Discontinuous Galerkin Methods for Surface Partial Differential Equations

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Introduction

- Partial differential equations (PDEs) on hypersurfaces have become an active area of research in recent years.
- Ubiquitous in fluid dynamics and material science, but have arised more recently in areas as diverse as image inpainting.

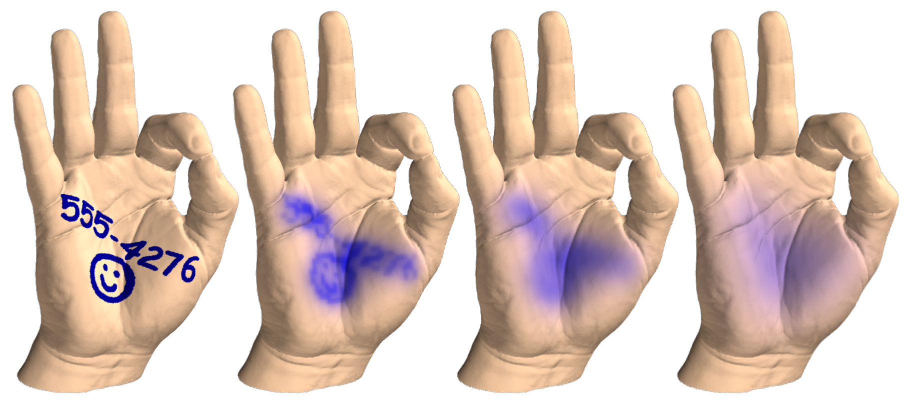

Figure: Image inpainting on a surface [Macdonald, 2009].

- Finite element methods (FEM) have been succesfully extended to surfaces from both a theoretical and numerical point of view.
- However, it is well-known that there are a number of situations where FEM may not be the appropriate numerical method.
- Comparatively little done to investigate alternative numerical methods that solve such issues on hypersurfaces.

Discontinuous Galerkin Methods

- Discontinuous Galerkin (DG) methods are a class of numerical methods that have been succesfully applied to hyperbolic, elliptic and parabolic PDEs arising from a wide range of applications. See Arnold et al. [2002].

Some of its main advantages compared to 'standard' finite element methods include:

- Capturing solution discontinuities (namely those arising in advection driven equations) sharply in a given mesh.
- Less restriction on grid structure and refinement (i.e. works with non-conforming grids).
- Less restriction on choice of basis functions.
- Easily parallelisable.

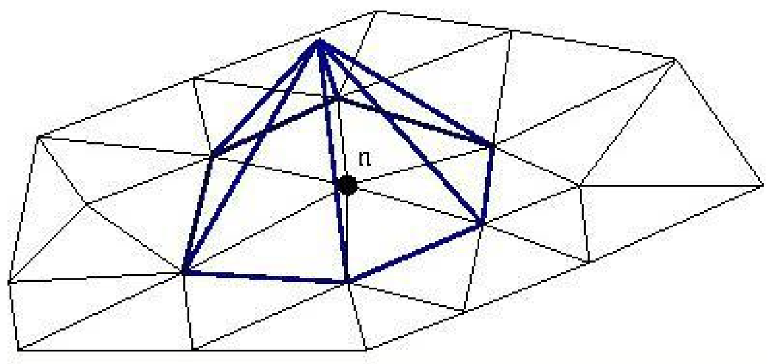

Figure: Linear finite element basis function.

The main idea of DG methods is to lift the requirement of continuity of the solution accross elements, in contrast to standard FEM.

Problem Formulation

- Let Γ be a compact smooth connected and oriented hypersurface in $\mathbb{R}^3$ with outward unit normal ν, to which we assign a signed distance function d which is well-defined in a sufficiently thin open tube U around Γ.
- We also assume that the map $a(x) : U \to \Gamma$ given by

$$a(x) = x - d(x)\nu(a(x))$$

is bijective.

- We are interested in approximating solutions to the Helmholtz equation on a hypersurface, whose variational formulation is given by:
$(\mathbf{P}_\Gamma)$ Find $u \in V := H^1(\Gamma)$ such that

$$a_\Gamma(u, v) = \int_\Gamma fv \, dA, \ \forall v \in V$$

where

$$a_\Gamma(u, v) := \int_\Gamma (\nabla_\Gamma u \cdot \nabla_\Gamma v + uv) \, dA, \quad u, v \in V.$$

- This involves deriving a-priori error estimates for the approximation.

