

Titles and Abstracts for PMB10

Ken Archer (USA)

Bolzano's Beiträge and the Revival of the Classical Model of Science

Clearly the most consistent theme throughout the works of Bolzano, from his first publication in 1804 to his magisterial *Theory of Science* (TS) in 1837, is the distinction between proofs that prove and proofs that explain. While scientific and mathematical proofs in Bolzano's time proved scientific conclusions against standards of certainty and obviousness, Bolzano called for a new presentation of these conclusions that explained their grounds from a more objective point of view. Throughout, Bolzano mentions the classical model of science, with primary reference to Aristotle, as a key source for the types of scientific presentation needed in his day.

The classical heritage of Bolzano's theory of science is underdeveloped by most commentators, despite the reception by Bolzano's contemporaries such as Menelaos who remarked of TS, "throughout, the author assumes the old, strictly objective or dogmatic, viewpoint, in contrast to the contemporary, which is based on the psychological self-consciousness of the thinking mind".

An appreciation of the classical heritage of Bolzano's work helps to reveal and explain two critical elements of Bolzano's logic that are generally presented in terms of Fregean concerns alien to Bolzano. These two elements are subjective, psychologistic proofs and propositions-in-themselves. First, incorporation of subjective proofs into, and not expelling them from, logic was Bolzano's mission. This is contrary to most interpretations of Bolzano that see him attacking and expelling subjective, psychologistic proofs from logic, an interpretation that is truer of Frege than it is of Bolzano.

Second, this incorporation of subjective proofs into logic occurs when one analyzes and transforms the intuitively presented propositions of such proofs into propositions which have the character of propositions-in-themselves. Bolzano is often accused of Platonism with regard to his doctrine of propositions-in-themselves, ascribing to them a role similar to Frege's *Gedanken*. However, propositions-in-themselves are not different propositions from spoken propositions; rather, they are the intended meanings that constitute spoken propositions, whose full meanings are generally inaccessible to their speakers. In this regard, Bolzano answers this accusation as follows: "What is meant, then, when we assert that there are such truths? Nothing, I answer, but that certain propositions have the character of truths in themselves." It is through the act of transformation that a spoken proposition takes on this character and discloses its full meaning. Thus, while subjective propositions have existence, Bolzano goes out of his way to deny existence to propositions-in-themselves. They have neither *Sein*, nor *Dasein*, nor *Existenz*, nor *Wirklichkeit*.

Both misunderstandings of Bolzano reflect a Fregean reading of Bolzano that does not take seriously the classical references made by Bolzano throughout his corpus to help the reader better understand his doctrines of subjective proofs and objectives propositions-in-themselves. This paper seeks to present these two doctrines based on a close reading of Bolzano's work as well as an exploration of the classical references made by Bolzano for these doctrines.

Arianna Betti (Amsterdam)

What's wrong with Bolzano's tentative definition of grounding in WL §221?

Bolzano's *Wissenschaftslehre*, I shall assume in this talk, can be fruitfully read as an attempt at a formal characterization of grounding (*Abfolge*) – a notion whose importance and significance can be best understood when studied from the perspective of the *Beyträge*. The role that grounding plays in the *Beyträge* is a fundamental element to show that Bolzano's notion of grounding is a systematization of the technical notion of explanatory proof in the context of an axiomatic conception of (proper) science – where a paradigmatic example of proper science is mathematics.

By 'attempt at a formal characterization of grounding' in the *Wissenschaftslehre* I mean Bolzano's attempt at a definition of grounding as a special kind of derivability (*Ableitbarkeit*). As known, in some parts of the *Wissenschaftslehre* Bolzano takes grounding to be a primitive notion, but in §221 Bolzano does offer a tentative definition of grounding as "that order among truths in virtue of which from the smallest amount of simple premises the biggest amount of the remaining truths can be derived [*ableiten lassen*] as mere conclusion". It is unclear why Bolzano is dissatisfied with this definition. A key reason for his dissatisfaction, is, apparently, an argument he raises in WL§200 against the possibility of defining grounding in terms of derivability, i.e. against the possibility of reducing all cases of grounding to cases of *formal* grounding. As I shall point out in my talk, the argument, which is not easy to understand, is rather puzzling.

