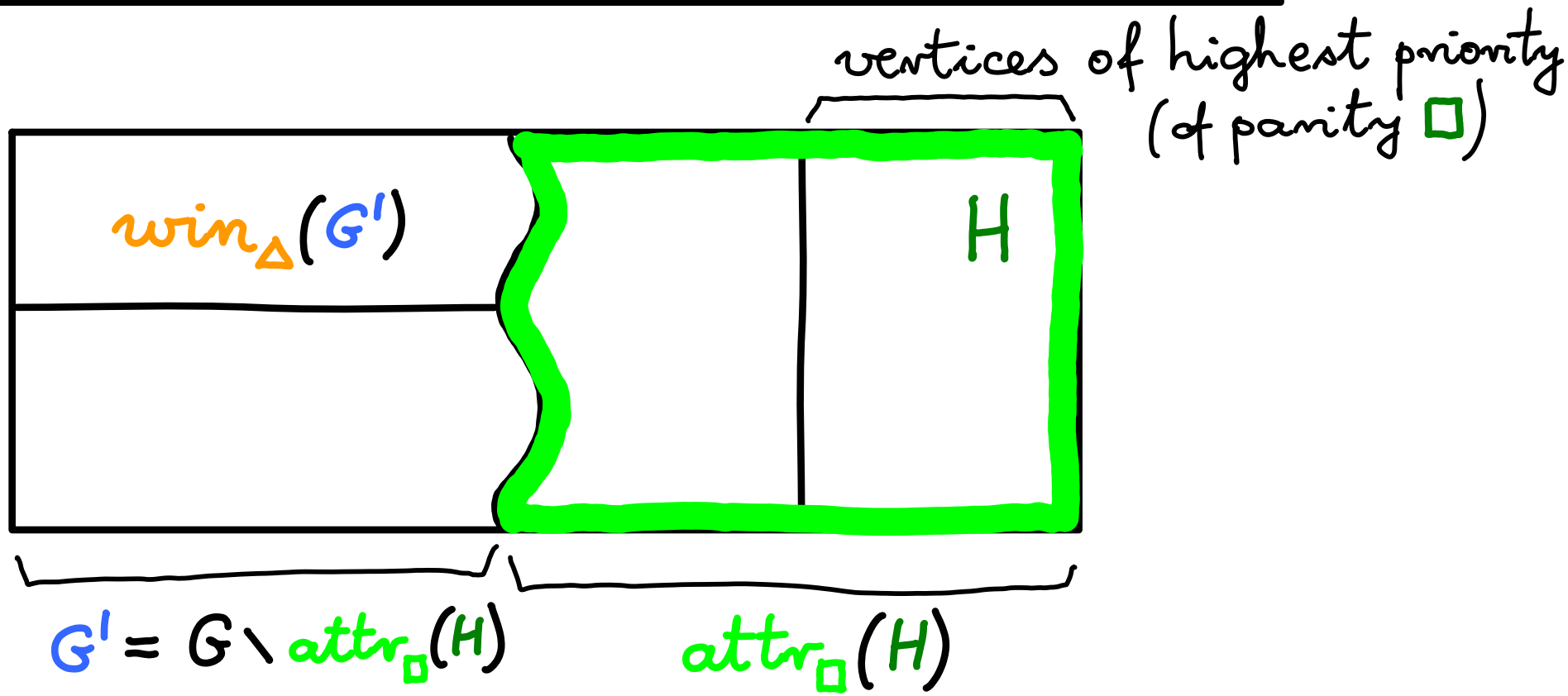
T-ATTRACTOR LEMMA



LEMMA

1. $win_{\triangle}(G') = V \setminus attr_{\square}(T) \subseteq win_{\triangle}(G)$

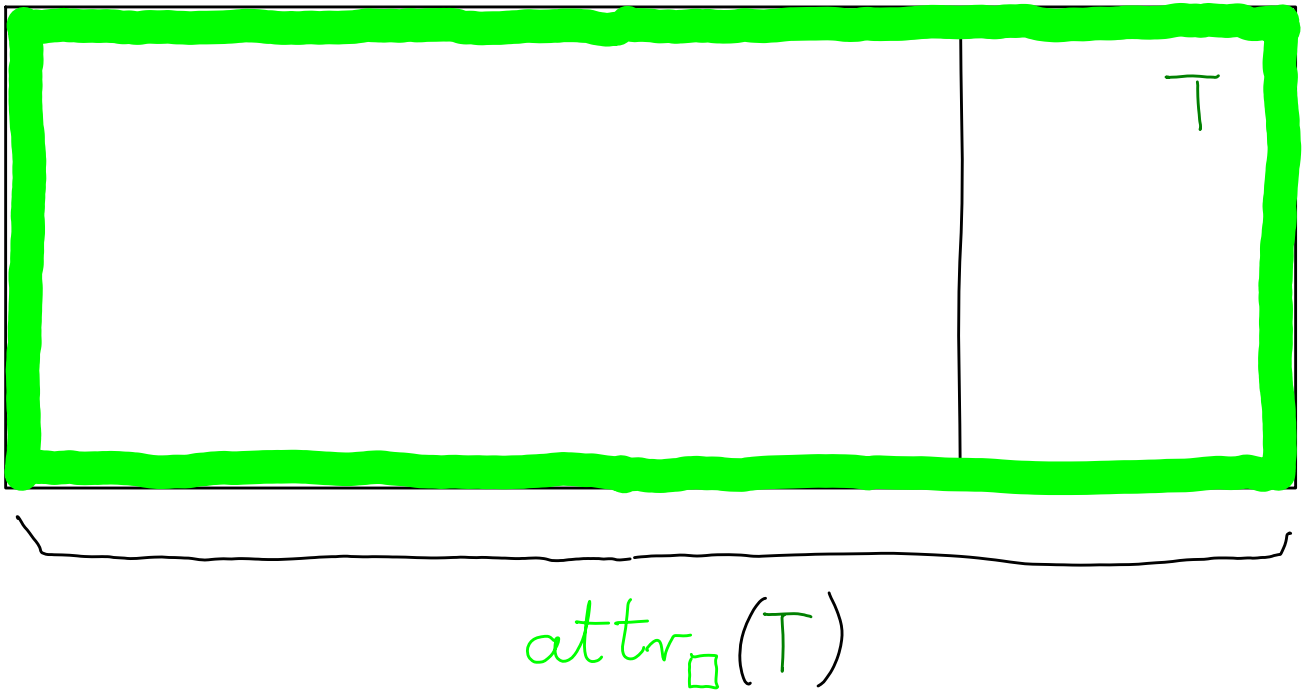
Highest-Priority attractor Lemma



Lemma

1. If $win_{\triangle}(G') \neq \emptyset$ then $win_{\triangle}(G') \subseteq win_{\triangle}(G)$

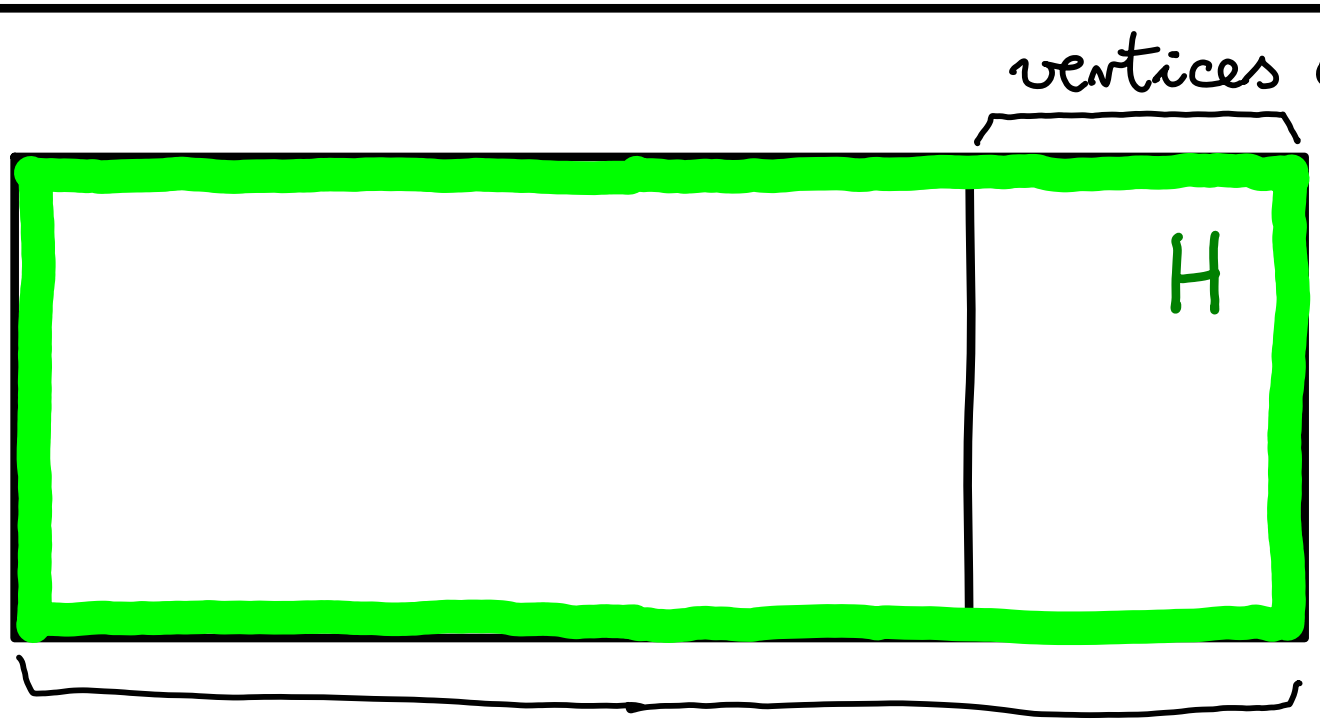
T-ATTRACTOR LEMMA



LEMMA

1. $win_{\triangle}(G') = V \setminus attr_{\square}(T) \subseteq win_{\triangle}(G)$
2. If $G' = \emptyset$ then $win_{\square}(G) = V$

Highest-Priority attractor Lemma

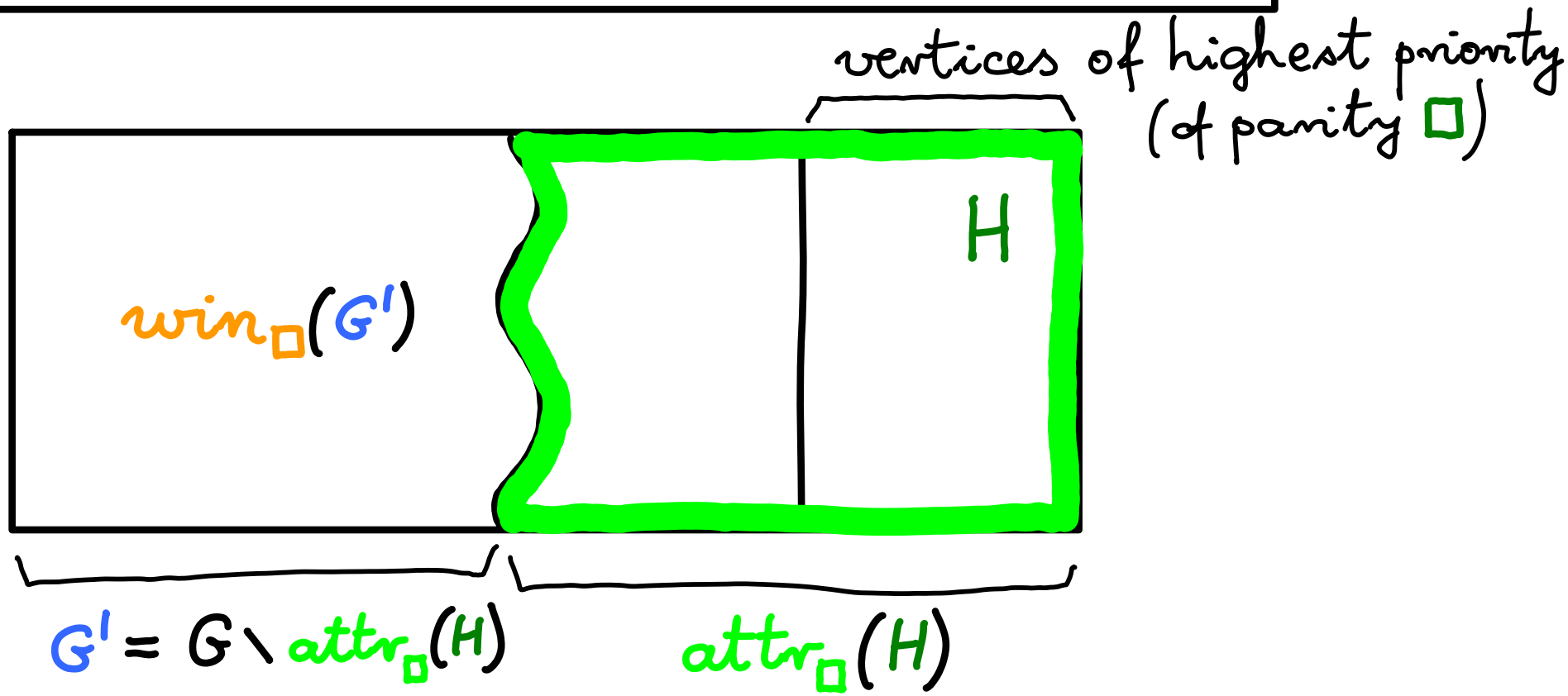


$G' = G \setminus attr_{\square}(H)$

Lemma

1. If $win_{\triangle}(G') \neq \emptyset$ then $win_{\triangle}(G') \subseteq win_{\triangle}(G)$
2. If $G' = \emptyset$ then $win_{\square}(G) = V$

Highest-Priority attractor Lemma



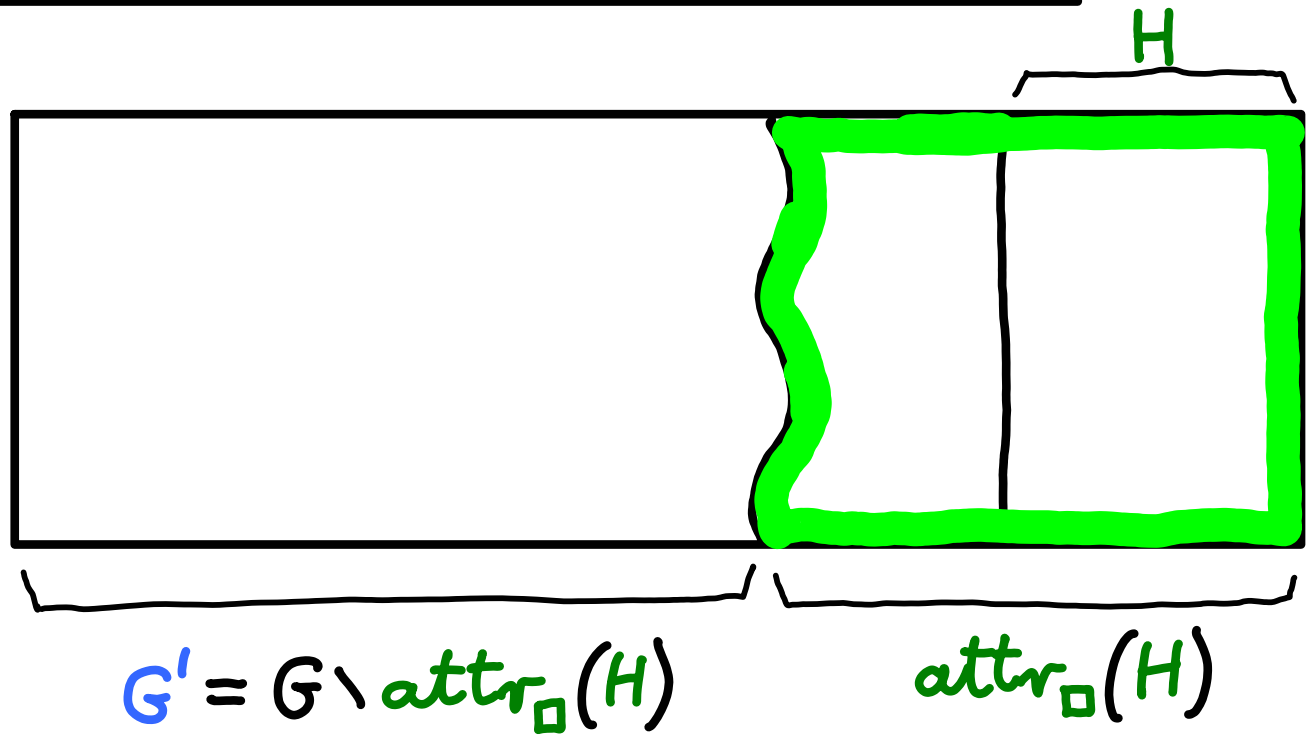
Lemma

1. If $win_{\triangle}(G') \neq \emptyset$ then $win_{\triangle}(G') \subseteq win_{\triangle}(G)$
2. If $win_{\triangle}(G') = \emptyset$ then $win_{\square}(G) = V$

Exercise 6

Prove the highest-priority attractor lemma.

