ES441 Advanced Fluid Dynamics Support 5 –Turbulence Similarity Solutions: Axisymmetric Wake & Flying Wing

2.14 **Axisymmetric Wake**

Assume profile $u = U_s f(\xi)$ where $\xi = \frac{y}{\ell}$

Take eddy viscosity for Reynolds stress of $R_T = 1$.

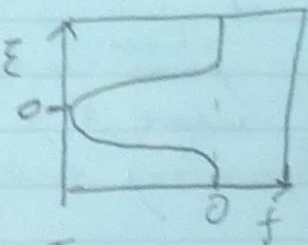


Fig A

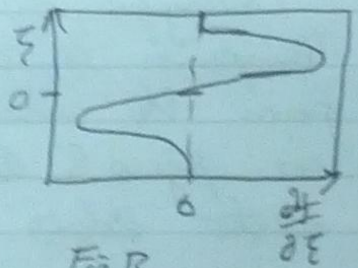


Fig B

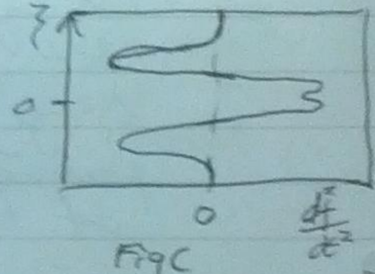


Fig C

Momentum Integral constraint

$$M = U_s \ell^2 \int_{-\infty}^{\infty} \xi f(\xi)\, d\xi = \text{const}$$

Similarity equation is same as for plane wake:

$$-U_0 \left[\frac{\ell}{U_s^2}\frac{dU_s}{dx}\right] f + U_0 \left[\frac{1}{U_s}\frac{d\ell}{dx}\right]\xi f' = g' = \frac{1}{R_T} f''$$

Assume $U_s(x) = Ax^m$, $\ell(x) = Bx^n$

(a)(i) How do U_s, ℓ depend on x?

__Answer__ For a self-similar solution we need $U_s \ell^2$ to be constant

& therefore $U_s \ell^2 = ABx^{m+2n} = ABx^0$

$$\Rightarrow m+2n=0$$

Also need $\frac{1}{U_s}\frac{d\ell}{dx}$ to be constant; so

$$\frac{1}{U_s}\frac{d\ell}{dx} = Ax^{-m}\,Bx^{n-1}n = ABn\,x^{0}$$

$$\Rightarrow \quad -m+n-1=0 \quad \Rightarrow n = m+1$$

then $\quad m + 2m + 2 = 0 \Rightarrow m = -\tfrac{2}{3}$

and $n = \tfrac{1}{3}$

$$U_s = Ax^{-2/3}, \quad \ell(x) = Bx^{1/3}$$

(ii) How does the Reynolds number change downstream?

Answer

Reynolds number $R_\ell = \frac{U_s \ell}{\nu} = ABx^{-1/3}$

(b) Energy budget for mean velocity balances advection and transport with drag:

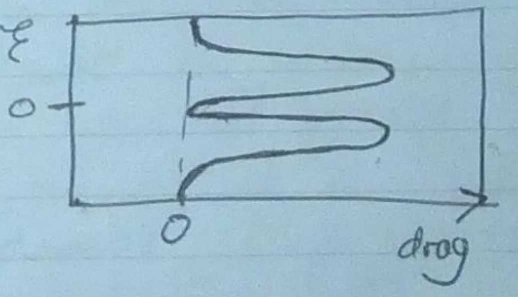
$$-U_0 \frac{d}{dx}\left(\tfrac{1}{2}U^2\right) - \frac{d}{dy}\left(\overline{uv}\,U\right) = -\overline{uv}\frac{dU}{dy}$$

(i) Sketch profile of drag

Answer We use eddy viscosity approximation

$\overline{uv} = -\nu_T \frac{dU}{dy}$ so the drag is

$$-\overline{uv}\frac{dU}{dy} = \nu_T\left(\frac{dU}{dy}\right)^2$$



Get this by squaring the profile for $\frac{df}{d\xi}$

(ii) Sketch profile for energy transport due to Reynolds stress.

Answer: Transport due to Reynolds stress is

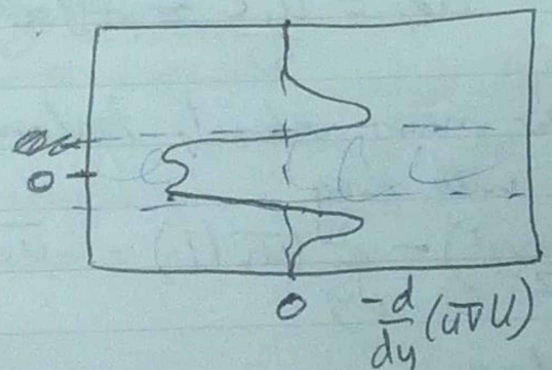
$$\frac{d}{dy}(\overline{uv}\,U) \underset{\text{product rule}}{=} \frac{d}{dy}(\overline{uv})\,U + \frac{dU}{dy}\,\overline{uv}$$