I shall argue that the §221 definition makes perfect sense in light of Bolzano's ideal of science as set out in the *Beyträge*, and, on the basis of an analysis of the WL§200 argument, I shall conjecture that the reason Bolzano is dissatisfied with his tentative definition is that he asks too much of grounding. More specifically, my hypothesis is that Bolzano's difficulties come from the fact that he problematically expects grounding to cover two different notions of explanation, which I shall call *internal* and *external* explanation – a fact that made it difficult for him to arrive at a rigorous way to dispose a number of sciences in relations of subordination (or presupposition).

Johan Blok (Netherlands)

On the Conception of the Beyträge: the Start of Bolzano's Philosophical Thinking

It is with the *Beyträge* that Bolzano took his first public step on the field of philosophy of mathematics and logic. It was the result of several years of study of many texts on logic, philosophy and mathematics. Two of the most important motivating topics are the state of mathematics and the distinction between analytic and synthetic judgments.

To start with the latter, the *Beyträge* offers the outcome of what can be seen as Bolzano's reaction to the famous Kant-Eberhard debate: science essentially consists of synthetic judgments. Since he did not yet develop his own version of the distinction, it is far from clear what synthetic exactly means in this context. Did Bolzano adhere to Kant's version and if so, what was his understanding of Kant's distinction?

In relation to the first motivating topic, the state of mathematics, the question rises whether

Bolzano was the first to complain about this in the eighteenth century, since generally mathematics was treated as the paradigmatic example of apodictic knowledge. What exactly raises these complaints and are they related to the then dominating treatment of and views on mathematics?

In this paper I will start answering these questions by investigating Bolzano's notes, among them the recently published notebooks of 1803-1810. These notes contain several relevant passages where Bolzano discusses other authors, like J.G.E. Maass and J.A.C. Michelsen. Maass is relevant insofar the synthetic/analytic distinction is described in terms of relations between ideas, as the Stanford Encyclopedia writes about the later *Wissenschaftslehre*.

The other relevant author to whom Bolzano refers is J.A.C. Michelsen, a reputable professor in mathematics and physics. Michelsen wrote a book entitled *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern* that was published in 1789. It is exceptional for the eighteenth century in which mathematics was generally regarded as the paradigmatic example of apodictic knowledge because of its topic: it is devoted to worries about the state of mathematics. I hope to show how this text contributed to Bolzano's complaints about the state of mathematics. Investigation of Bolzano's notes on these authors promises to reveal more about Bolzano's start as a philosopher and logician.

Anne-Sophie Brüggem (Tübingen)

Bolzano's method of variation and Frege's "unsaturated functions"

At the heart of Bernard Bolzano's method of variation lies the idea that the parts of a *Satz an sich* can systematically be replaced since they can be regarded as "changeable" ("veränderlich", *Wissenschaftslehre* § 147). This idea reminds of Gottlob Frege's notion of unsaturated language-functions such as "x is red" which contain "changeable" parts as well. This conception allowed Frege to treat language expressions as decomposable into a function term and an argument term like mathematical functions and to finally develop a systematic way of separating the form and content of propositions.

In my talk I would like to compare Bolzano's notion of variation and Frege's notion of unsaturated functions. Although the basic idea behind the two accounts seems to be the same I will show that there are fundamental differences between the two methods and explain why this is the case. Since Bolzano's concept of variation (as Bolzano himself remarks) actually is no more than a metaphoric description it therefore first of all needs to be spelled out in nonmetaphoric terms. I then will discuss two of Bolzano's central notions since the reasons for the differences between Bolzano's and Frege's approach can be traced back to at least these two concepts: Bolzano's notion of *Satz an sich* and his notion of a function, which shows not to be congruent with Frege's concept of a function.