A DIVIDE-AND-CONQUER ALGORITHM



Win(G): $(W'_{\square}, W'_{\triangle}) := \text{Win}(G')$

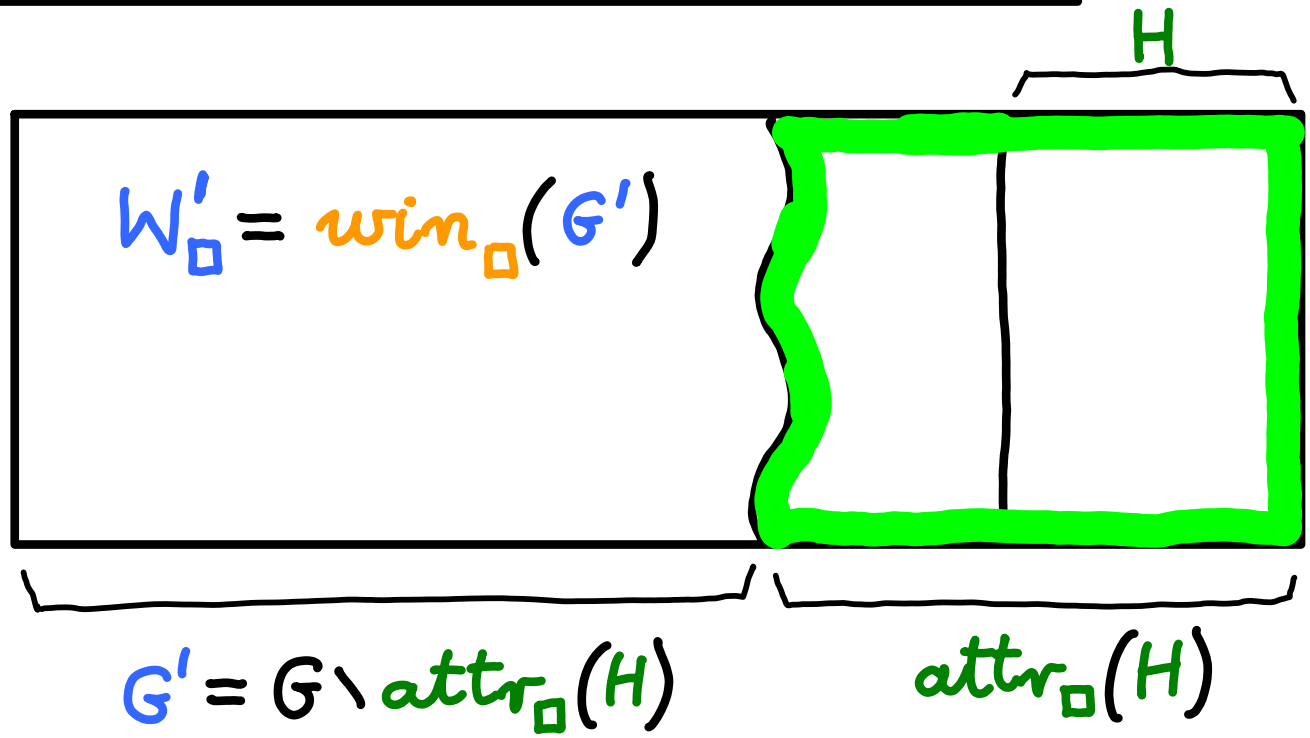
if $W'_{\triangle} = \emptyset$ then $(W_{\square}, W_{\triangle}) := (V, \emptyset)$

else $(W''_{\square}, W''_{\triangle}) := \text{Win}(G'')$

$(W_{\square}, W_{\triangle}) := \left(W''_{\square},\ attr_{\triangle}(W'_{\triangle}) \cup W''_{\triangle}\right)$

return $(W_{\square}, W_{\triangle})$

A DIVIDE-AND-CONQUER ALGORITHM



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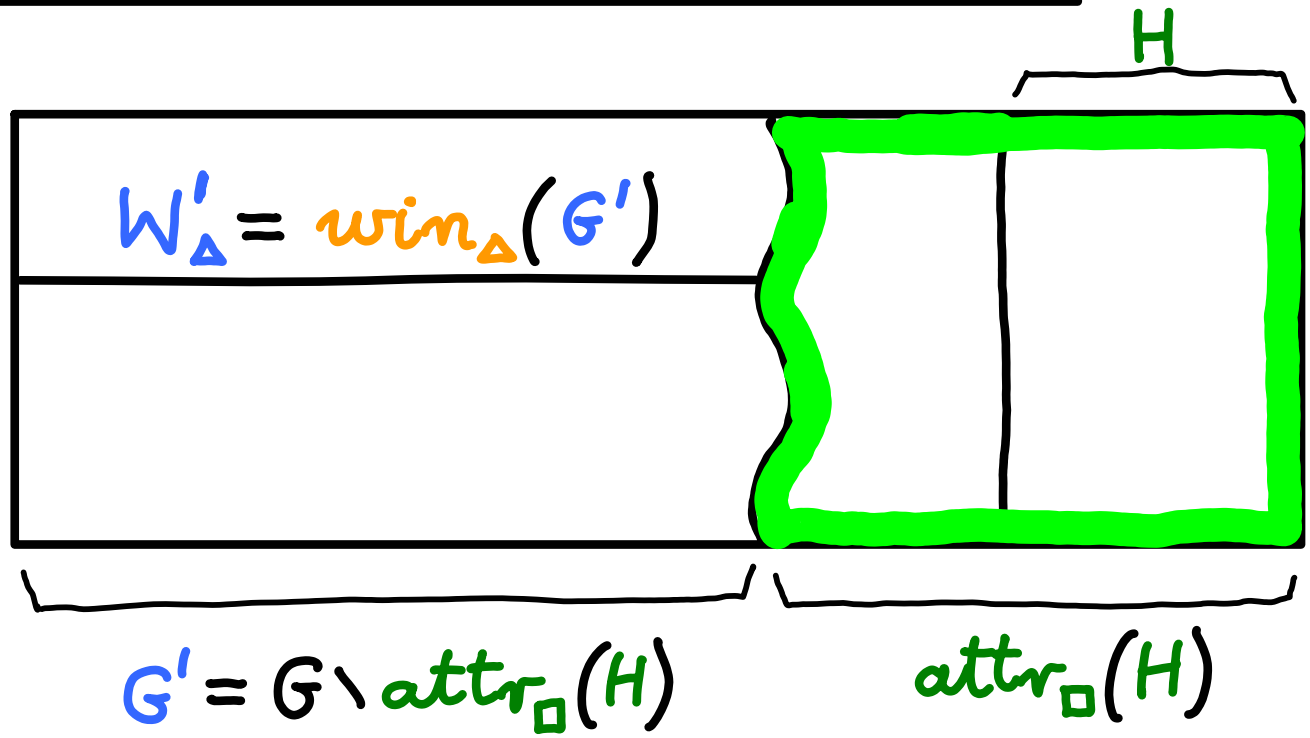

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$(W_\square, W_\triangle) := \left(W''_\square,\ attr_\triangle(W'_\triangle) \cup W''_\triangle\right)$

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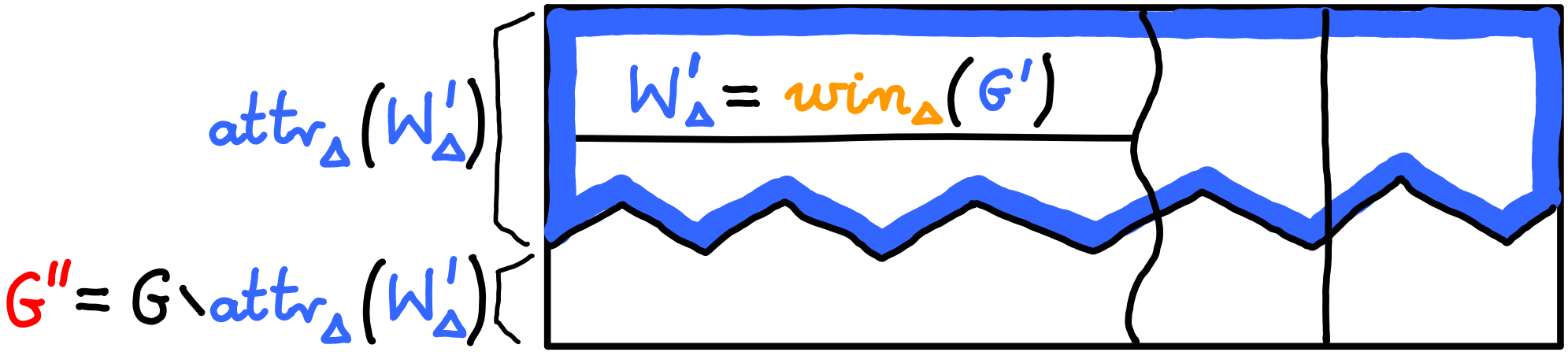
A DIVIDE-AND-CONQUER ALGORITHM



WIN(G):

$(W'_\square, W'_\Delta) := \text{WIN}(G')$

if $W'_\Delta = \emptyset$ then $(W_\square, W_\Delta) := (V, \emptyset)$

else $(W''_\square, W''_\Delta) := \text{WIN}(G'')$

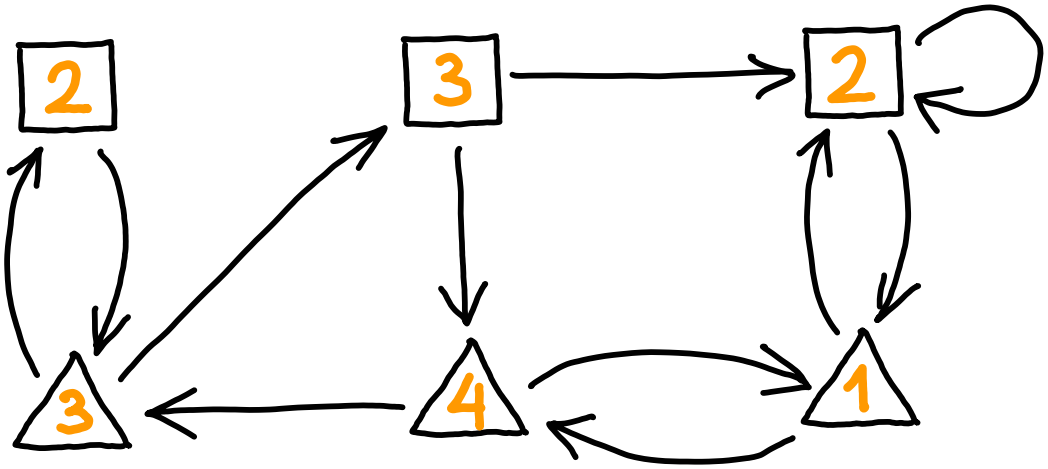
$(W_\square, W_\Delta) := \left(W''_\square,\ attr_\Delta(W'_\Delta) \cup W''_\Delta\right)$

return $(W_\square, W_\Delta)$

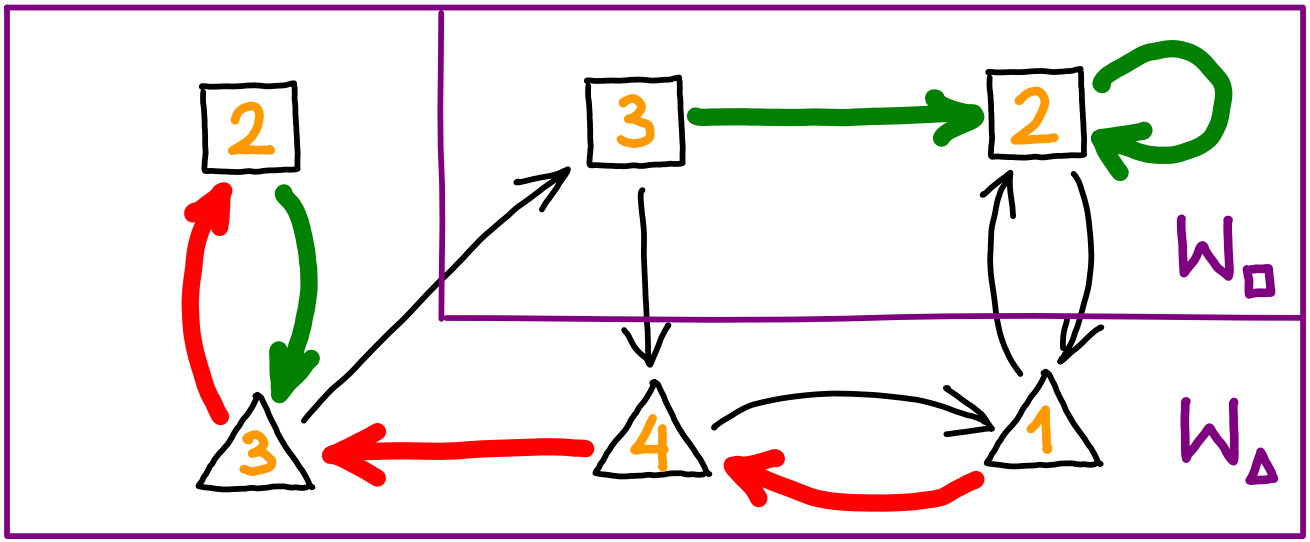
Exercise 7

Run the divide-and-conquer algorithm on the following parity game.

Indicate the winning sets for both players, and their positional winning strategies.



Positional Determinacy



Thm [Emerson, Jutla ; Mostowski 1991]

Parity games are **positionally determined**

Corollary

(Deciding the winner in) parity games is in $NP \cap co\text{-}NP$

Exercise 8

- Use the oracle and highest-priority attractor lemmas, and induction on the size of the game graph, to prove the positional determinacy for parity games.

- Design a polynomial-time algorithm for checking if a given positional strategy of a player in a parity game is a winning strategy for her.

Running time of the divide-and-conquer algorithm

$T(n)$

Win(G): $(W'_\square, W'_\triangle) :=$ **Win**(G') — $T(n-1)$

  if $W'_\triangle = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$

  else $(W''_\square, W''_\triangle) :=$ **Win**(G'') — $T(n-1)$

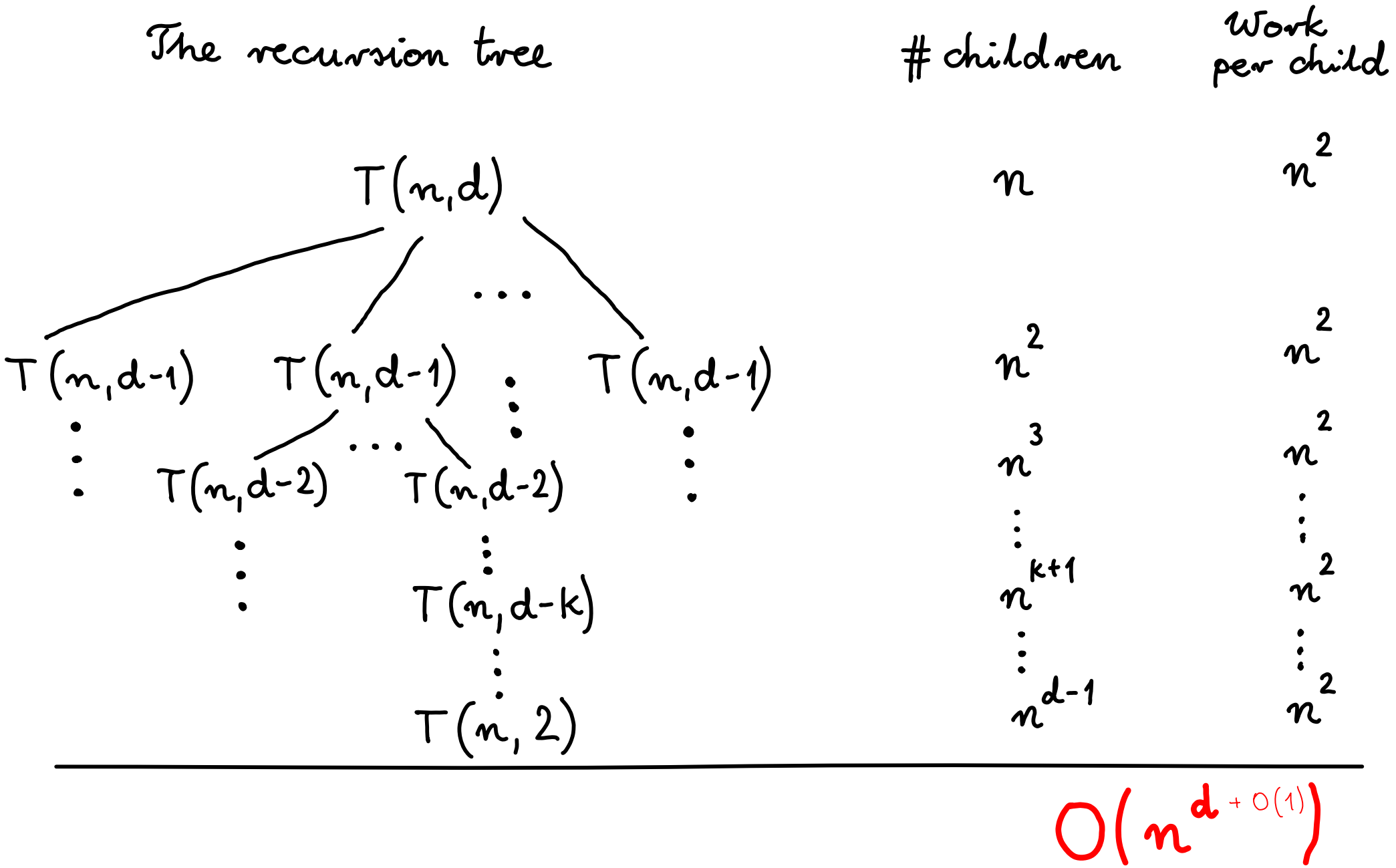
  $(W_\square, W_\triangle) := \left(W''_\square, \mathrm{attr}_\triangle(W'_\triangle) \cup W''_\triangle\right)$

  return $(W_\square, W_\triangle)$

Recurrence: $T(n) \leq 2 \cdot T(n-1) + O(n^2)$

Solution: $T(n) = O(2^n)$

Running time of the divide-and-conquer algorithm



Exercise 9

- Observe that the first recursive call **WIN**(G') is on the game G' with less than d distinct priorities.