Surface Finite Elements

- The theory for surface FEM, including a-priori error estimates, was first introduced in Dziuk [1988].
- The smooth surface Γ and its associated triangulation T_h composed of curved triangles K is approximated by a polyhedral surface $\Gamma_h \subset U$ whose associated triangulation $\tilde{T}_h$ is composed of planar triangles K_h.
- The vertices of the planar triangles are taken to sit on Γ so that Γ_h is a linear interpolation of Γ.

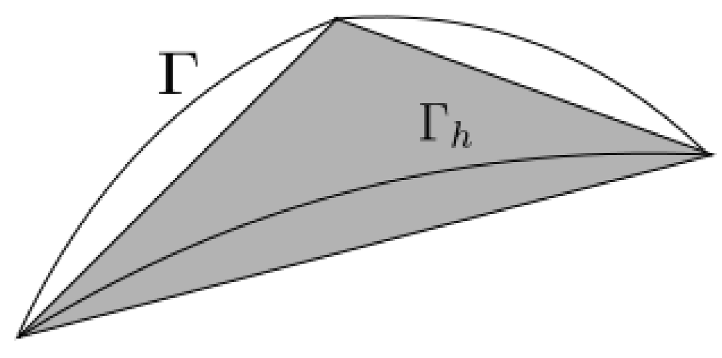

Figure: Linear interpolation of curved triangular element $K \subset \Gamma$ by planar triangular element $K_h \subset \Gamma_h$.

- Note that the triangulation $\tilde{T}_h$ is well-defined by the bijection property of the map $a(x)$.

Surface FEM Approximation

- For FEM approximations we consider the finite-dimensional space

$$V_h := \{\xi \in C^0(\Gamma_h) \ : \ \xi|_{K_h} \in P^1(K_h) \ \forall K_h \in \tilde{T}_h\}.$$

- $(\mathbf{P}_{\Gamma_h})$ Find $u_h \in V_h$ such that

$$a_{\Gamma_h}(u_h, v_h) = \int_{\Gamma_h} f^l v_h \, dA_h \ \forall v_h \in V_h$$

where

$$a_{\Gamma_h}(u_h, v_h) := \int_{\Gamma_h} (\nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h + u_h v_h) \, dA_h, \quad u_h, v_h \in V_h.$$

- Want to compare u satisfying $(\mathbf{P}_\Gamma)$ with u_h satisfying $(\mathbf{P}_{\Gamma_h})$ but they do not live on the same space.
- For any function ξ defined on Γ_h we define the lift onto Γ by

$$\xi^l(a) := \xi(x(a)), \ a \in \Gamma, x \in \Gamma_h.$$

- This lift allows us to define the lifted approximation u_h^l on Γ.
- The lifted finite element space is

$$V_h^l := \{\xi_h^l \in C^0(\Gamma) \ : \ \xi_h^l(a) = \xi_h(x(a)) \text{ with some } \xi_h \in V_h\} \subset V.$$

Theorem (Surface FEM A-priori Error Estimate)

Let $u \in V$ and $u_h \in V_h$ denote the solutions to $(\mathbf{P}_\Gamma)$ and $(\mathbf{P}_{\Gamma_h})$, respectively. Denote by $u_h^l \in V_h^l$ the lift of u_h onto Γ. Then

$$\|u - u_h^l\|_{L^2(\Gamma)} + h\|u - u_h^l\|_V \leq Ch^2\|f\|_{L^2(\Gamma)}.$$

- Would like to derive error estimates for the surface DG approximation on hypersurfaces in a similar way.

Surface DG Approximation I

- For DG approximations we consider the finite-dimensional space

$$V_h := \{\xi \in L^2(\Gamma_h) \ : \ \xi|_{K_h} \in P^1(K_h) \ \forall K_h \in \tilde{T}_h\}.$$

- DG space has no continuity requirement accross elements.
- Let $e_h \in \mathcal{E}_h$ be an edge shared by neighbouring elements K_h^+ and K_h^-, with n_h^+ and n_h^- the corresponding conormals. In addition, $v_h^{+/-} := v_h|_{\partial K_h^{+/-}}$ for every $v_h \in V_h$.