Paola Cantù (Aix-Marseilles)

Changes in Bolzano's definition of mathematics

The paper will discuss some changes in Bolzano's definition of mathematics by comparing some passages from the *Beyträge*, *Wissenschaftslehre* and *Grössenlehre*: is mathematics a theory of forms or a theory of quantities? Relevant changes between 1810 and 1848 will be analysed with respect to the distinction between analytic and synthetic propositions, and to

Kant's influence. It will be argued that the change in Bolzano's views can be better explained if one interprets Bolzano's philosophical stance in the *Beyträge* as deeply influenced by Kant's philosophy: notwithstanding the harsh criticism to be found in the *Appendix*, Bolzano is still somehow Kantian, or at least more than he thinks. It is only in the *Wissenschaftslehre* and in the *Grössenlehre*, where Bolzano develops a new logical theory and better defines some mathematical concepts that he finally distances himself from Kant, and moves from a foundational perspective that is external to mathematics to a philosophical analysis that is related to the solution of certain specific mathematical problems.

The aim of the paper is to explain Bolzano's changes in the definition of mathematics, which are less radical than one might think. Several issues are common to the three works: 1) the definition of mathematics does not depend on its objects; 2) mathematics is exclusively conceptual; 3) its propositions are partly analytic and partly synthetic; 4) its objects are not just numbers, and concrete continuous magnitudes, or magnitudes that can be measured. Notwithstanding some similarities, there are relevant issues that change in the three works: 1) mathematical objects are possible entities in 1810 but in 1837 and in 1848 they are conceived as ideas in themselves, which cannot be possible, because they can never become actual; 2) mathematics is defined as a theory of forms in the *Beyträge* and as a science of quantities in the *Grössenlehre*; 3) quantity is considered as synonymous with number or measurable magnitude in 1810 and as a magnitude (or quantity) in general later on; 4) in the *Beyträge* Bolzano remarks that analytic truths should not occur in a scientific treatment, given that they are only conventions used to recall the concept designated by a certain word, but in the *Wissenschaftslehre* analytic truths are considered as relevant enough to deserve a place in a scientific system, and even a proof.

These changes in Bolzano's account of mathematics are interpreted as a move from philosophy to logic, from the point of view of a philosopher who is still deeply influenced by Kant and is thus mainly interested in the discussion on the foundations of mathematics, to the perspective of a mathematician who develops some logical tools to solve specific problems arising in the mathematical practice (as in the case of objectless ideas).

Stefania Centrone (Hamburg)

Bolzano and the Canon of Reciprocity

The claim that intension and extension stand in an inverse relation, the so called *Canon of Reciprocity*, says that the extension of a concept A is properly included in the extension of a concept B if, and only if, the conceptual components of B constitute a proper part of the conceptual components of A. In his monumental *Wissenschaftslehre* Bolzano traces back the *Canon* to the *Logic of Port Royal* (it has been however observed in the literature that Bolzano's contention is not tenable, cp. W. Künne, *Versuche über Bolzano*, Academia Verlag 2008, 213n.)

In the present paper I will analyse those passages from Bolzano's *Theory of Science* and from his 1810 booklet *Contribution to a more well-founded presentation of mathematics* that are explicitly dedicated to the *Canon* and to questions related with its refutation, in particular §64 (*Whether the parts of an idea are the same as the ideas of the parts of its object*) and §120 (*Concerning the claim that intension and extension stand in an inverse relation*) of the *Theory of Science*, as well as *Contribution II A (On descriptions, definitions and classification)*, §5.

Bolzano's refusal of the *Canon* in his more mature work leads him to accept the following:

- [*] There are concepts C such that all objects that fall under C have a certain property P which, however, does not appear among the conceptual components of C.

For instance, all actual objects are possible objects, but, according to Bolzano, [possible] does not happen to be a component of the concept [real] (I follow here the convention usually adopted in the literature to indicate the idea expressed by the word "F" by means of "[F]"). As Wolfgang Künne correctly points out (cp. Künne *cit.*, 213 f.), in his *Contributions* Bolzano explicitly states that

- [**] the *genus proximus* is always a component of a *species* which is subordinated to it

which more or less amounts to the negation of [*]. Thus, Bolzano appears to accept the *Canon* in his *Contribution*. However, not much is said on the circumstance that Bolzano clearly perceives the tension caused by his acceptance of [**] on the one hand, and his recognition that there are *simple* species-concepts, on the other hand.

On the basis of a close reading of the indicated texts, my paper pursues three complementary goals:

- (i) to bring to light and reconstruct Bolzano's arguments in the *Theory of Science*, supporting his refusal of the *Canon*;
- (ii) to bring to light the above mentioned tension in the *Contribution* between his alleged acceptance of the *Canon* (in particular of its consequence [**]) and his acceptance of species-concepts that are simple;
- (iii) and finally, in this context, to discuss some critical aspects concerning Bolzano's way of conceiving extensions of concepts as aggregates (*Inbegriffe*) of objects.