- Argue that if the second recursive call **WIN**(G'') occurs then the game G'' has less vertices of priority d than G.

- Conclude that the following recurrence:

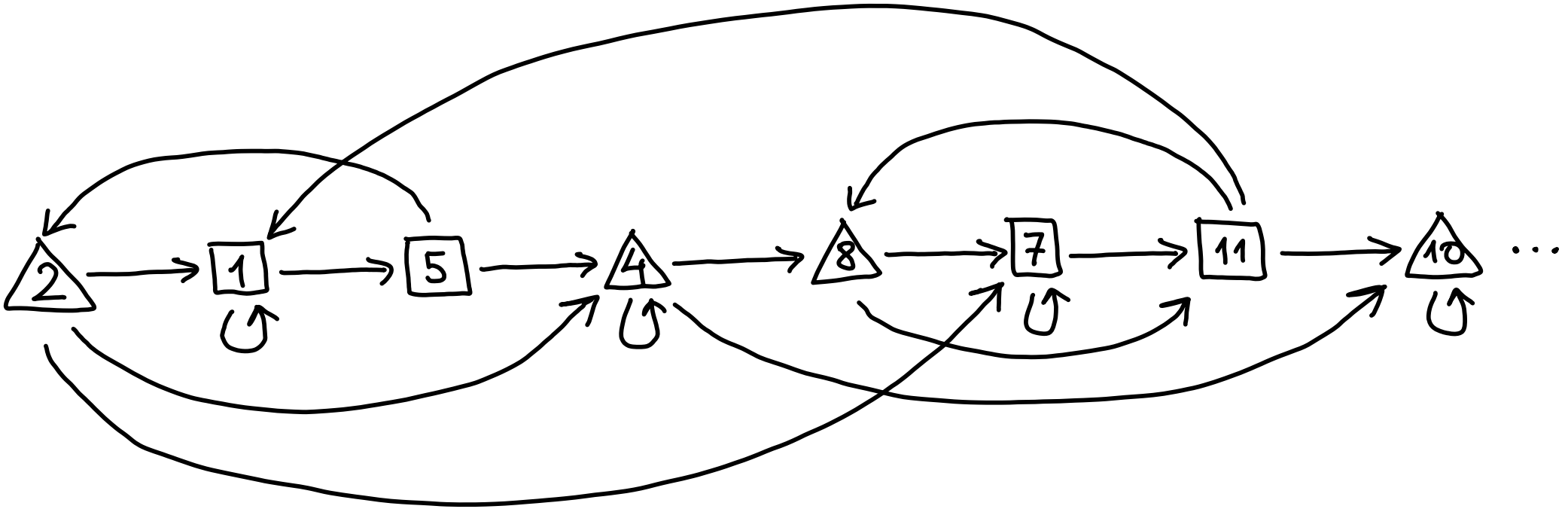
$$\begin{cases} T(n,1) = O(n) \\ T(n,d) \leq (n_d+1) \cdot T(n,d-1) + O(n_d \cdot m) \end{cases}$$

characterizes the running time of the algorithm.

- Prove that $T(n,d) = O\left(m \cdot \left(\frac{n+d}{d}\right)^d\right) = O\left(n^{d+O(1)}\right)$

Exercise 10

What is the running time of the divide-and conquer algorithm on this parity game?



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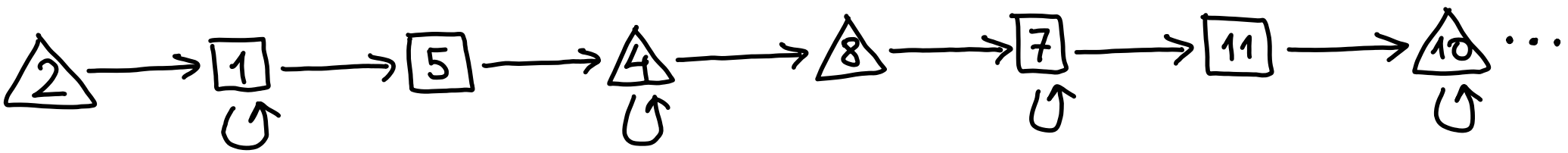

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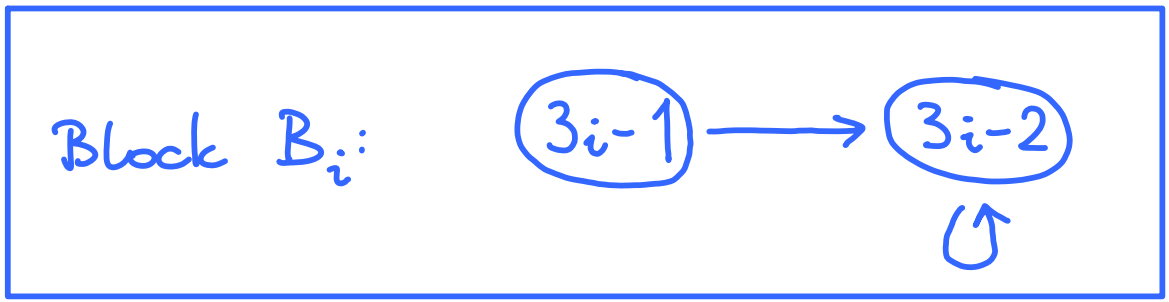
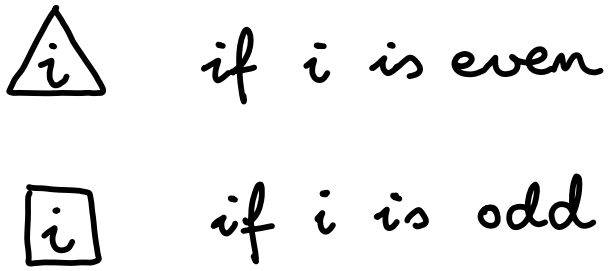
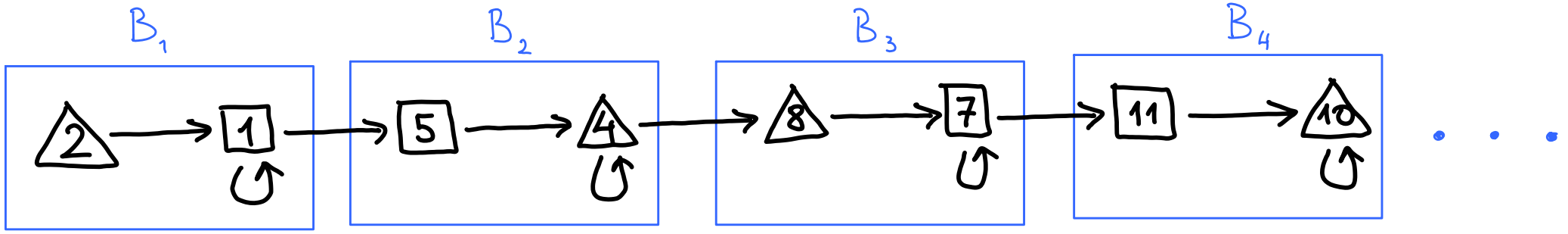
What is the running time of the divide-and conquer algorithm on this parity game?







PLAN

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. Parity games
6. **Better algorithms for parity games**
7. Search complexity of computing optimal strategies

Better algorithms

Many priorities

$d = \Omega(n^{\frac{1}{2}+\varepsilon})$

Randomized

Expected $n^{O(\sqrt{n})}$

Random Facet

Deterministic

$n^{O(\sqrt{n})}$

[J., Paterson, Zwick 2008]

Few priorities

$d = O(n^{1/2})$

$O(n^{\frac{d}{2}+O(1)})$

[Browne et al. 1997, Seidl 1996, J. 2000]

$O(n^{\frac{d}{3}+O(1)})$

[Schewe 2007]

Better algorithms

Many priorities

$d = \Omega(n^{\frac{1}{2}+\varepsilon})$

Randomized

Expected $n^{O(\sqrt{n})}$

Deterministic

$n^{O(\sqrt{n})}$

Few priorities

$d = O(n^{1/2})$

Randomized

$O(n^{\frac{d}{2}+O(1)})$

Deterministic

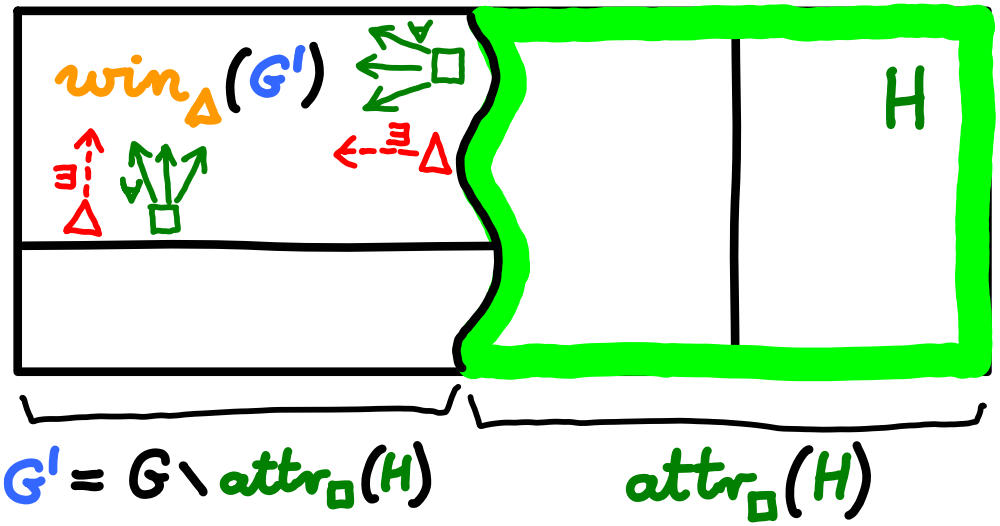
$O(n^{\frac{d}{3}+O(1)})$

Dominions

Def $D \subseteq V$ is an **dominion** for Δ if

Δ has a **trapping** strategy in D

that is **winning** for her



Fact $win_\Delta(G')$ is a dominion for Δ in G

Fact A dominion of size $\leq \ell$ can be found in $n^{O(\ell)}$ time

Exercise 11

Design a brute-force algorithm for detecting dominions of size $\leq \ell$ that runs in time $O(n^\ell)$.

A Subexponential Algorithm

$T(n)$

Win(G): If there is a dominion $D \neq \emptyset$ of size $\leq \ell$

then return $\textsc{Win}(G \setminus D) \oplus D$

else

$(W'_\square, W'_\triangle) := \textsc{Win}(G')$

if $W'_\triangle = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$

else $(W''_\square, W''_\triangle) := \textsc{Win}(G'')$

$(W_\square, W_\triangle) := \left(W''_\square, \mathrm{attr}_\triangle(W'_\triangle) \cup W''_\triangle\right)$

return $(W_\square, W_\triangle)$

$n^{O(\ell)}$

$T(n-1)$

$T(n-1)$

$T(n-\ell)$

Recurrence: $T(n) \leq n^{O(\ell)} + T(n-1) + T(n-\ell)$

Solution: $\boxed{T(n) = n^{O(\sqrt{n})}}$ if $\ell = \sqrt{n}$

Exercise 12

- Define labelled tree τ_n
  - root labelled n
  - if $n > 3$ then the root has:
    - * left subtree τ_{n-1}
    - * right subtree $\tau_{n-\lceil\sqrt{2n}\rceil}$
  - if $n = 1, 2,$ or 3 then the root has no children
- Prove that on every path from root to leaf in τ_n there are at most $\lfloor\sqrt{2n}\rfloor$ right turns.

  Hint: Use induction and the fact that

  $$\text{if } \tfrac{1}{2}j^2 < n \leq \tfrac{1}{2}(j+1)^2 \text{ then } n - \lceil\sqrt{2n}\rceil \leq \tfrac{1}{2}j^2$$

- Conclude that τ_n has $n^{O(\sqrt{n})}$ leaves.

Better algorithms

Many priorities

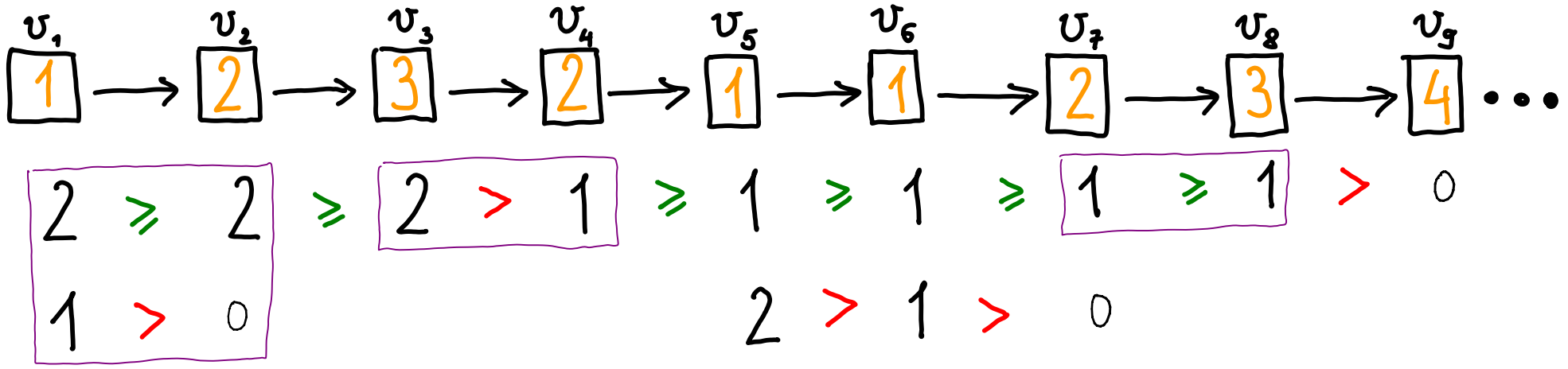
$d = \Omega(n^{\frac{1}{2}+\varepsilon})$

Few priorities

$d = O(n^{1/2})$

	Randomized	Deterministic
Expected	$n^{O(\sqrt{n})}$	$n^{O(\sqrt{n})}$
	$O(n^{\frac{d}{2}+O(1)})$	$O(n^{\frac{d}{3}+O(1)})$

A SUFFICIENT CONDITION FOR WINNING PLAYS



FACT

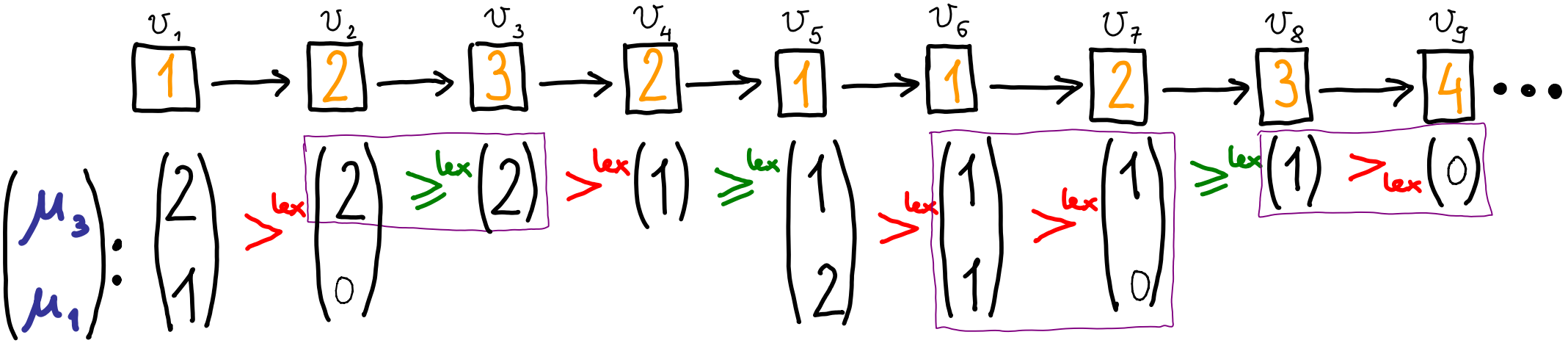
$v_1, v_2, v_3, \ldots$ is winning for $\square$

if

there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $\mu_k(v_i) \geqslant \mu_k(v_{i+1})$ for $p(v_i) < k \leq d-1$
- $\mu_k(v_i) > \mu_k(v_{i+1})$ for $k = p(v_i)$

A SUFFICIENT CONDITION FOR WINNING PLAYS



FACT

$v_1, v_2, v_3, \ldots$ is winning for $\square$

if

there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v_i)} \end{pmatrix}(v_i) \geq^{lex} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v_i)} \end{pmatrix}(v_{i+1})$$

$>^{lex}$ if $p(v_i)$ is odd!