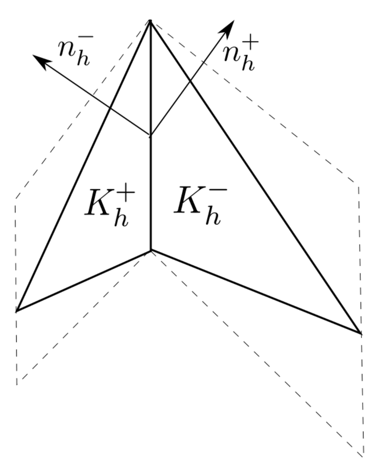

Figure: Conormals on K_h^- and K_h^+.

Surface DG Approximation II

- $(\mathbf{P}_{\Gamma_h}^{DG})$ Find $u_h \in V_h$ such that

$$a_{\Gamma_h}^{DG}(u_h, v_h) = \int_{\Gamma_h} f^l v_h \, dA_h \ \forall v_h \in V_h$$

where

$$a_{\Gamma_h}^{DG}(u_h, v_h) := \sum_{K_h \in \tilde{T}_h} \int_{K_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h + u_h v_h \, dA_h$$
$$- \sum_{e_h \in \tilde{\mathcal{E}}_h} \int_{e_h} (u_h^+ - u_h^-)\frac{1}{2}(\nabla_{\Gamma_h} v_h^+ \cdot n_h^+ - \nabla_{\Gamma_h} v_h^- \cdot n_h^-)$$
$$+ (v_h^+ - v_h^-)\frac{1}{2}(\nabla_{\Gamma_h} u_h^+ \cdot n_h^+ - \nabla_{\Gamma_h} u_h^- \cdot n_h^-) \, ds_h$$
$$+ \sum_{e_h \in \tilde{\mathcal{E}}_h} \int_{e_h} \beta_{e_h}(u_h^+ - u_h^-)(v_h^+ - v_h^-) \, ds_h.$$

with β_{e_h} being a penalty parameter which imposes continuity in a weak sense as the mesh size h tends to zero.

- Want to compare $u \in H^2(T_h)$ satisfying the DG formulation of $(\mathbf{P}_\Gamma)$, call it $(\mathbf{P}_\Gamma^{DG})$, with $u_h \in V_h$ satisfying $(\mathbf{P}_{\Gamma_h}^{DG})$.
- The lifted DG space is given by

$$V_h^l := \{v_h^l \in L^2(\Gamma) \ : \ v_h^l(a) = v_h(x(a)) \text{ with some } v_h \in V_h\} \subset H^2(T_h).$$

Theorem (Surface DG A-priori Error Estimate)

Let $u \in H^2(T_h)$ and $u_h \in V_h$ denote the solutions to $(\mathbf{P}_\Gamma^{DG})$ and $(\mathbf{P}_{\Gamma_h}^{DG})$, respectively. Denote by $u_h^l \in V_h^l$ the lift of u_h onto Γ. Then

$$\|u - u_h^l\|_{L^2(\Gamma)} + h\|u - u_h^l\|_{DG} \leq Ch^2\|f\|_{L^2(\Gamma)}$$

where $\|\cdot\|_{DG}$ is the DG norm.

References

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P. Bastian, M. Blatt, A. Dedner, Ch. Engwer, J. Fahlke, C. Gräser, R. Klöfkorn, M. Nolte, M. Ohlberger, and O. Sander. DUNE Web page, 2011. http://www.dune-project.org.

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Test Problem on Sphere

- All simulations have been perform using the Distributed and Unified Numerics Environment (DUNE). Further information about DUNE can be found in Bastian et al. [2011].
- We solve the Helmholtz equation

$$-\Delta_\Gamma u + u = f \quad (1)$$

on the unit sphere

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}.$$

- Choose right-hand side f such that

$$u(x) = \cos(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3)$$

is the exact solution.

- Confirm theoretical error estimates numerically for both conforming and non-conforming grids.