Anita Kasabova (Sofia)

Bolzano and/or Meinong on chimeras

Non-existing entities or chimeras are a problem, in the sense that we talk, write and sing about them, as well as reading numerous texts about them, from mythology to mathematics. We understand chimeras, but how can that be, since chimeras do not exist? I argue that chimeras interact with *realia* as objects of communication between utterers or narrators and addressees. There are causal relations between things which do not exist, existing things and our representations of them and their properties in the world in which we live. Consider the symbolic capital in stock markets and real investments in chimerical interests, investments which result in real profits or losses.

Bolzano and Meinong develop semantics of chimeras as possible or impossible *entia* in order to provide answers to the epistemological question: are chimeras knowable *a priori* and, if so, how? This epistemological question is important for previsions or visualizing what will or should be, as well as for making predictions about whether stocks will raise or fall – that is, for making statements about future (or past) existence. For Bolzano, the object of a presentation is its referent (although he also accepts presentations or names without referents). He analyzes the status of non-pictorial and objectless presentations

(*gegenstandslose Vorstellungen*) such as [a 997 year old man], [quark] or [round square] and the referential relation between them and their objects. Meinong, on the other hand, examines the objectual status (*Gegenständlichkeit*) of entities such as nymphs, quarks or round squares which he classifies as intentional 'higher-order' entities that subsist (*bestehen*), for example as objectives (*Zielgegenstände*), although they are not real.

I show that both authors contribute to the discussion about possible and impossible objects by using the late scholastic notions of *suppositio* or assumption and *ampliatio* or extension. Past and future tenses, as well as participles such as *wished*, *thought*, *cited*, or *remembered* have an ampliative function in relation to things which do not exist at the time of the utterance. In addition, the presentations of non-existing objects are components of valid judgments or assumptions and existential judgments follow from assumptions or presuppositions about eventual or questionable, possible or impossible objects.

Anita Konzelmann Ziv (Basel)

"Wise Self-Deception" – An Epistemic Challenge

In one of his "*Erbauungsreden*" for the students of philosophy at the University of Prague, Bolzano defends the view that self-deception is not only a necessary but also a highly virtuous practice in the pursuit of truth. Given that deception, delusion and the possibility of error are the main obstacles of knowledge that epistemology must deal with, a rationalist's public call for self-deception seems at least surprising. The aim of my talk is to consider, on the one hand, the extent to which self-deception can be rationally motivated, and, on the other hand, the possibility of a person voluntarily believing *p* whilst actually believing non-*p*. I examine Bolzano's analysis of self-deception, outlining some of the relations it bears to his notions of judging, believing and knowing. I then discuss how the cognitive dissonance and instability resulting from self-deception can be accommodated in the conception of responsible epistemic agency.

Wolfgang Künne (Hamburg)

Some Notes on Bolzano's Semiotic

Abstract to follow.

Sandra Lapointe (Kansas State University/University of Leipzig)

On Begründungen

Bolzano's views on proof rest on a crucial distinction between three notions: grounding (*Abfolge*), objective justification (*objective Erkenntnisgrund*) and what we may call objective demonstrations or proofs and which Bolzano calls *Begründungen*. This tripartite distinction in itself testifies to Bolzano's acute sense of the differences between logical, epistemological and pragmatic concerns: grounding is a relation between propositions (not facts or states of

affairs), objective justification between beliefs (i.e. certain types of mental states) and *Begründungen* are linguistic objects that are meant, according to Bolzano, to cause objectively justified knowledge. I present these three notions and explain how they are related in order to stress the specificity of Bolzano's views on demonstrations in mathematics.

Edgar Morscher (Salzburg)

The Development of Bolzano's Logical Ideas –from Beytraege (1810) to Wissenschaftslehre (1837)

In the first part of my paper I will summarize Bolzano's logic in his *Beytraege* of 1810.