Exercise 13

- Prove the two facts.

- Argue that the following holds for the first characterization:

  if $\mu_k(v) = l$

  then, starting from v,

  at most l vertices of priority k will occur,

  before a vertex of priority $> k$ occurs.

- Does a similar property hold for the second characterization?

Characterizing existence of winning plays

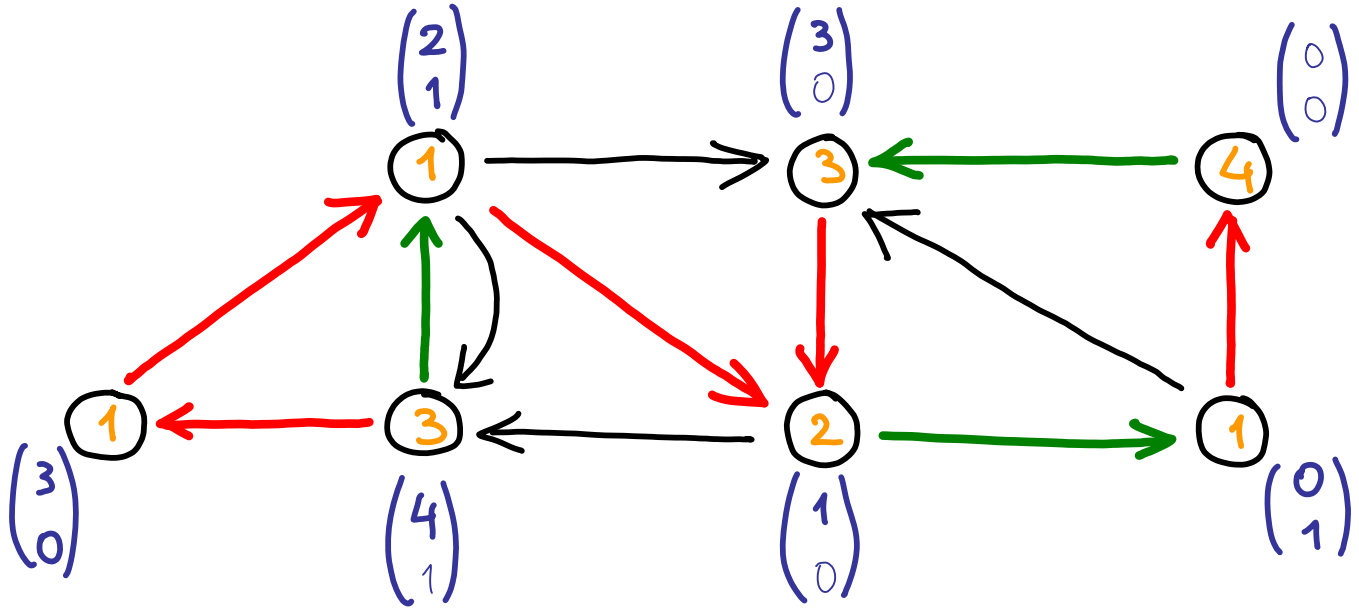
Lemma

There is a winning play (from every vertex) for $\square$ if there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{\text{lex}} \min_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$$

$>^{\text{lex}}$ if $p(v)$ is odd!

Characterizing existence of winning plays



$$\mu_3(v) >^{lex} \min_{(v,w)\in E} \mu_3(w) \quad \text{if } p(v) = 3$$

$$\mu_3(v) \geq^{lex} \min_{(v,w)\in E} \mu_3(w) \quad \text{if } p(v) = 2$$

$$\begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) >^{lex} \min_{(v,w)\in E} \begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) \quad \text{if } p(v) = 1$$

CHARACTERIZING UNIVERSALITY OF WINNING PLAYS

LEMMA

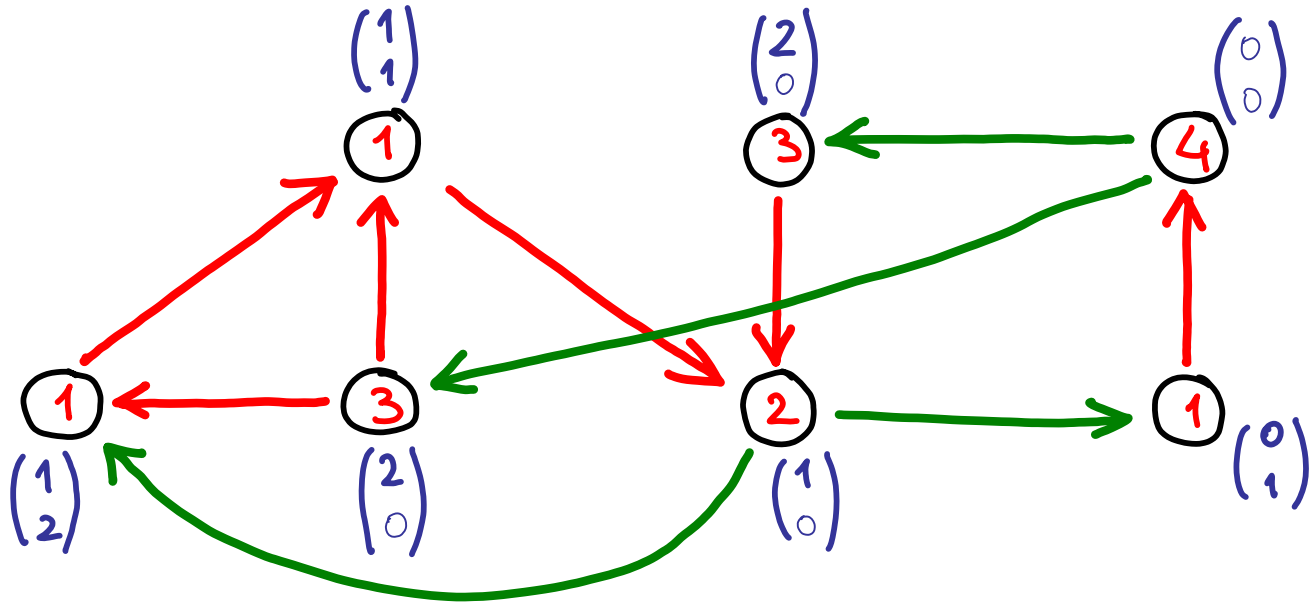
ALL plays (from every vertex) are winning for $\square$

if there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{\text{lex}} \max_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$$

$>^{\text{lex}}$ if $p(v)$ is odd!

Characterizing universality of winning plays



$$\mu_3(v) >^{lex} \max_{(v,w)\in E} \mu_3(w) \qquad \text{if } p(v) = 3$$

$$\mu_3(v) \geq^{lex} \max_{(v,w)\in E} \mu_3(w) \qquad \text{if } p(v) = 2$$

$$\begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) >^{lex} \max_{(v,w)\in E} \begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) \qquad \text{if } p(v) = 1$$

Characterizing existence of winning strategies

Theorem

There is a winning strategy for $\Box$ (from every vertex)

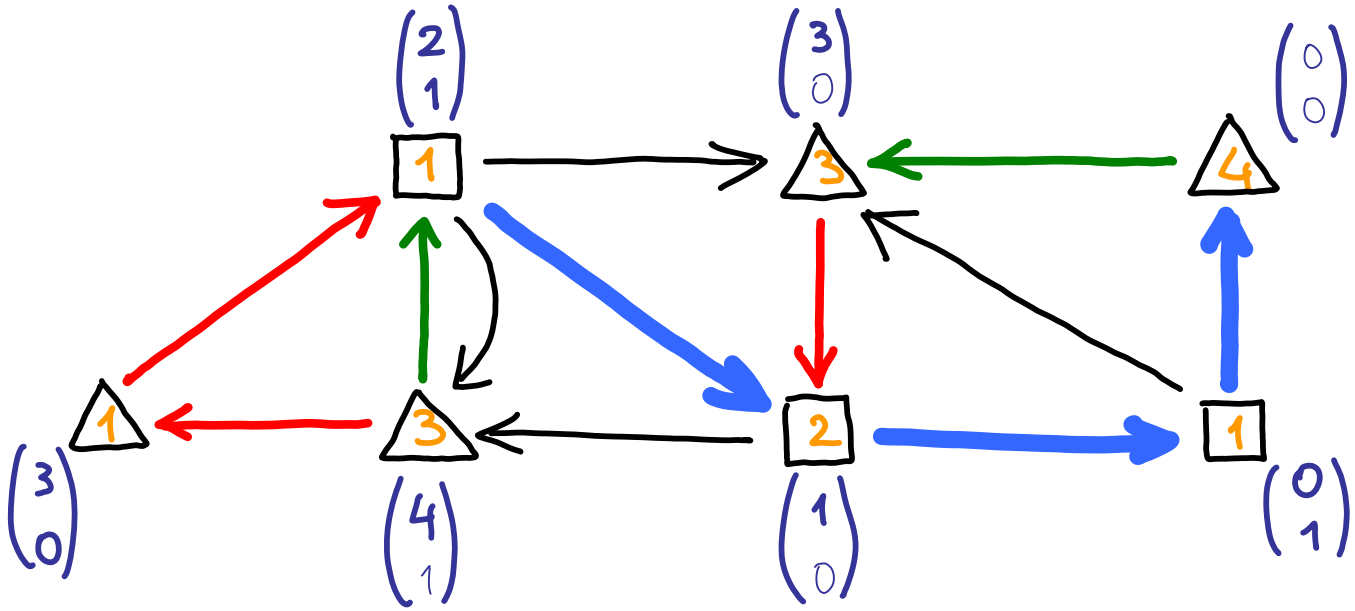
iff

there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{lex} \min_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$$ $>^{lex}$ if $p(v)$ is odd! for $v \in V_\Box$

- $$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{lex} \max_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$$ $>^{lex}$ if $p(v)$ is odd! for $v \in V_\Diamond$

Characterizing existence of winning strategies



$v \in V_\square$

$$\mu_3(v) >^{lex} \min_{(v,w)\in E} \mu_3(w)$$

$$\mu_3(v) \geqslant^{lex} \min_{(v,w)\in E} \mu_3(w)$$

$$\begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) >^{lex} \min_{(v,w)\in E} \begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v)$$

$v \in V_\triangle$

$$\mu_3(v) >^{lex} \max_{(v,w)\in E} \mu_3(w) \quad \text{if } p(v) = 3$$

$$\mu_3(v) \geqslant^{lex} \max_{(v,w)\in E} \mu_3(w) \quad \text{if } p(v) = 2$$

$$\begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) >^{lex} \max_{(v,w)\in E} \begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}(v) \quad \text{if } p(v) = 1$$

SMALL PROGRESS MEASURES

THEOREM

There is a winning strategy for $\Box$ (from every vertex)

iff

there are $\mu_1, \mu_3, \ldots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{\text{lex}} \min_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$ $\quad >^{\text{lex}}$ if $p(v)$ is odd! $\quad$ for $v \in V_\Box$

- $\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(v) \geq^{\text{lex}} \max_{(v,w)\in E} \begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_{p(v)} \end{pmatrix}(w)$ $\quad >^{\text{lex}}$ if $p(v)$ is odd! $\quad$ for $v \in V_\Diamond$

and $\mu_k : V \to \{0, 1, 2, \ldots, n_k\}$ where $n_k = \left| p^{-1}(k) \right|$

Exercise 14

Prove the small progress measure theorem.

Hint: Use positional determinacy of parity games and induction on the size of the winning set similar to that in the proof of correctness of the divide-and-conquer algorithm.

SMALL PROGRESS MEASURES

LEMMA

The number of tuples

$$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_1 \end{pmatrix} \in \mathbb{N}^{d/2}$$

s.t. $\mu_k \in \{0, 1, 2, \ldots, n_k\}$ where $n_k = \left| p^{-1}(k) \right|$

is $\boxed{\leq \left(\frac{n}{d/2}\right)^{d/2}}$

$$= O\left(n^{\frac{d}{2}}\right)$$

Progress Measure Lifting Algorithm

1. Start with $\mu : V \ni v \mapsto (0, 0, \ldots, 0)$

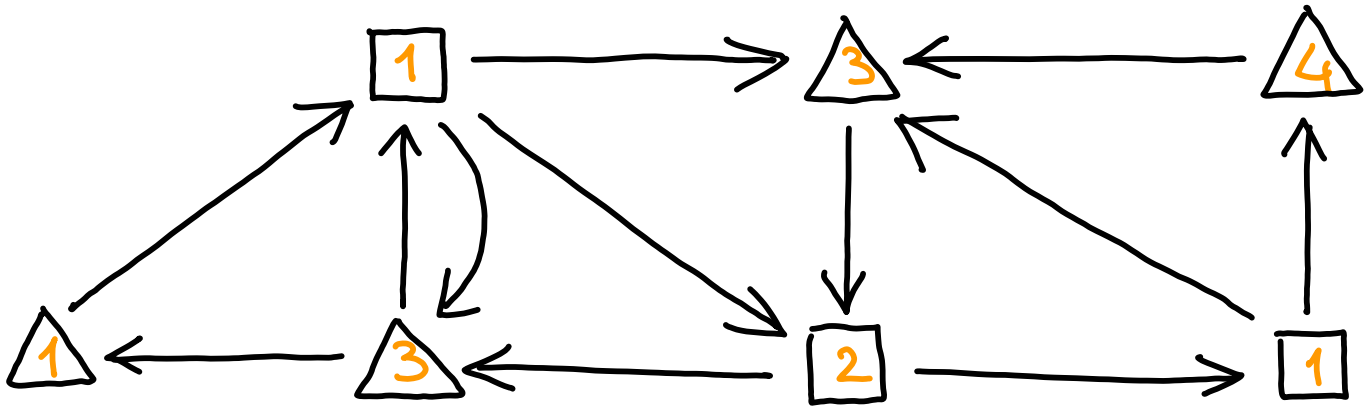
2. <u>While</u> μ **violates the progress inequality** at some $v \in V$, lift $\mu(v)$ (minimally) so that it doesn't

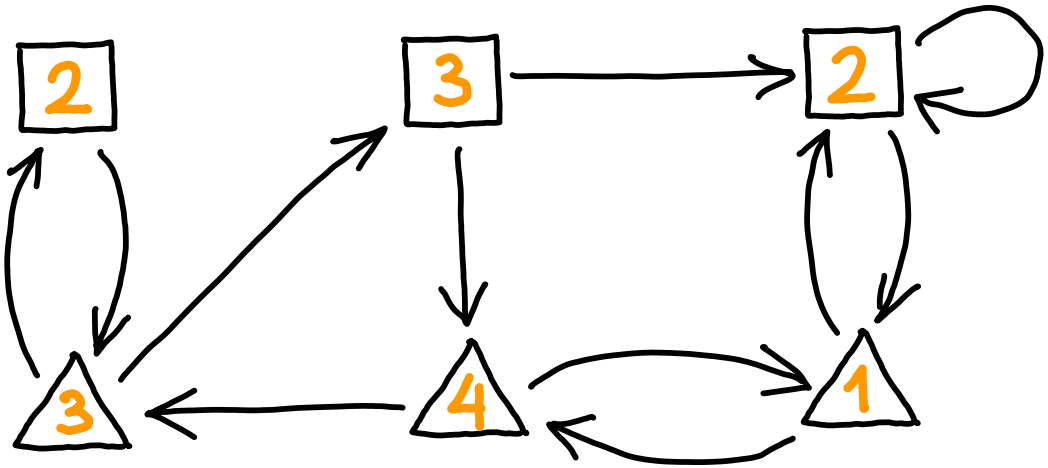
Running time: $\leqslant dm \cdot n^{\frac{d}{2}}$

$= O\left(n^{\frac{d}{2} + O(1)}\right)$

Exercise 15

Run the progress-measure-lifting algorithm on the following parity games:





Better algorithms

Many priorities

$$d = \Omega\left(n^{\frac{1}{2}+\varepsilon}\right)$$

Randomized	Deterministic
Expected $n^{O(\sqrt{n})}$	$n^{O(\sqrt{n})}$

Few priorities

$$d = O(n^{1/2})$$

Randomized	Deterministic
$O\left(n^{\frac{d}{2}+O(1)}\right)$	$O\left(n^{\frac{d}{3}+O(1)}\right)$

Smaller Progress Measures for Dominions

Thm If $D \subseteq V$ is a dominion of size $\leq \ell$, then there is a progress measure $\mu : V \to M_G^{\infty}(\ell)$, s.t. $\mu(D) \subseteq M_G(\ell)$,

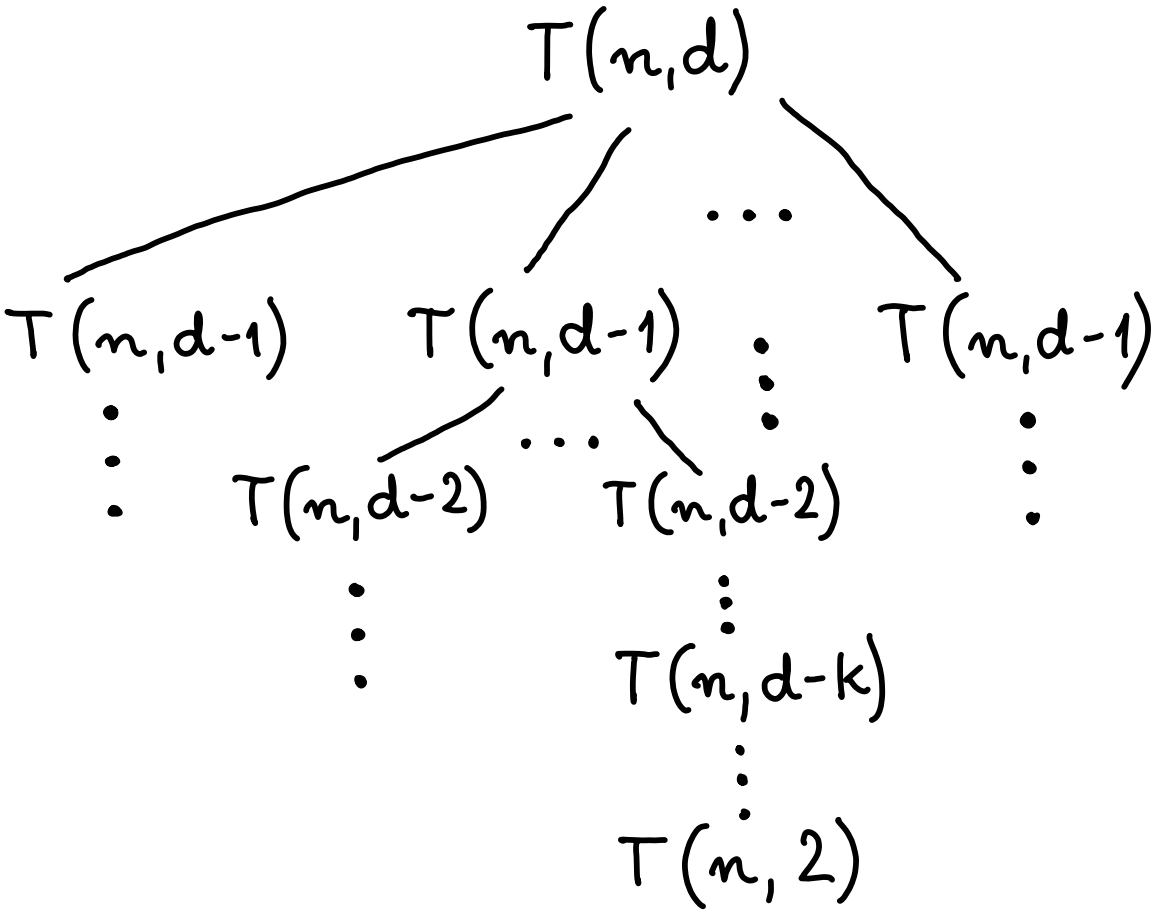
where $$M_G(\ell) = \left\{ \mu \in \mathbb{N}^{d/2} : \sum_{k=1}^{d/2} \mu_k \leq \ell \right\}$$

Cor

1. *All* dominions of *size* $\leq \ell/2$ can be found in *time* $O\binom{\ell + \frac{d}{2}}{\frac{d}{2}}$

2. *All* dominions of *size* $\leq n^{2/3}$ can be found in *time* $O(n^{d/3})$

The Recursion Tree for Schewe's Algorithm



# children	Dominion preprocessing
$\leq n^{1/3}$	$n^{d/3}$
$\leq n^{2/3}$	$n^{(d-1)/3}$
$\leq n^{3/3}$	$n^{(d-2)/3}$
$\leq n^{(k+1)/3}$	$n^{(d-k)/3}$
$\leq n^{(d-1)/3}$	

$$O\left(n^{\frac{d}{3}+O(1)}\right)$$

Exercise 16

- Prove that

$$\underbrace{\left|\left\{\mu \in \mathbb{N}^{d/2} : \sum_{k=1}^{d/2} \mu_k \leq \ell\right\}\right|}_{M_G(\ell)} = \binom{\ell + \frac{d}{2}}{\frac{d}{2}} = \prod_{i=1}^{d/2} \frac{\ell + i}{i}$$

- Argue that if $\ell = 2 \cdot \lceil n^{2/3} \rceil$ then $|M_G(\ell)| \leq n^{d/3}$ for all $d \geq 8$.

- Prove that if the dominion returned by the progress measure lifting algorithm, using $M_G(\ell)$, is of size $\leq \ell/2$ then there is no other dominion of size $\leq \ell/2$ in G.

EXERCISE 17

- Argue that the running time of Schewe's algorithm can be characterized by the recurrence:

$$\begin{cases} T(n,1) = O(n) \\ T(n,d) \leq T(n,d-1) + T\left(n - \lceil n^{2/3} \rceil, d\right) + O\left(n^{d/3 + O(1)}\right) \end{cases}$$

and hence also by:

$$\begin{cases} T(n,1) = O(n) \\ T(n,d) \leq n^{1/3} \cdot T(n,d-1) + O\left(n^{d/3 + O(1)}\right) \end{cases}$$

- Prove that $T(n,d) = O\left(n^{d/3 + O(1)}\right)$.

Exercise 18

- Give algorithms with the smallest asymptotic worst-case running time bound as you can find, for parity games with 2, 3, 4, 5, 6, 7, and 8 priorities.

PLAN

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. Parity games
6. Better algorithms for parity games
7. Search complexity of computing optimal strategies

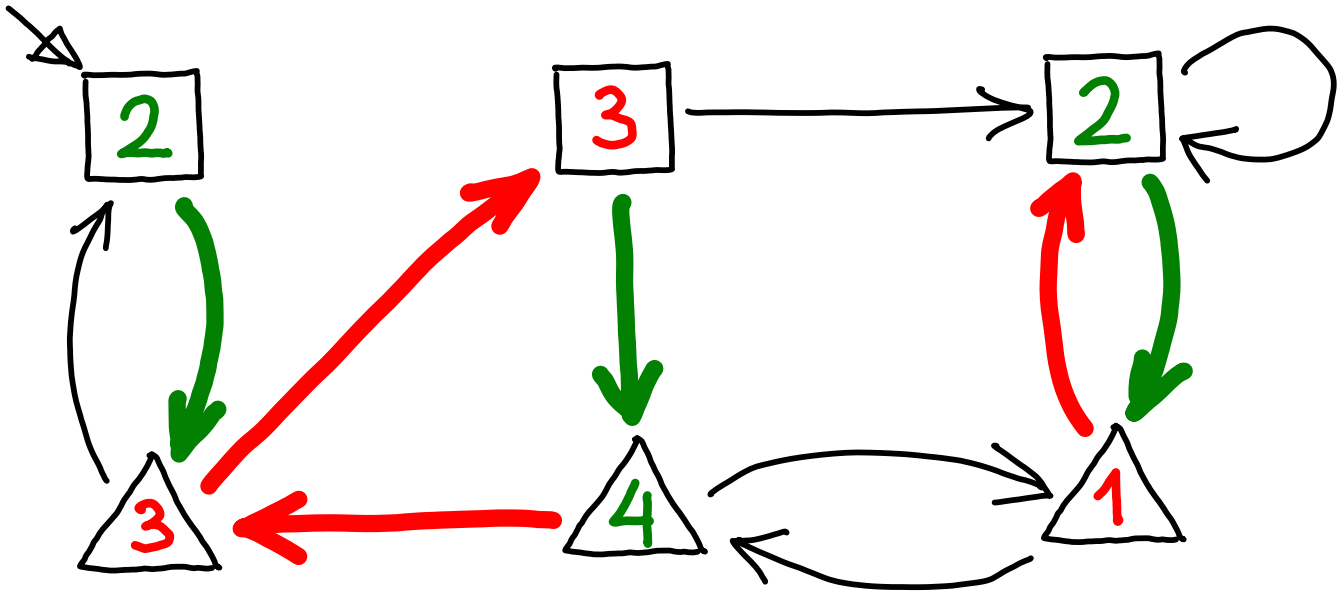
Parity Games

Players:

n vertices, m edges, d priorities

Even = □ = 0

Odd = △ = 1



Winner of an infinite play:

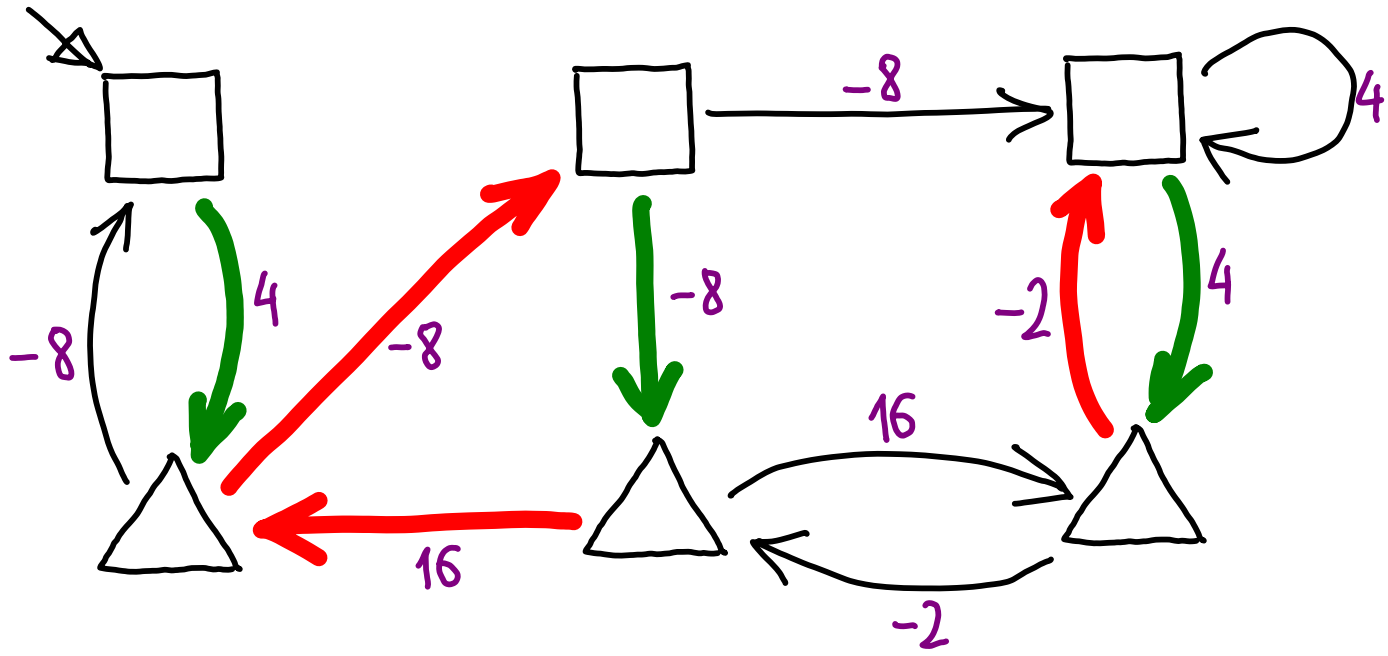
parity of the highest priority occurring infinitely often

Mean-Payoff Games

Players:

MAX = □

MIN = △



Winner of an infinite play $\pi = \langle v_0, v_1, v_2, \dots \rangle$

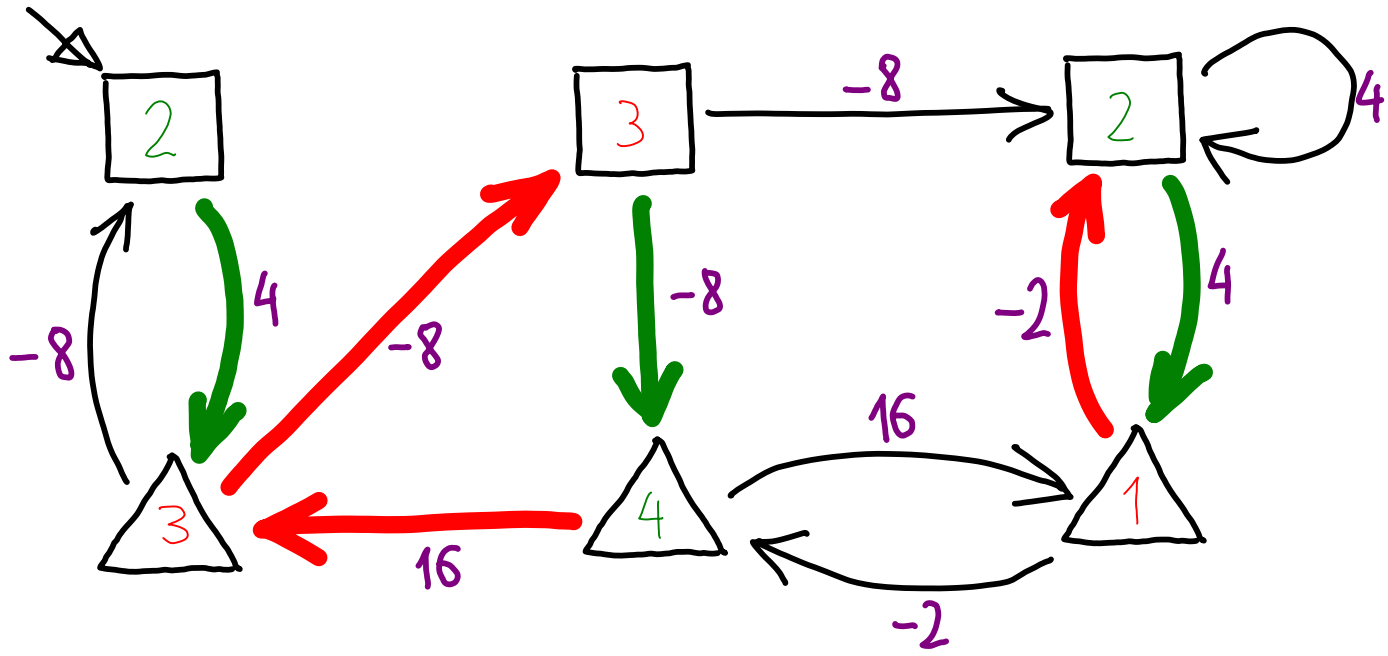
MAX if $\lim_{n \to \infty} \left(\frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1}) \right) \geqslant 0$

Mean-Payoff Games

Players:

Max = □

Min = △



$$(v, w) \longmapsto (-n)^{p(v)}$$

here 2 was good enough

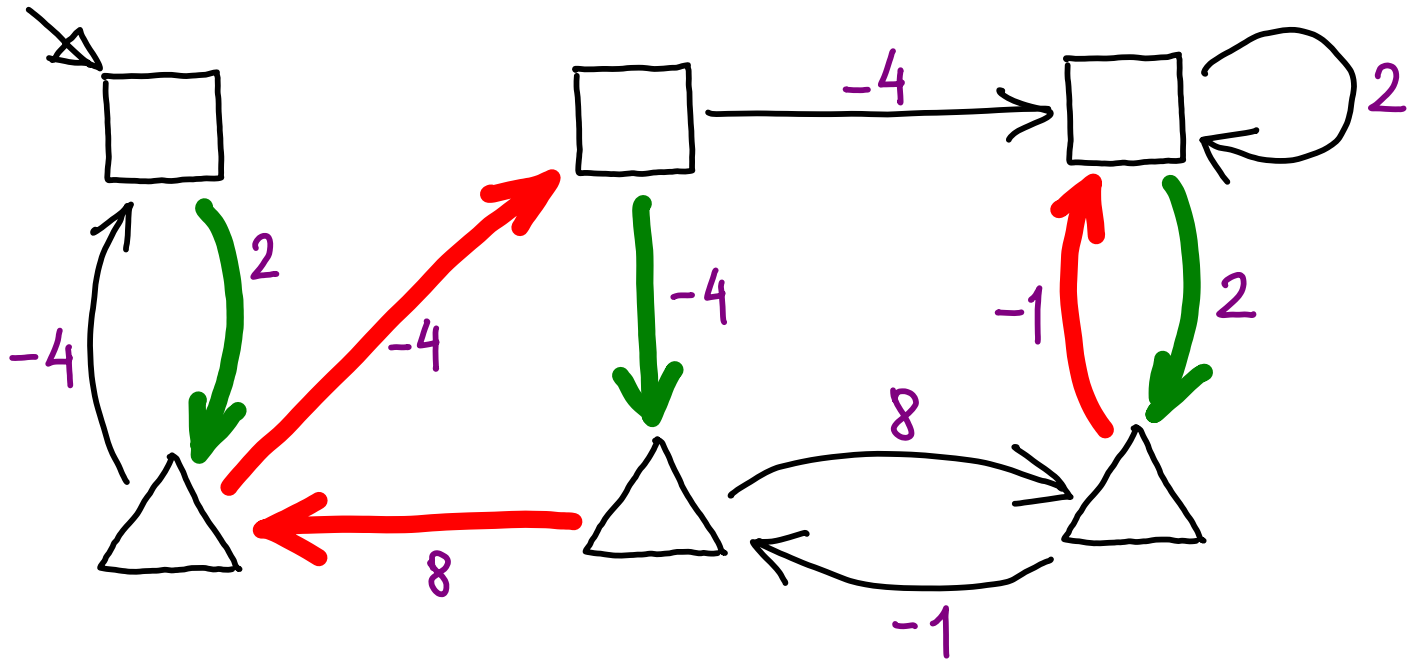
Discounted Games

Players:

MAX = □

MIN = △

$0 < \delta < 1$



Winner of an infinite play $\pi = \langle v_0, v_1, v_2, \dots \rangle$

MAX if $\sum_{i=0}^{\infty} \delta^i \cdot r(v_i, v_{i+1}) \geq 0$

Payoff, Value, Determinacy

$$\pi = \langle s_0, s_1, s_2, \ldots \rangle$$

Mean payoff : $$\mathcal{A}(\pi) = \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} r(s_i, s_{i+1})$$

Discounted total: $$\mathcal{D}_\delta(\pi) = \sum_{i=0}^{\infty} \delta^i \cdot r(s_i, s_{i+1})$$

Lower value: $$Val_*(s) = \sup_{\chi \in \Sigma_{Max}} \inf_{\mu \in \Sigma_{Min}} \mathcal{P}(Play(s, \mu, \chi))$$

Upper value: $$Val^*(s) = \inf_{\mu \in \Sigma_{Min}} \sup_{\chi \in \Sigma_{Max}} \mathcal{P}(Play(s, \mu, \chi))$$

Determinacy: $$Val(s) = Val_*(s) = Val^*(s)$$

COMPARING COMPUTATIONAL COMPLEXITY OF INFINITE GAMES

THM [PURI 1995]

There is a polynomial-time reduction
from parity games to mean-payoff games

THM [1960's]

There is a polynomial-time reduction
from mean-payoff games to discounted games

THM [HARDY LITTLEWOOD 1930's]

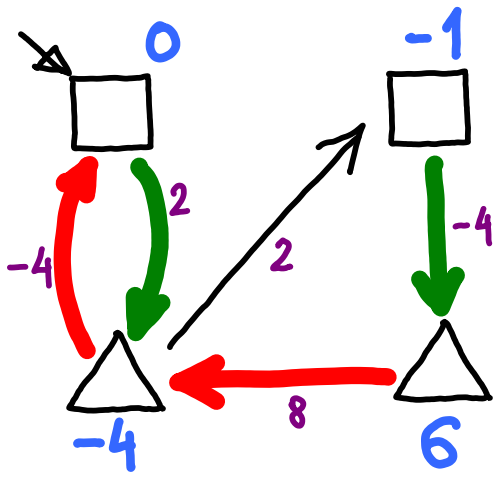
If $\lim_{n\to\infty}\left[\frac{1}{n}\cdot\sum_{i=0}^{n-1} a_i\right]$ exists

then $\lim_{n\to\infty}\left[\frac{1}{n}\cdot\sum_{i=0}^{n-1} a_i\right] = \lim_{\delta\nearrow 1}\left[(1-\delta)\cdot\sum_{i=0}^{\infty}\delta^i\cdot a_i\right]$

Optimality equations for discounted games

Opt(G):

$$v_s = \begin{cases} \max\limits_{(s,t)\in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Max} \\ \min\limits_{(s,t)\in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Min} \end{cases}$$



$\delta = \frac{1}{2}$

Lemma If $V \vDash Opt(G)$ then $V = Val^G$
and positional strategies choosing optimal successor are optimal

Optimality equations for discounted games

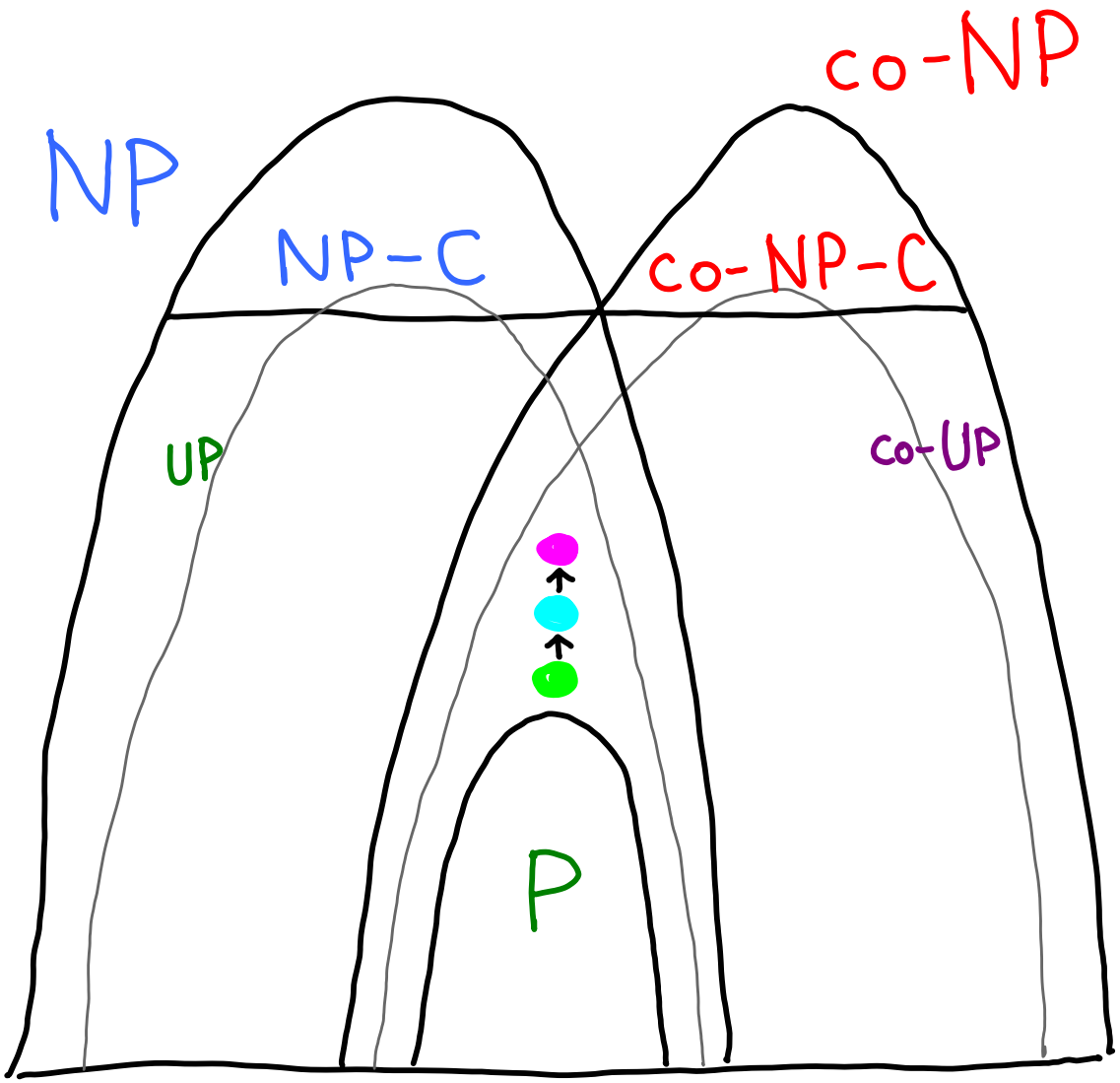
$$F : v_s \longmapsto \begin{cases} \max\limits_{(s,t)\in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Max} \\ \min\limits_{(s,t)\in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Min} \end{cases}$$

Fact $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction

Corollary

- F has a **unique fixed point**, i.e., Opt(G) has a solution
- Discounted, mean-payoff, and parity games:
  - are positionally determined
  - are in UP ∩ co-UP

COMPUTATIONAL COMPLEXITY OF SOLVING INFINITE GAMES

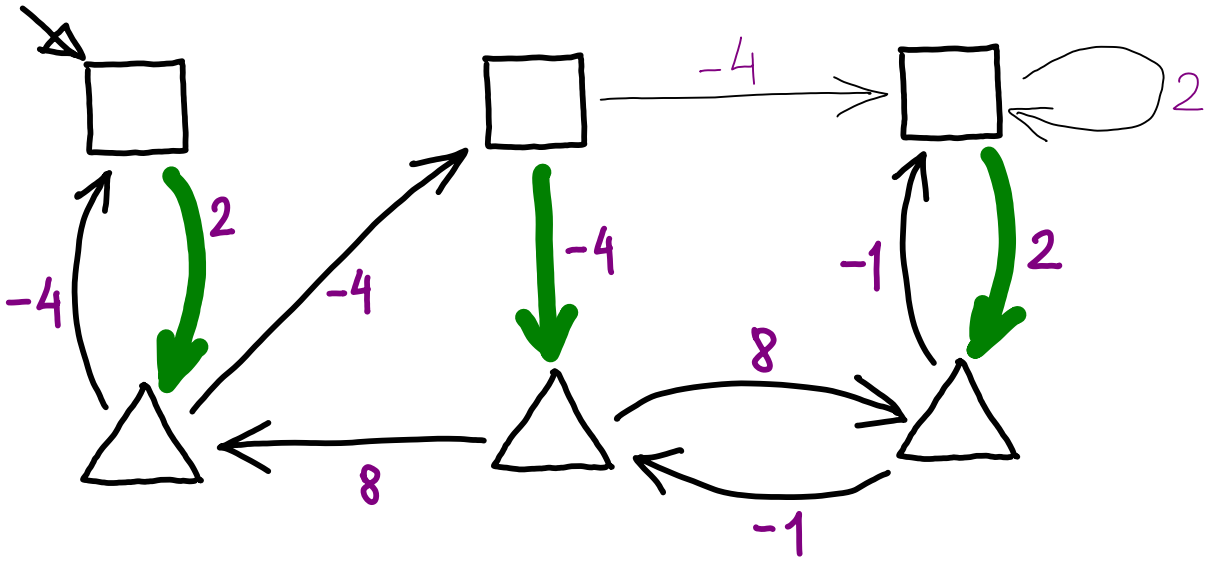


DISCOUNTED GAMES

AVERAGE-REWARD GAMES

PARITY GAMES

2-player Strategy Improvement



$\delta = \frac{1}{2}$

0. Pick $\chi \in \Pi_{max}$

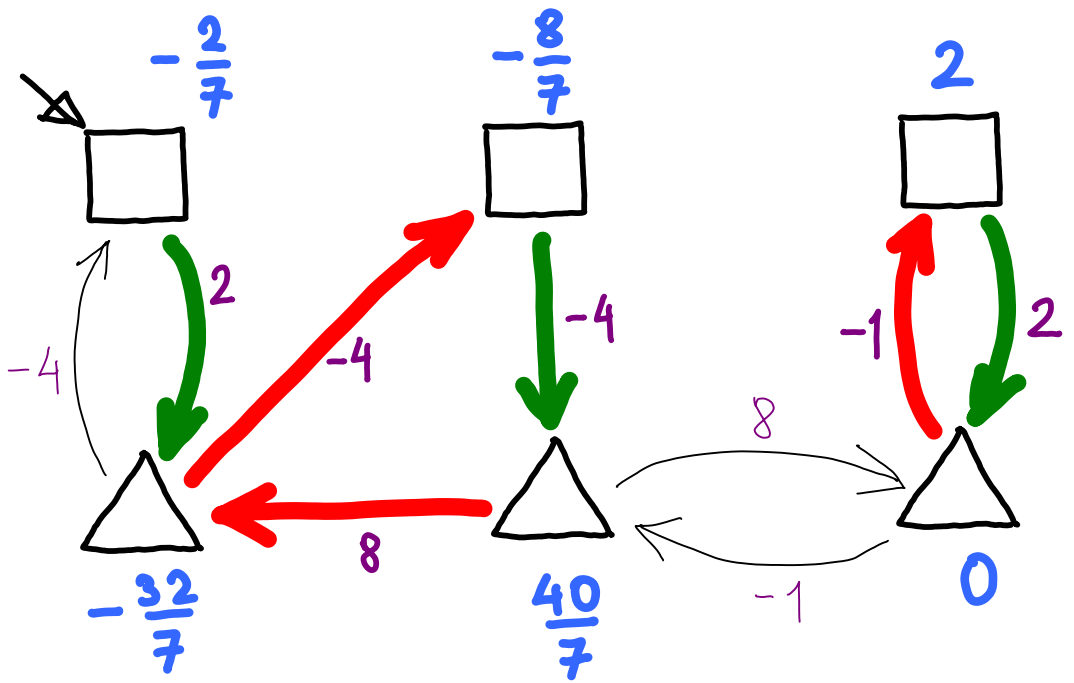
1. Find $V : S \to \mathbb{R}$, such that $V \models \mathrm{OPT}(\Gamma^{\chi})$

2. If $V \not\models \mathrm{OPT}(\Gamma)$
   then $\chi := \mathrm{Improve}(\chi, V)$; goto 1.

Compute the best response μ to χ for Min

Improve locally w.r.t. the best response

2-player Strategy Improvement



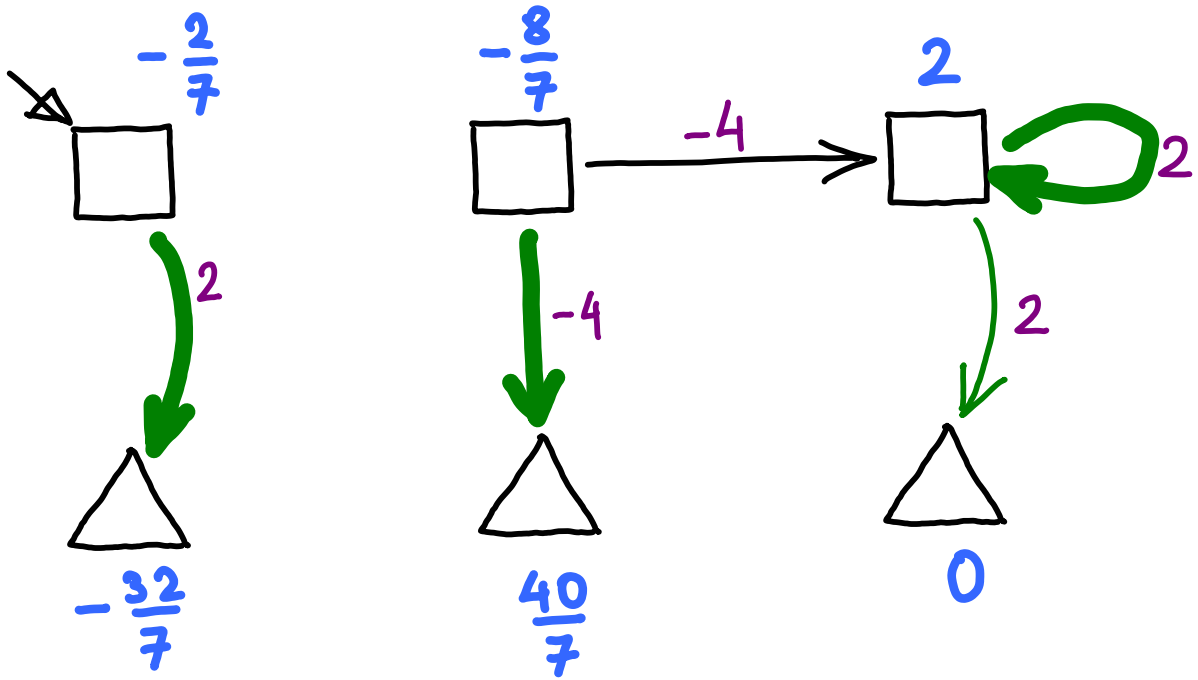
$\delta = \frac{1}{2}$

0. Pick $\chi \in \Pi_{Max}$
1. **Find $V: S \to \mathbb{R}$, such that $V = \mathrm{OPT}(G^{\chi})$**
2. If $V \neq \mathrm{OPT}(G)$
   then $\chi := \mathrm{Improve}(\chi, V)$; goto 1.

