ES441 Advanced Fluid Dynamics Support 9 – 2D Turbulence

5.5 Kolmogorov spectrum for two dimensions

KE spectrum of 2D turbulence is

$$E(k) = \frac{k}{(2\pi)^2} \int \underbrace{\langle u(\underline{x}) \cdot u(\underline{x}+\underline{r}) \rangle}_{\text{average}} e^{i\underline{k}\cdot\underline{r}} \, d\underline{r} \quad (1)$$

where $k = |\underline{k}|$ wavevector magnitude. This spectrum $E(k)$ describes distribution of energy over length scales $l = 2\pi/k$.

(a) Find the physical dimensions of $E(k)$ and dissipation rate ε:

Answer: Dimensions of $E(k)$ can be found from the formula above (1),

$$[E(k)] = [k][u]^2 \underbrace{[L]^2}_{\text{from 2D integral}}$$

$$= [L]^{-1}[U]^2[L]^2 = [U^2 L].$$

ε is the energy dissipated per unit time per unit volume so

$$[\varepsilon] = \underbrace{[u^2]}_{KE} \underbrace{[T^{-1}]}_{\text{per unit time}} = [U^3][L^{-1}]$$

$$\left([U] = \frac{[L]}{[T]}\right)$$

(b) Use dimensional argument to find KE spectrum assuming the only dimensional quantities it can depend on are ε and k.

Answer: Assume $[E(k)] \sim [\varepsilon]^a [k]^b$

so $[U^2][L] \sim \cancel{[U^3]^a} \left[\frac{1}{L}\right]^b$ ~~$\Rightarrow 3a = 2 \Rightarrow a = -2/3$, $a - b = 1 \Rightarrow b = -5/3$~~

$$\sim \left[\frac{U^3}{L}\right]^a \left[\frac{1}{L}\right]^b \Rightarrow -a - b = 1$$

$$3a = 2 \Rightarrow a = 2/3$$

$$\text{so } b = -5/3$$

$[E(k)] = [\varepsilon^{2/3} k^{-5/3}]$ (Close to Kolmogorov but the constant is different $C = 7$, $C_K = 0.5$)

(c) What are the dimensions of enstrophy spectrum $Z(k) = \omega^2$ and palenstrophy $PZ = (\nabla\omega)^2$?

<u>Answer</u> Enstrophy Spectrum:

$$[Z(k)] = \frac{[\omega^2]}{[k]} = [(\nabla\times u)^2][L]$$

$$= \frac{[u^2][L]}{[L^2]} = \left[\frac{u^2}{L}\right] = \left[\frac{L}{T^2}\right]$$

Palenstrophy $= (\nabla\omega)^2$

$$[PZ] = [(\nabla\omega)^2] = \left[\frac{1}{L^2}\right][\omega^2] = \left[\frac{u^2}{L^4}\right] = \left[\frac{1}{T^2L^2}\right]$$

(d) What is the predicted ~~enstrophy~~ enstrophy spectrum?

<u>Answer</u> Assume $[Z(k)] \sim [PZ]^a [k]^b$

$$\Rightarrow \left[\frac{u^2}{L}\right] = \left[\frac{u^2}{L^4}\right]^a \left[\frac{1}{L}\right]^b \Rightarrow a = 1$$

$$b = 1 - 4a \Rightarrow b = -3$$

$$Z(k) \sim PZ\, k^{-3}$$

For the energy spectrum: let dissipation of enstrophy be η then

$$[\eta] = \underbrace{\left[\frac{u^2}{L^2}\right]}_{\text{enstrophy}} \underbrace{\left[\frac{1}{T}\right]}_{\text{per unit time}} = \left[\frac{1}{T^3}\right]$$

<u>Assume</u> $E(k) \sim \eta^a k^b$

$$\Rightarrow [u^2L] = \left[\frac{1}{T^3}\right]^a \left[\frac{1}{L}\right]^b \Rightarrow 3 = -b \Rightarrow b = -3$$

$$2 = 3a \Rightarrow a = 2/3$$

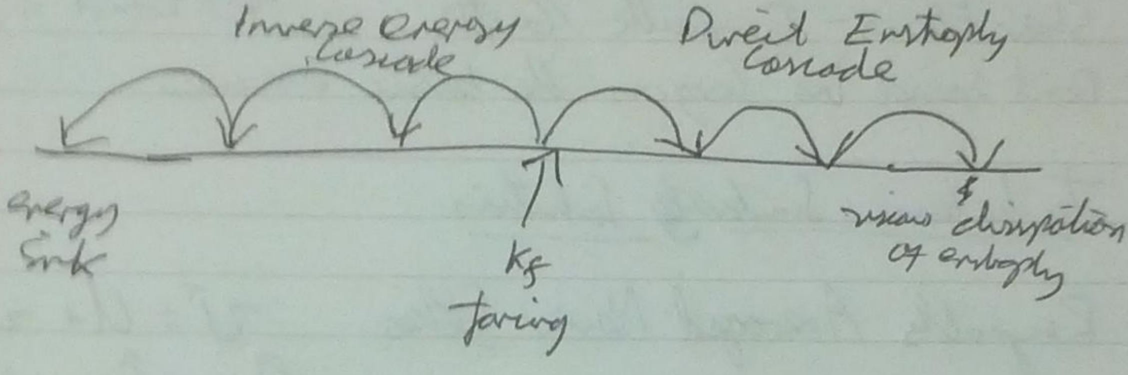
$$= \left[\frac{L^3}{T^2}\right]$$

So $E(k) \sim \eta^{2/3} k^{-3}$

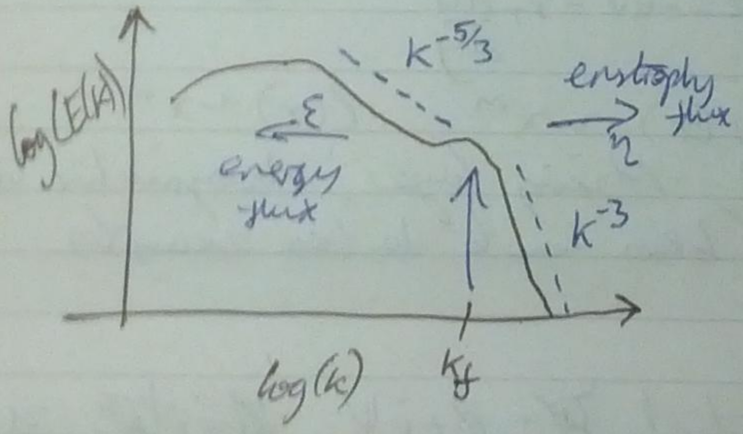
(e) ~~What~~ Consider 2D turbulence with energy sinks at both large scales and small scales, what does the energy spectrum look like?

Answer At low wavenumbers $E(k) \sim k^{-5/3}$ where there is an energy cascade to the large scale energy sink.

At large wavenumber $E(k) \sim k^{-3}$ where enstrophy cascades to small scale viscous dissipation.



In 2D energy cascades in opposite direction to the Kolmogorov energy cascade in 3D, hence it is called an inverse energy cascade.



This happens in 2D because vortex stretching term is zero hence enstrophy (ω^2) is quadratic invariant as is KE (u^2). This causes a dual cascade behaviour.