In the second part I will compare it with the logic as presented in Bolzano's manuscripts *Etwas aus der Logik* (dated around 1812) and *Logische Vorbegriffe* (dated around 1815). In the third part I will present a short survey on Bolzano's main contributions to formal logic in his *Wissenschaftslehre*; almost none of these contributions can be found in his *Beytraege* or in the two manuscripts on logic mentioned before. In the final part I will try to make a diagnosis of why Bolzano, in spite of his great merits, has not become the founder of modern logic.

Kevin Mulligan (Geneva)

Bolzano's Mind

The descriptive psychology of Brentano and his heirs and in particular Brentano's account of intentionality mark the beginning of modern anglophone philosophy of mind which gets going when Stout introduces the Cambridge of Moore and Russell to descriptive psychology. The aim of my talk is to consider the extent to which Bolzano's analysis of mental episodes and his psychology of signs anticipate the descriptive psychologies of Brentano and his heirs. I sketch the relations between Bolzano's account of presenting, judging, wishing, desiring, willing and acting and knowledge and the accounts given by his Austrian successors and then examine in more detail (a) the relation between Bolzano's account of mental "Adhärenzen", Husserl's account of mental "Momente" and Jonathan Lowe's recent account of mental "tropes" ; (b) the relation between the propositional theory of judging given by Bolzano and that given by Husserl in the fifth of his "Logical Investigations" and in "Experience and Judgment"; (c) the relation between the psychology of signs of Bolzano and of Brentano's Minister for the Philosophy of Language, the Swiss-Prague philosopher, Anton Marty.

Michael Otte (Bielefeld)

The Analytic/Synthetic Distinction in Kant and Bolzano (with an eye on Peirce).

Bolzano endorsed, like Leibniz, a kind of Platonic theory of concept, conceiving of a concept as a combinatorial ensemble of simple concepts. From this he derived the existence of synthetic propositions. Neither Kant nor Peirce admitted the possibility of simple concepts and derived the existence of synthetic positions from the continuum instead. Peirce essentially claims that besides predicative or conceptual generality, there exists generality in perception. Usually one believes that what we perceive are singular facts or individuals, whereas the strength of perception and intuition lies, in fact, in its grasping of the global and general. Generality and continuity become synonymous, meaning just that the general is like the continuous, that is, the not (yet) fully specified and determined. And Peirce, differently from Kant or Bolzano, considers arithmetic as analytic. Peirce's evolutionary realism in addition renders the analytic/synthetic distinction relative.

[Much extended version also received.]

Gaëtan Pégny (Berlin)

Bolzano and the Kantian classification of categories

Abstract and full paper to follow.

Stefan Roski (Amsterdam)

Bolzano On Ground and Consequence – from the Beyträge to the Wissenschaftslehre

The relation of ground and consequence (German: 'Grund und Folge' or, for short, 'Abfolge') played an important role in Bolzano's work from his early *Betrachtungen über einige Gegenstände der Elementargeometrie* (1804) to his *opus magnum Wissenschaftslehre* (1837). Notably it is important also in his strictly mathematical and even his theological writings. Throughout his scientific career, Bolzano struggled with the question of how the propositions of a science have to be presented in order to give an insight of *why* they are true (independently of how knowledge of them actually has been acquired), i.e. what kind of relation obtains between the real, objective *grounds* and their *consequences*. He was, however, never able to give a comprehensive treatment of that notion that fully satisfied him. A first extended attempt to give an account of the relation of ground and consequence can be found in his *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810). In this programmatic work Bolzano seemed to be confident that the relation can be characterized by some general, structural characteristics of propositions. The leading intuition was that the grounds of a proposition are always *simpler* or at least *more general* than its consequences. The account given, basically consists of a short list of very simple schematic rules or forms of inference by which such an order can be established. With the development of his method of variation the picture changes in Bolzano's later work. The account of the *Wissenschaftslehre* is far more differentiated. Ground-consequence now appears as one relation among propositions besides others, namely the relations of derivability and of (conditional) probability. By the aid of the method of variation, Bolzano investigates those latter relations systematically at great length and with remarkable precision and generality. With respect to the relation of ground and consequence, however, he seems to be much more skeptical. The leading intuition that the grounds of a truth always have to be more general than and at least as simple as their consequences is still in place, but what is left from the clear cut schematic

rules of the *Beyträge* are “nothing but a few conjectures” (WL II, 384). In particular, Bolzano seems to be undecided whether it might be possible to make use of the method of variation in clarifying the ground-consequence relation, i.e. whether ground-consequence can be defined as a special case of derivability – as: “that order among truths in virtue of which from the smallest amount of simple premises the biggest amount of the remaining truths can be derived as mere conclusions.” (WL II, 388). At least tentatively he rejects this idea because of some particular counterexamples (cf. WL, §200).