Compute the best response μ to χ for Min

Improve locally w.r.t. the best response

2-player Strategy Improvement



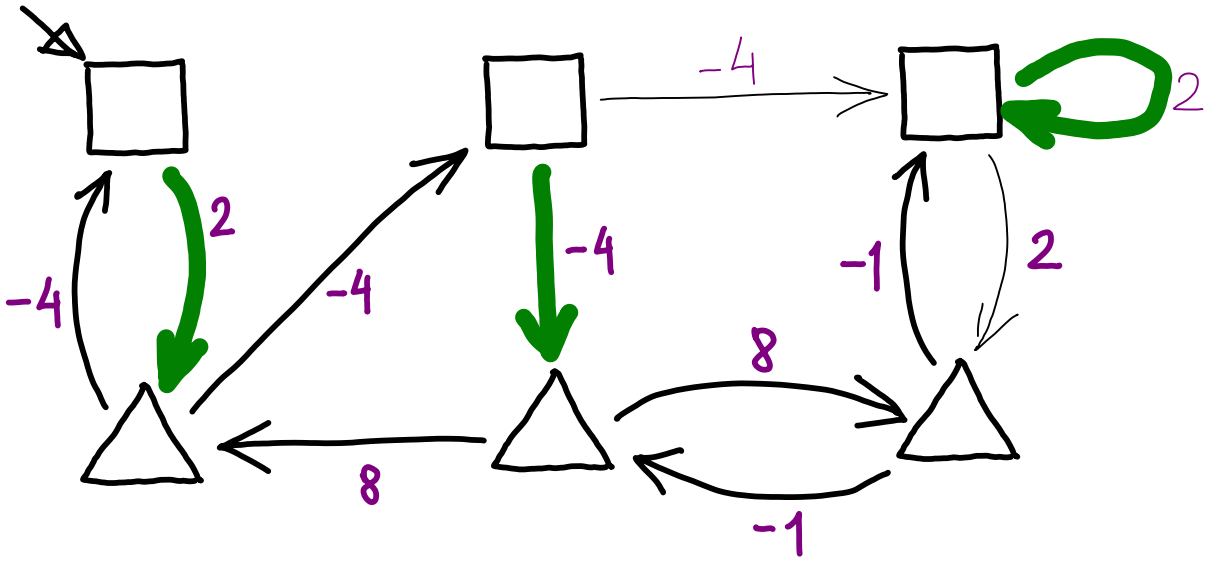
$\delta = \frac{1}{2}$

0. Pick $\chi \in \Pi_{Max}$
1. Find $V : S \to \mathbb{R}$, such that $V \models Opt(G^{\chi})$
2. **If** $V \not\models Opt(G)$
   then $\chi := Improve(\chi, V)$; goto 1.

Compute the best response μ to χ for Min

Improve locally w.r.t. the best response

2-player Strategy Improvement



$\delta = \frac{1}{2}$

0. Pick $\chi \in \Pi_{Max}$
1. **Find $V: S \to \mathbb{R}$, such that $V = \text{Opt}(G^{\chi})$**
2. If $V \neq \text{Opt}(G)$
   then $\chi := \text{Improve}(\chi, V)$; goto 1.

Compute the best response μ to χ for Min

Improve locally w.r.t. the best response

Correctness of 2-player Strategy Improvement

Improvement Lemma

If $V \vDash \mathrm{Opt}(G^{x})$,

$x' = \mathrm{Improve}(x, V)$, and

$V' \vDash \mathrm{Opt}(G^{x'})$

then $V' \geq V$, and

$V' > V$ if $x' \neq x$.

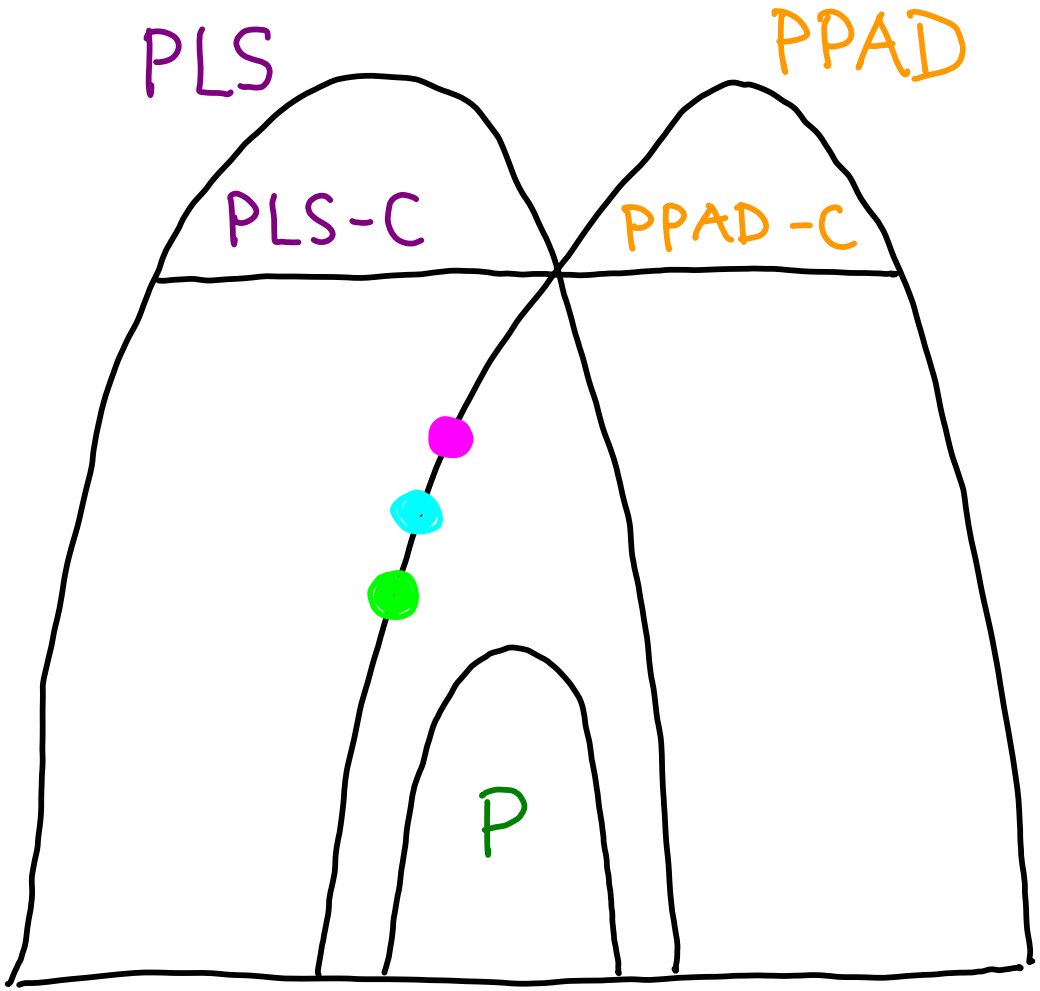
Termination Lemma

Strategy improvement terminates (in $\leq |\Pi_{\mathrm{Max}}|$ steps)

and returns $V \vDash \mathrm{Opt}(G)$.

Corollary Computing the value of discounted, average-reward, and parity games is in PLS

COMPUTATIONAL COMPLEXITY OF SOLVING INFINITE GAMES



DISCOUNTED GAMES

AVERAGE-REWARD GAMES

PARITY GAMES

Linear Complementarity Problem

Given: $M \in \mathbb{R}^{n \times n}$

$q \in \mathbb{R}^n$

Find: $z, w \in \mathbb{R}^n$

such that

linear
$$\begin{cases} z \geq 0 \\ w \geq 0 \\ w = Mz + q \end{cases}$$

complementarity $\{ \; z \perp w$ i.e., $z^T \cdot w = 0$

Fact

If $q \geq 0$

then $(\underset{z}{0}, \underset{w}{q}) \models LCP(M, q)$

Complementary cones of LCP

$$z \geq 0 \quad \perp \quad w = Mz + q \geq 0$$

iff

$$z \geq 0 \perp w \geq 0 \quad \text{and} \quad q = -Mz + Iw$$

iff

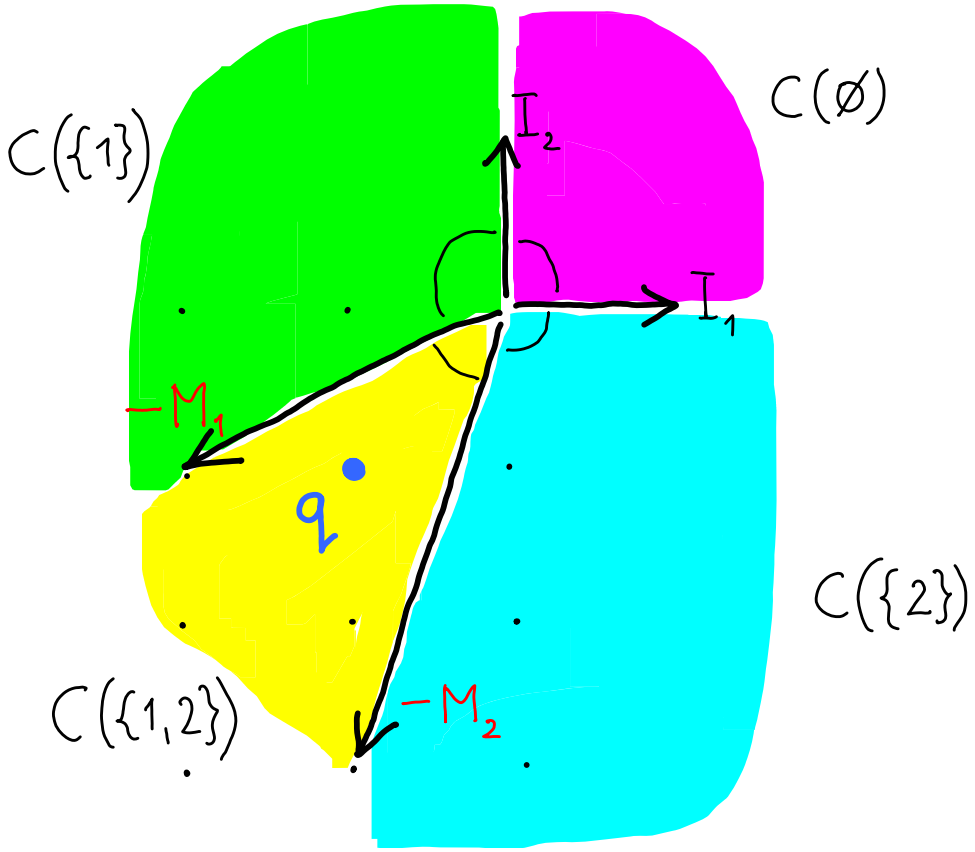
$$q \in C(\alpha) = \text{cone}\left(\{-M_k : k \in \alpha\} \cup \{I_k : k \notin \alpha\}\right)$$

for some $\alpha = \{k : w_k = 0\} \subseteq \{1, \ldots, n\}$

Complementary cones of a matrix

$$z \geqslant 0 \perp w \geqslant 0 \quad \text{and} \quad q = -Mz + Iw$$

$$M = \begin{pmatrix} \overset{M_1}{2} & \overset{M_2}{1} \\ 1 & 3 \end{pmatrix} \qquad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



$q \in C(\{1,2\})$

$\Downarrow$

$(z_1, z_2, 0, 0) \in LCP(M, q)$

where $q = -Mz$

P-MATRICES

DEF $M \in \mathbb{R}^{n \times n}$ is a **P-matrix**
if all its **principal minors** are **positive**

THM M is a P-matrix
iff $LCP(M,q)$ has a **unique solution** for every $q \in \mathbb{R}^n$.

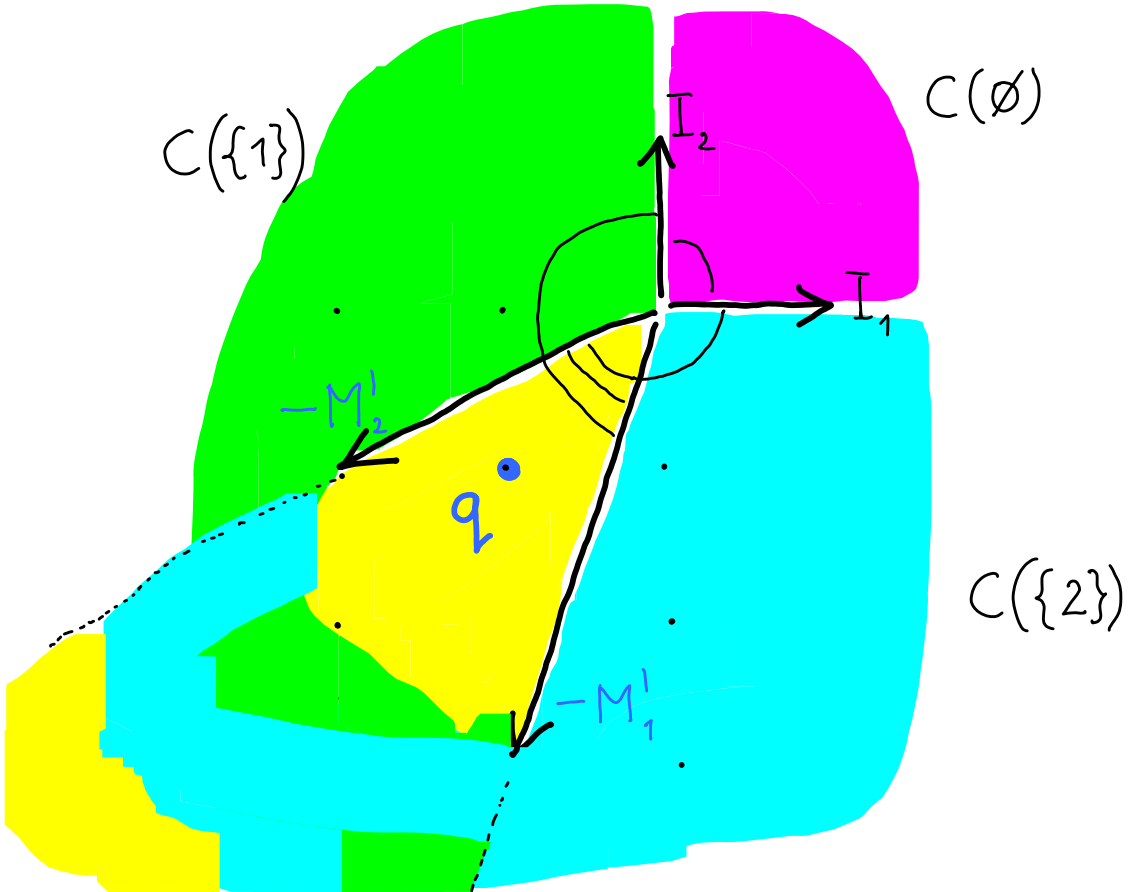
EXAMPLE $M = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $M' = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

- M is a P-matrix: $\det(M_{11}) = 2$
  $\det(M_{22}) = 3$
  $\det(M) = 5$
- M' is not a P-matrix: $\det(M') = -5$

Complementary cones of a non P-matrix

$$w \geqslant 0 \perp z \geqslant 0 \quad \text{and} \quad q = -Mz + Iw$$

$$M' = \begin{pmatrix} \overset{M'_1}{1} & \overset{M'_2}{2} \\ 3 & 1 \end{pmatrix} \qquad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



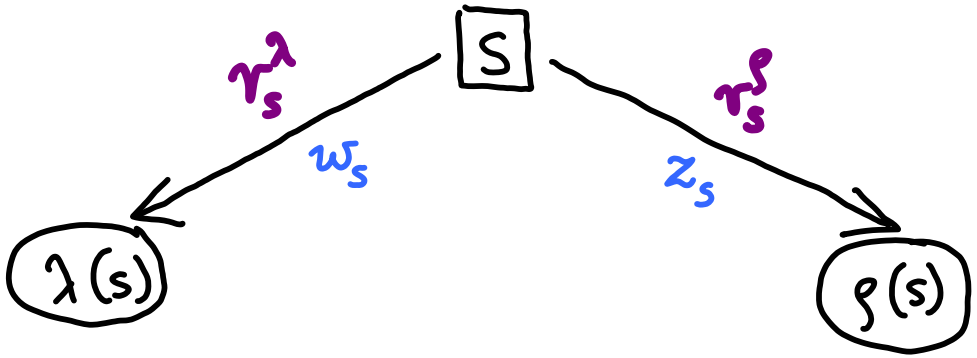
$q \in C(\{1,2\})$

$q \in C(\{1\})$

$q \in C(\{2\})$

Optimality equations and comlementarity

$$v_s = \max \left\{ r_s^\lambda + \delta \cdot v_{\lambda(s)} \, , \; r_s^\rho + \delta \cdot v_{\rho(s)} \right\}$$



slacks

$$v_s = w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)}$$

$$v_s = z_s + r_s^\rho + \delta \cdot v_{\rho(s)}$$

$$z_s, w_s \geq 0 \qquad z_s \cdot w_s = 0$$

complementarity

OPTIMALITY EQUATIONS AND COMLEMENTARITY

$$s \in S_{Max}: \quad v_s = \max\left\{ r_s^\lambda + \delta \cdot v_{\lambda(s)} \,,\; r_s^\rho + \delta \cdot v_{\rho(s)} \right\}$$

$$s \in S_{Min}: \quad v_s = \min\left\{ r_s^\lambda + \delta \cdot v_{\lambda(s)} \,,\; r_s^\rho + \delta \cdot v_{\rho(s)} \right\}$$