EOC for DG Approximation on Unit Sphere

h	L_2-error	L_2-eoc	DG-error	DG-eoc
0.112141	0.0528817		2.64273	
0.0560925	0.0146074	1.86	1.3151	1.01
0.028049	0.00378277	1.95	0.653612	1.01
0.0140249	0.000957472	1.98	0.325961	1.00
0.00701247	0.000240483	1.99	0.162822	1.00

Table: Errors and orders of convergence using conforming grid.

h	L_2-error	L_2-eoc	DG-error	DG-eoc
0.112141	0.146369		4.24728	
0.0560925	0.0402358	1.86	2.11183	1.01
0.028049	0.0104518	1.94	1.04316	1.02
0.0140249	0.0026346	1.99	0.516816	1.01
0.00701247	0.000658561	2.00	0.25718	1.01

Table: Errors and orders of convergence using non-conforming grid.

Sphere Problem Visualisation

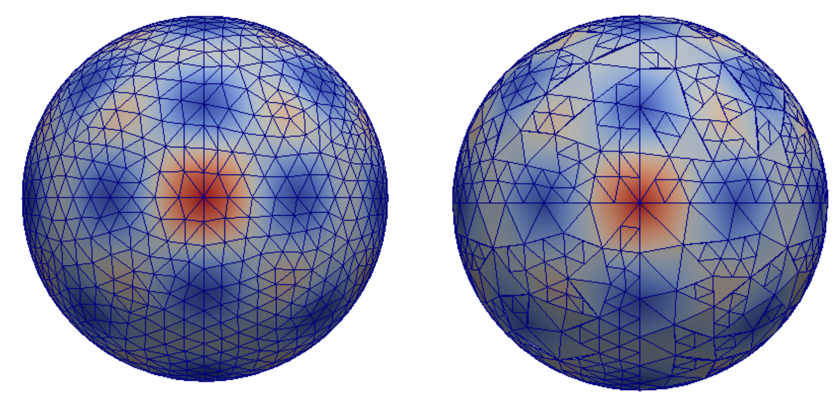

Figure: DG approximation to (1) on the unit sphere for respectively a conforming grid and a non-conforming grid.

Test Problem on Dziuk Surface

- We now solve (1) on the Dziuk surface, given by

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ (x_1 - x_3^2)^2 + x_2^2 + x_3^2 = 1\}.$$

- Choose right-hand side f such that

$$u(x) = x_1 x_2$$

is the exact solution.

- Aim to confirm numerically that the a-priori error estimates hold for more complicated hypersurfaces.

EOC for DG Approximation on Dziuk Surface

h	L_2-error	L_2-eoc	DG-error	DG-eoc
0.27298	0.37683		0.841075	
0.136976	0.102478	1.88	0.26595	1.66
0.068555	0.0276256	1.89	0.096890	1.46
0.0342854	0.00709917	1.96	0.041448	1.22
0.0171432	0.00178764	1.99	0.019655	1.07
0.0085714	0.0004477	2.00	0.0096852	1.02

Table: Errors and orders of convergence using conforming grid.

h	L_2-error	L_2-eoc	DG-error	DG-eoc
0.27298	1.04311		1.96926	
0.136976	0.331642	1.65	0.640044	1.62
0.068555	0.0945755	1.81	0.210186	1.60
0.0342854	0.0251866	1.91	0.0782745	1.42
0.0171432	0.00644021	1.97	0.0339022	1.21
0.0085714	0.00162702	1.98	0.0161914	1.06

Table: Errors and orders of convergence using non-conforming grid.

Dziuk Surface Problem Visualisation

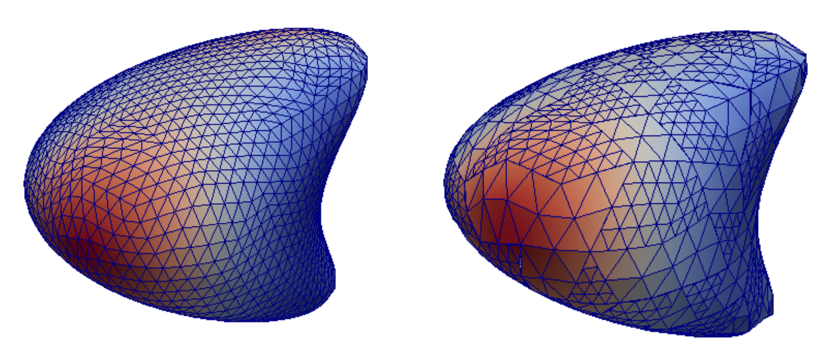

Figure: DG approximation to (1) on the Dziuk surface for respectively a conforming grid and a non-conforming grid.

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