This seems to be a rather unsatisfying result. Given the rather straightforward intuitions Bolzano starts with (grounds are always more general than and at least as simple as their consequences), one might expect him to be able to make use of the rich apparatus of his logic of variation in clarifying the notion. My paper will be devoted to understand and to evaluate Bolzano’s reasons to the contrary. In order to do this, I will at first sketch the ‘simple’ intuitive picture of the *Beyträge* and explain why Bolzano was not able to accept it anymore in this form in the *Wissenschaftslehre*. After that I will give a sketch of the account given in the *Wissenschaftslehre* and how it differs from that in the *Beyträge*. Finally I will be concerned with the arguments Bolzano gives against explicating ground-consequence as a special kind of derivability. I will close with a discussion of the question of whether it might be possible – despite Bolzano’s doubts – to give a more precise characterization of the relation within the framework of Bolzano’s logic.

Paul Rusnock (Ottawa)

A quiet revolution in mathematics

Bolzano intended the *Beyträge* to be but the first of many installments of a massive work on the foundations of mathematics. Thus it is not only appropriate but also essential for understanding to see Bolzano’s later mathematical work as a continuation of his project—perhaps even the *Größenlehre*, but certainly the manuscript on universal mathematics and the three works published in 1816-17 (The Binomial Theorem, Purely Analytic Proof, Three Problems). To read the *Beyträge* without consulting the latter is like reading Descartes’ *Discours* without looking at the appendices, which present the purported fruits of the method (a practice that is unfortunately widespread among philosophers). In my talk, I will draw attention to some of the key features of Bolzano’s approach to foundations as set out in the *Beyträge* (objective and subjective orders, the nature of the primitive elements of axiomatic systems, the possibility of carrying out foundational research locally, etc.) and illustrate their application by considering Bolzano’s mathematical works of the early period, principally the Binomial Theorem, where Bolzano develops one of the first rigorous proofs in the theory of power series.

Steve Russ (Warwick)

From the Betrachtungen to the Beyträge

This paper could have the subtitle, ‘from mathematics to philosophy and back again’. We shall trace some of the specifically mathematical themes which seem to have provoked or promoted the general ground-consequence relation (*Abfolge*) which emerges explicitly for the first time in the *Beyträge* (BD). While there are few mathematicians who found – in 1804 or

subsequently – Bolzano’s re-organisation of elementary geometry either attractive or interesting as a contribution to geometry, there are nevertheless several concepts or insights in this first published work of Bolzano that are of wider interest and importance. For example, the identification of the distinct and independent concepts of distance and direction in any ‘system of two points’ (in Part II of the *Betrachtungen* (BG)) was a pre-requisite for the development of vectors. The diagnosis of a profound disorder in the Euclidean development of geometry was clearly a powerful motivation for his entire re-thinking of geometry in BG. This disorder is a theme he emphasises again in the section (BD II. §9) immediately preceding his discussion in BD (II. §§10 – 12) of scientific proof and the ‘objective grounding’ such proof should reflect. His extensive use of the (Leibnizian) notion of ‘determination’ as – among other things – a replacement for the empirical notion (in Bolzano’s view) of congruence, also remains of interest today as a way of characterizing, or constituting, an abstract object. We shall sketch how each of these (and other) themes from BG may have influenced his vision and grasp of the objective *Abfolge* relationship, which in its turn had (I claim) a strong influence on the many mathematical concepts and proofs he elaborated in the rest of his work. We shall further illustrate the interaction between mathematics and philosophy in Bolzano’s thought in finer detail by drawing on material now published, thanks to the late Bob van Rootselaar, in the *Miscellanea Mathematica* series Bd. 2B 2/1 and 2/2 of the *Bernard Bolzano Gesamtausgabe*.