Replace max/min with slacks and complementarity

$$s \in S_{Max}: \quad \begin{aligned} v_s &= w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)} \\ v_s &= z_s + r_s^\rho + \delta \cdot v_{\rho(s)} \end{aligned}$$

slacks

$$s \in S_{Min}: \quad \begin{aligned} v_s &= -w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)} \\ v_s &= -z_s + r_s^\rho + \delta \cdot v_{\rho(s)} \end{aligned}$$

$$s \in S: \quad w_s \geqslant 0 \;\perp\; z_s \geqslant 0$$

complementarity

Reduction to LCP: rewrite

$s \in S_{max}$:

$$v_s = w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)}$$
$$v_s = z_s + r_s^\rho + \delta \cdot v_{\rho(s)}$$

slacks

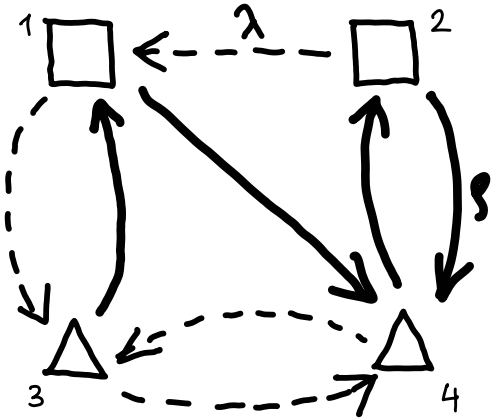
$s \in S_{min}$:

$$v_s = -w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)}$$
$$v_s = -z_s + r_s^\rho + \delta \cdot v_{\rho(s)}$$

$$\rightsquigarrow$$

$$(\hat{I} - \delta \cdot \hat{T}^\lambda) v = w + \hat{I} r^\lambda$$
$$(\hat{I} - \delta \cdot \hat{T}^\rho) v = z + \hat{I} r^\rho$$

Example



$$\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\hat{T}^\rho = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Reduction to LCP: eliminate v

$$\left(\hat{I} - \delta \cdot \widehat{T^{\lambda}}\right) v = w + \hat{I} r^{\lambda}$$

$$\left(\hat{I} - \delta \cdot \widehat{T^{s}}\right) v = z + \hat{I} r^{s}$$

Eliminate v:

$$w + \hat{I} r^{\lambda} = \underbrace{\left(\hat{I} - \delta \cdot \widehat{T^{\lambda}}\right) \cdot \left(\hat{I} - \delta \cdot \widehat{T^{s}}\right)^{-1}}_{M} \cdot \left(z + \hat{I} r^{s}\right)$$

$$w = Mz + q$$

$$w \geq 0 \quad \perp \quad z \geq 0$$

$$M = \left(\hat{I} - \delta \cdot \widehat{T^{\lambda}}\right) \cdot \left(\hat{I} - \delta \cdot \widehat{T^{s}}\right)^{-1}$$

$$q = M\left(\hat{I} r^{s}\right) - \left(\hat{I} r^{\lambda}\right)$$

Reduction to LCP: eliminate v

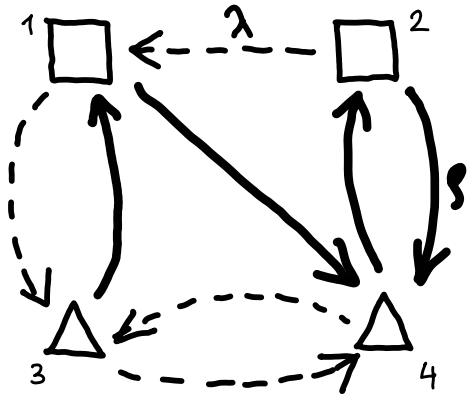
$$w = Mz + q$$

$$w \geq 0 \quad \perp \quad z \geq 0$$

$$M = (\hat{I} - \delta \cdot \widehat{T^{\lambda}}) \cdot (\hat{I} - \delta \cdot \widehat{T^{s}})^{-1}$$

$$q = M(\hat{I} r^{s}) - (\hat{I} r^{\lambda})$$

Example



$$\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\widehat{T^{s}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$(\hat{I} - \delta \cdot \widehat{T^{s}}) = \begin{pmatrix} 1 & 0 & 0 & -\delta \\ 0 & 1 & 0 & -\delta \\ \delta & 0 & -1 & 0 \\ 0 & \delta & 0 & -1 \end{pmatrix}$$

STRICTLY DIAGONALLY DOMINANT MATRICES

FACT [Levy-Desplanques]

If $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant, then A is non-singular.

FACT $(\hat{I} - \delta \cdot \hat{T}^{\lambda})$ and $(\hat{I} - \delta \cdot \hat{T}^{\beta})$ are s.d.d.

COROLLARY $M = (\hat{I} - \delta \hat{T}^{\lambda}) \cdot (\hat{I} - \delta \hat{T}^{\beta})^{-1}$ is well-defined

P-MATRICES OF THE FORM $B \cdot C^{-1}$

THM [Johnson-Tsatsomeros '95]

Let $M = B \cdot C^{-1}$, where $B, C \in \mathbb{R}^{n \times n}$

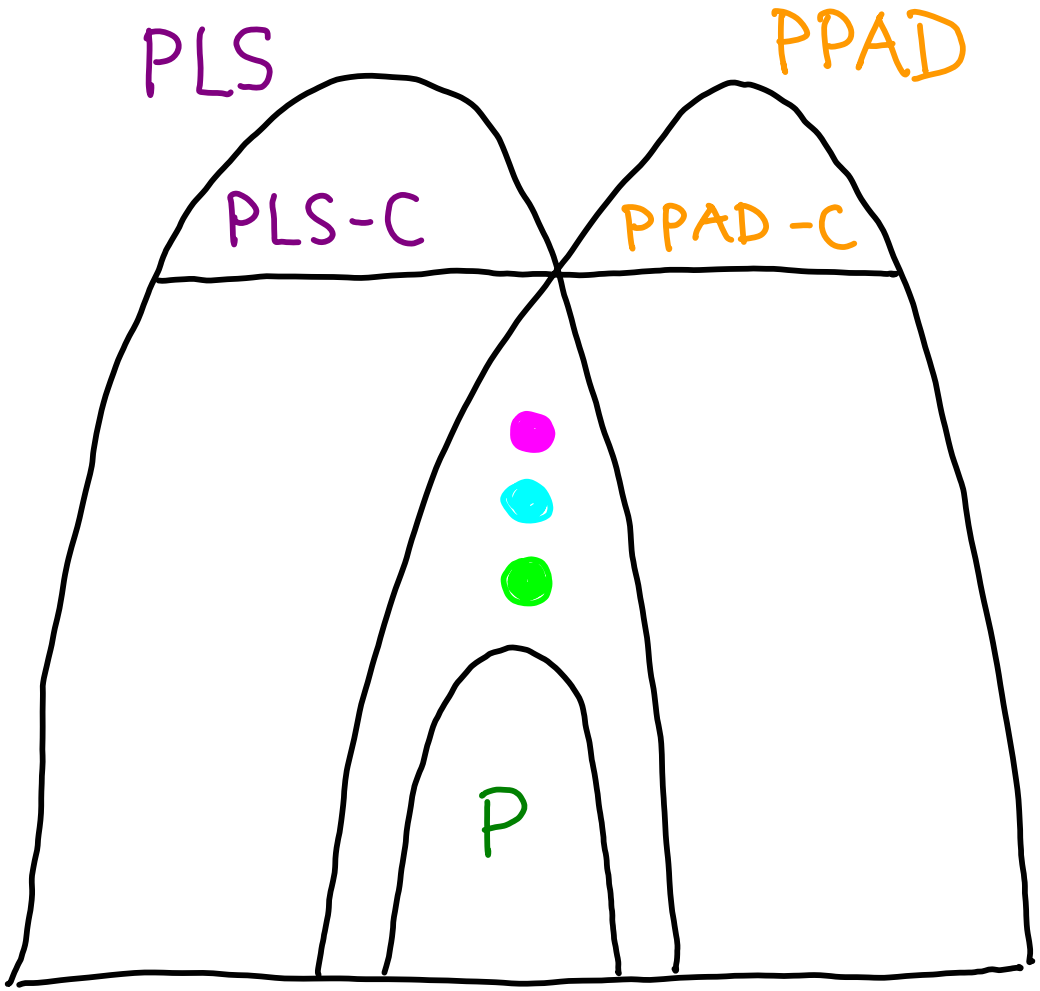
Then, M is a **P-matrix**

if $T \cdot B + (I - T) \cdot C$ is non-singular for all $T \in [0, I]$

COROLLARY

- $M = (\hat{I} - \delta \cdot \hat{T}^{\lambda}) \cdot (\hat{I} - \delta \cdot \hat{T}^{s})^{-1}$ is a **P-matrix.**
- Computing the value of discounted, average-reward, and parity games is in **PPAD**

COMPUTATIONAL COMPLEXITY OF SOLVING INFINITE GAMES



DISCOUNTED GAMES

AVERAGE-REWARD GAMES

PARITY GAMES

What is q?

Lemma

Let v^ρ be the vector of values of strategy pair ρ, i.e.,

$$v^\rho = r^\rho + \delta \cdot T^\rho \cdot v^\rho$$

Then:

$$q = \hat{I}\left(\left(r^\rho + \delta \cdot T^\rho \cdot v^\rho \right) - \left(r^\lambda + \delta \cdot T^\lambda \cdot v^\rho \right) \right)$$

take "right" edge and then follow ρ

take "left" edge and then follow ρ

Algorithm 1: Strategy improvement revisited

Def. $s \in S_{Max}$ is switchable (for strategy pair ρ) if

$$r_s^{\rho} + \delta \cdot v_{\rho(s)}^{\rho} < r_s^{\lambda} + \delta \cdot v_{\lambda(s)}^{\rho}$$

---

1. Start with arbitrary ρ, λ.
2. While there is a switchable vertex (for ρ) do
3. Find ρ' such that:
   a) $\rho' \restriction S_{Max} = \rho \restriction S_{Max}$
   b) no $s \in S_{Min}$ is switchable for ρ'

   } $\rho' \restriction S_{Min}$ is a best response to $\rho \restriction S_{Max}$
4. Switch all (or some) switchable vertices in S_{Max}

ALGORITHM 2: MURTY'S "LEAST-INDEX" METHOD

1. Fix a permutation ("indexing") of states.
2. While there is a switchable vertex do
3. Switch the switchable vertex with **least index**.

THM Murty's algorithm terminates (in $\leq 2^n$ steps).

"Nested" strategy improvement: 

New algorithm: 

Algorithm 3: Cottle-Danzig

1. For $s = 1, 2, \ldots, n$ do
2. If s is switchable then
3. [Drive r_s^ρ until s becomes **indifferent**
4. While driving r_s^ρ,
   switch a vertex in $\{1, 2, \ldots, s-1\}$
   when it becomes **indifferent**
5. Switch s and restore r_s^ρ

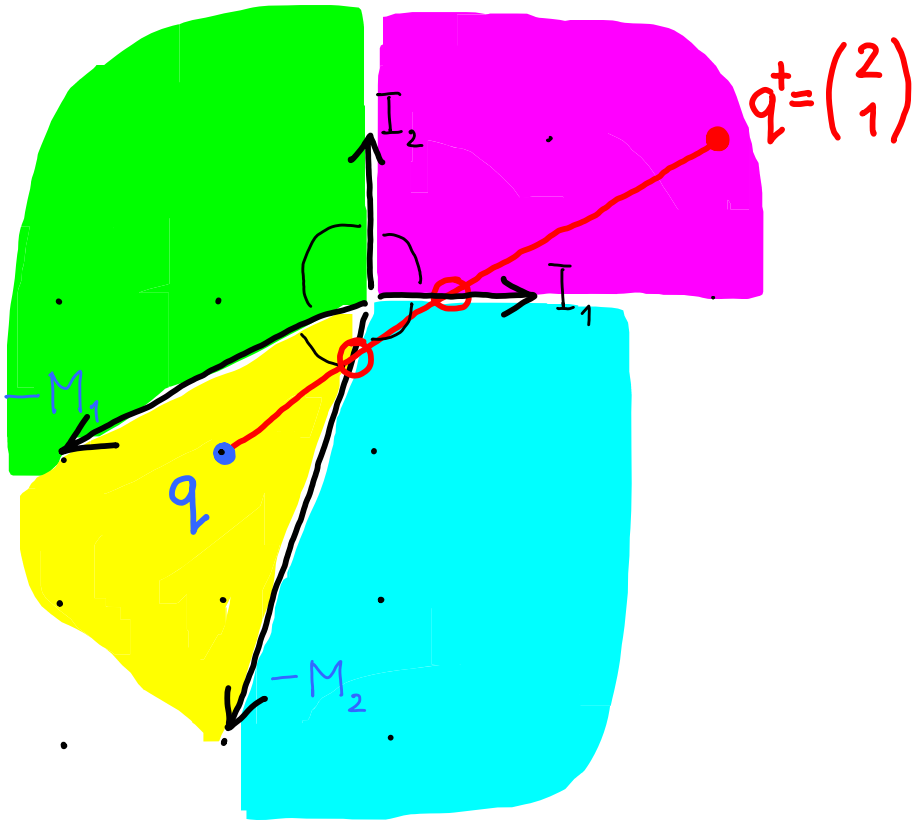
Fact Cottle-Danzig terminates (in $\leq 2^n$ steps)

Algorithm 4 : Lemke

1. Drive r^λ ("linearly"), until no vertex is switchable.
2. Drive r^λ ("linearly") back, until original r^λ is restored.

   While driving r^λ back,
   switch vertices when they become indifferent.

$$q = M(\hat{I} r^s) - (\hat{I} r^\lambda)$$



EXPONENTIAL TIME LOWER BOUNDS

THM [SAVANI, VON STENGEL 2004]

Lemke-Howson algorithm needs exponential time to find a Nash equilibrium

THM [FEARNLEY, J., SAVANI 2009]

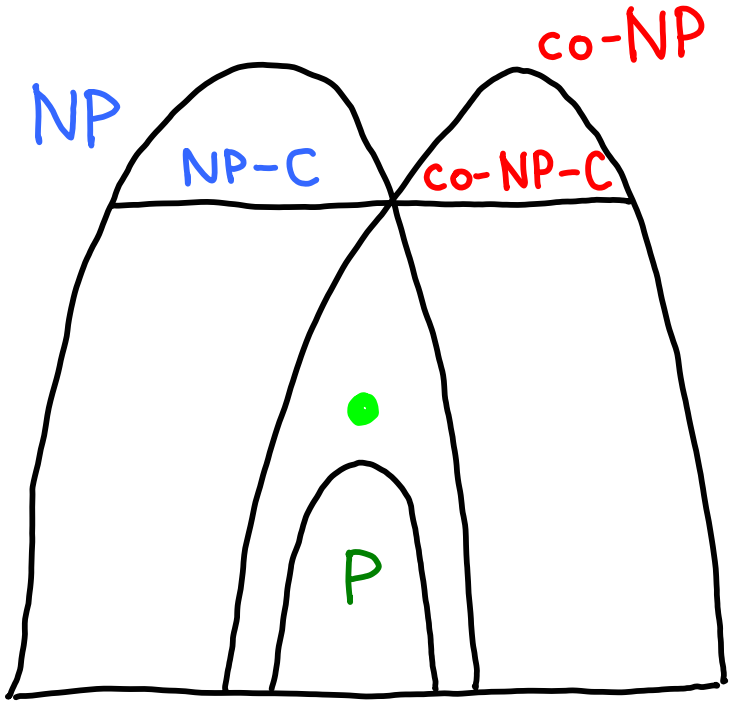
Murty's, Cottle-Danzig, and Lemke's algorithms need exponential time to solve parity games

LEMKE CAN BE FAST ?

THM [ADLER, MEGGIDO 1985]

There is an implementation of Lemke's algorithm that terminates in quadratic number of steps on random linear programs

How difficult is it to find a winning strategy?



P-MATRIX LCP
↑
COMPUTING OPTIMAL STRATEGIES