from eddy viscosity:

$$\overline{uv} = -\nu_T \frac{dU}{dy}, \quad \text{so} \quad \frac{d(\overline{uv})}{dy} = -\nu_T \frac{d^2U}{dy^2}$$

$$\text{so} \quad -\frac{d}{dy}(\overline{uv}\,U) = +\nu_T \frac{d^2U}{dy^2}\,U \;\cancel{\approx}\; + \nu_T\left(\frac{dU}{dy}\right)^2$$

$$\sim +\nu_T\left(\frac{d^2f}{d\xi^2}\cdot f + \left(\frac{df}{d\xi}\right)^2\right)$$

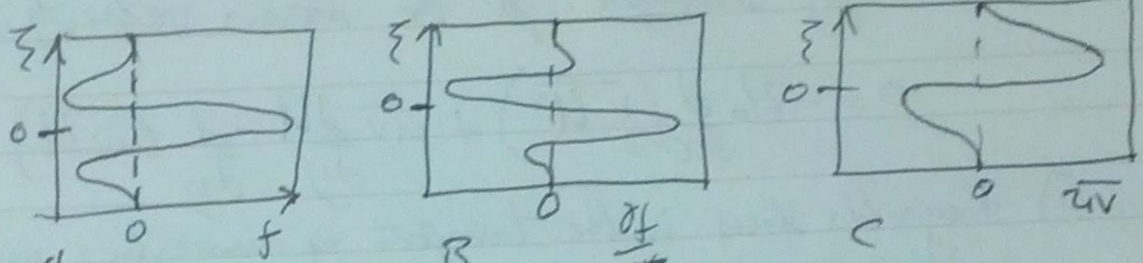


(c) Which way does the energy flux point?

Answer: The energy flux $-\frac{d}{dy}(\overline{uv}U) < 0$ within the wake and positive outside the wake. So ~~transport~~ energy is transported out of the wake.

2.16 Flying Wing

Consider the wake of a self propelled plane projectile (ie flying wing). Assume the profile is $U = U_s f(\xi)$, $\xi = y/\ell$



A B C

Momentum integral is
$$U_s \ell^3 \int_{-\infty}^{\infty} \xi^2 f(\xi)\, d\xi = \text{const}$$

Similarity equation is same as for a plane wake:
$$-U_0 \left[\frac{\ell}{U_s}\frac{dU_s}{dx}\right] f + U_0 \left[\frac{1}{U_s}\frac{d\ell}{dx}\right] \xi f' = g' = \frac{1}{R_T} f''$$

(a)(i) Determine how U_s and ℓ depend on x.

<u>Answer</u> We assume $U_s = A x^a$, $\ell = B x^b$,

From the similarity equation we have

$\frac{1}{U_s}\frac{d\ell}{dx}$ is constant therefore

$\frac{b}{A} x^{-a} x^{b-1} = \text{const} \Rightarrow -a + b - 1 = 0$

$\Rightarrow a = b - 1$

Also from the momentum integral

$U_s \ell^3$ is constant therefore

$A B^3 x^a x^{3b} = A B^3 x^{b-1} x^{3b} = \text{const}$

$\Rightarrow b - 1 + 3b = 0$

$\Rightarrow b = 1/4$, so $a = -3/4$

$U_s(x) \sim x^{-3/4}$ $\ell(x) \sim x^{1/4}$

(ii) Show that the Reynold number is $R_\ell \propto x^{-1/2}$.

Answer

$$R_\ell = \frac{U_s \ell}{\nu} = \underbrace{\frac{AB}{\nu}}_{\text{const}} x^{-3/4} x^{1/4} \sim x^{-1/2}$$

(b) Energy budget for the mean velocity is

$$0 = -U_0 \frac{d}{dx}\left(\frac{1}{2} \overline{U^2}\right) + \overline{uv}\,\frac{dU}{dy} - \frac{d}{dy}\left(\overline{uv}\,U\right)$$

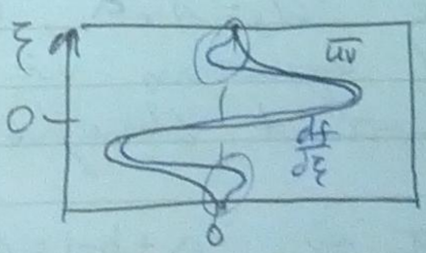
(i) Reynolds stress is the transport of momentum, would it make sense for the momentum flux to point into the centre of the wake?

Answer No, momentum moves away from the centre of the wake

(ii) Why does Reynolds stress not follow eddy viscosity prediction?

Answer Eddy viscosity would mean

$$-\overline{uv} = \nu_T \frac{dU}{dy}$$



but profile of $\frac{df}{d\xi}$ would suggest momentum flux pointing into centre of wake coming from just outside the wake.

(c) Drag $= -\overline{uv}\,\frac{dU}{dy}$