Jan Sebestik (CNRS Paris)

The significance of Bolzano’s Beyträge zu einer begründeteren Darstellung der Mathematik

Two of the first Bolzano’s publications have permanent interest : the *Rein analytischer Beweis* (1817) which inaugurates the arithmetization of analysis, and the *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810). While the first one was noticed by Weierstrass, the second one, going against the spirit of the dominant Kantian and post-Kantian philosophy, was completely neglected.

Written in the Leibnizian spirit, the *Beyträge* develop a new philosophy of mathematics and should influence the philosophy of the XIXth century in a similar way as Wittgenstein’s *Tractatus* did in the XXth century. His main innovations consist in a new definition of mathematics, the claim of the objectivity of mathematics, the first modern study of an axiomatic system, an innovative theory of definition, a proof theory, the criticism of Kant’s philosophy of mathematics and, in the second unfinished part, published only in 1975, the first outline of his doctrine of collections, worked out later in the *Grössenlehre*. In the *Beyträge*, Bolzano sets up not only some of his fundamental concepts, but the structure of his work that will be expanded in the *Wissenschaftslehre* and in *Über die Mathematische Lehrart*.

Göran Sundholm (Leyden and Nancy)

WL §220

Observing analogues in Leibniz and Frege, I discuss the interpretation of Bolzano's *Begründungsbäume* and his intriguing diagram. The possibility of using notions from current proof theory as interpretative tools will be explored:

- a) normal natural deduction derivations;
- b) cut-free sequent calculus derivations;
- c) computation-trees for ("Curry-Howard") proof-objects.

Kateřina Trlifajová (Prague)

Bolzano's measurable numbers revisited: between standard and non-standard interpretation

Bolzano begins his *Beitraege* by stressing the need to improve foundations of mathematics. He claims that mathematics 'is only an incomplete work'. As an example, he mentions problems in the arithmetic theory of negative numbers, ambiguities in the theory of irrational and imaginary numbers, and defects in higher algebra and differential and integral calculi.

It is well-known that Bolzano did a lot for improving foundations of mathematics. One of the key contributions was his study of the continuum. At that time arithmetic (or 'analytic' in Bolzano words) representation of the geometric line and of the continuum in general, would have been desirable for two reasons. First, to provide an important bridge connecting arithmetic and geometry. Second, to serve as a rigorous basis for differential and integral calculi.

Bolzano had been returning to the problem of continuum several times. Already in *Beitraege*, he writes that the straight line and the plane are composed of innumerably many points. But the most important answer to the problem was his theory of measurable numbers. We think that Bolzano was here successful in achieving the first goal but partially unsuccessful in the second goal, providing a basis of the infinitesimal calculus.

Bolzano introduces infinite number concepts which are created from integers by using infinitely many arithmetic operations. He defines that some of the infinite concepts are measurable, some are infinitely small, and other infinitely large. Two measurable numbers are equal, if their difference is infinitely small. The structure of measurable numbers with equality has all desired properties of real numbers.

Bolzano's theory, after several minor corrections, can be interpreted as a rigorous theory of standard real numbers. This has been already persuasively demonstrated. But thus we lose the richness of non-archimedean Leibniz continuum with infinitesimals that corresponds better Bolzano's measurable numbers.

We interpret measurable numbers (with the same minor corrections) as a non-standard extension of the rational numbers. From the algebraic point of view, they form a partially ordered commutative ring. Infinitely small numbers constitute its maximal ideal. Bolzano's introduction of equality corresponds to factorization of measurable numbers modulo infinitely small numbers. Thus we obtain the desired linearly ordered field of the real numbers.

If this structure should also serve as a basis for infinitesimal calculus, a transfer principle between measurable numbers and their factorization would be needed. However, such

transfer is not possible. Therefore Bolzano's measurable numbers are not suitable for this purpose, though they comprise infinitely small numbers.

Bolzano apparently recognized these difficulties. In his sequel work, *Theory of Functions*, he returned to his original idea of building calculus on the basis of quantities which can become smaller than any given quantities - an idea rather close to the later Weierstrass' " ϵ - δ calculus". Nevertheless he did not give up the concept of infinitely small and infinitely large quantities and devoted them his last book *Paradoxes of Infinity*.