ES441 Advanced Fluid Dynamics Handout 2 - Lift

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1 Kelvin's Circulation Theorem

Theorem 1. *In an ideal flow with a conservative force, let $C(s,t)$ be a closed material contour. Then the* circulation

$$\Gamma = \oint_{C(s,t)} \boldsymbol{u} \cdot dx = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n} dS, \tag{1.1}$$

is independent of time.

This is an important theorem in fluid dynamics. Note that this only holds for non-viscous fluids.

2 Complex Potential

If a flow is 2D, incompressible and irrotational then the velocity field can be represented as

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \tag{2.1}$$

where ϕ is the velocity potential and ψ is the streamfunction. These are the Cauchy-Riemann equation, which in Complex Analysis implies there is an analytic function called the *complex potential*,

$$\chi = \phi + i\psi, \tag{2.2}$$

which is a function of $z = x + iy$, then $\partial_z \chi = u - iv$ is the complex velocity.

3 Irrotational Flow Around a Cylinder

We are given that the complex potential of uniform flow of speed U_0 in the x-direction around a cylinder of radius a is

$$\chi = U_0 \left(z + \frac{a^2}{z} \right). \tag{3.1}$$

This comes from something called Milne-Thomson's circle Theorem, (Acheson §4.4, §4.5). If the cylinder has circulation Γ then

$$\chi = U_0 \left(z + \frac{a^2}{z} \right) - \frac{i\Gamma}{2\pi} \log z, \tag{3.2}$$

(see complex potential of point vortex). To find velocities u_r and u_θ we use the polar form $z = re^{i\theta}$ to get

$$\chi = U_0 \left(re^{i\theta} + \frac{a^2 e^{-i\theta}}{r} \right) - \frac{i\Gamma}{2\pi} \left(\log r + i\theta \right),$$

and using the identity $e^{i\theta} = \cos\theta + i \sin\theta$,

$$\chi = \underbrace{U_0 \left(r + \frac{a^2}{r} \right) \cos\theta + \frac{\Gamma\theta}{2\pi}}_{\phi} + i \underbrace{\left(U_0 \left(r - \frac{a^2}{r} \right) \sin\theta - \frac{\Gamma}{2\pi} \log r \right)}_{\psi}. \tag{3.3}$$

Then using

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, u_\theta = -\frac{\partial \psi}{\partial r} \text{ or } u_r = \frac{\partial \phi}{\partial r}, u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

we get

$$u_r = U_0 \left(1 - \frac{a^2}{r^2}\right) \cos\theta, u_\theta = -U_0 \left(1 + \frac{a^2}{r^2}\right) \sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.4}$$

Theorem 2. *(Bernoulli's Theorem) In a steady flow of an ideal fluid,*

$$H = \frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \varphi \tag{3.5}$$

is constant along streamlines. ($\boldsymbol{g} = \nabla\varphi$).

This is an important theorem, the main consequence is that we if we know either one of pressure or velocity, then we may obtain the other.

We now calculate the lift on a rotating cylinder. In lectures it was shown that $r = a$ is a streamline, hence $p + \frac{1}{2}\rho\boldsymbol{u}^2 = \text{const}$. Using this and $u_r = 0$, $u_\theta = -2U_0 \sin\theta + \frac{\Gamma}{2\pi a}$ at $r = a$ to get pressure. Then integrate the y-component of force on a small element $d\theta$ which is $-pa\sin\theta d\theta$ to get $F_y = -\rho U_0 \Gamma$, this is the lift on a rotating cylinder.

4 Kutta-Zhukovski Lift Theorem

Theorem 3. *For a steady flow, which is uniform at infinity with speed U_0 in the x-direction, past a 2D body, the cross section of which is some simple closed curve. Let circulation around the body be Γ. Then the force on the body will be $\boldsymbol{F} = (0, -\rho U_0 \Gamma)$.*

We have already proved this for the case of the cylinder, the full proof requires Blasius' Theorem and conformal mappings. This theorem provides the basis for calculating lift on aerofoils, hence is important for flight.

5 Conformal Mapping

A *conformal mapping* $Z = f(z)$ preserves local shapes and angles, with inverse $z = F(Z)$ and is an analytic function of $Z = X + iY$. The complex potential in the Z -plane is $\mathbb{X}(Z) = \chi(F(Z)) = \Phi(X,Y) + i\Psi(X,Y)$.

Conformal mappings allow us to work on the simple case of a cylinder and transform to more useful bodies like an aerofoil. The *Zhukovsky transformation* is $Z = z + \frac{1}{z}$.

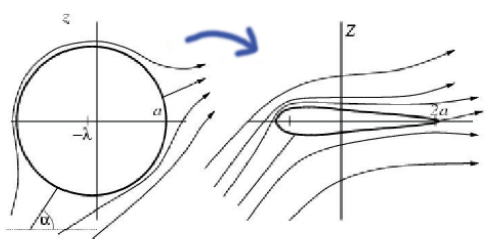

Figure 1: Zhukovsky transformation transforms a circle to an aerofoil. Moving the center of the circle generates different aerofoil shapes.

However the problem with this is that we get singularities at the tail end of the aerofoil. This can be resolved by changing the circulation Γ so that the flow leaves the aerofoil smoothly and parallel to the horizontal axis. This particular value of the circulation is given below.

Theorem 4. *(Kutta-Zhukovsky Hypothesis) The critical value of circulation Γ_k for a wing in flow of speed U_0 at an angle of attack α and length L is $\Gamma_k = -\pi L U_0 \sin\alpha$.*

Thus the total lift on a wing is $F_y = \pi {U_0}^2 \rho L \sin\alpha$.

6 D'Alembert's Paradox

We have $F_x = 0$ ie. no drag. This contradicts the wakes in experimental observations, the reason is we have not taken viscosity into account.

What happens in reality is we get vortex shedding from the wing tip. Note that the circulation of the flow around an aerofoil is zero when at rest, hence by Kelvin's Circulation Theorem it should remain zero. In fact it does, since the circulation around the aerofoil and the trailing vortex cancel as seen in Figure 2. Vortex shedding continues until the circulation around the aerofoil reaches the Kutta-Zhukovsky value and flow leaves the aerofoil smoothly, generating lift.

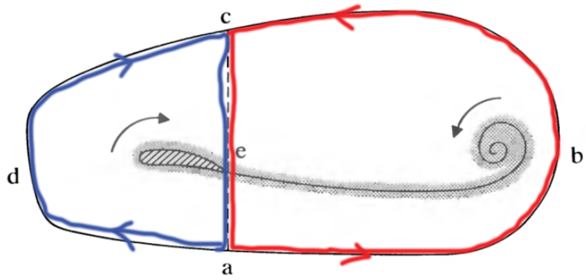

Figure 2: Net circulation is zero.

Example 1. *(Problem 4.1 - Airplane Lift and Trailing Vortices) (a) Explain why the vortex near the wing surface (responsible for the circulation round the aerofoil) must continue into a trailing vortex through the wing tip. What is the relation between the circulation around the wing and around its trailing vortex?*

Answer: *Near the wing surface, velocity is nearly tangential to the surface and variations are in the normal direction to the surface, so vorticity is parallel to the wing (vorticity is the curl of velocity). Vortex lines cannot end within the fluid since the velocity field is solenoidal (divergence free), ie. fluid is incompressible. They must continue beyond the wing tips which means vorticity flux remains the same. Hence the circulation around the trailing vortices is equal with opposite sign to the circulation around the wing.*

(b) Consider an airplane of mass M, speed V with air density ρ. Each wing is L meters long. Find the circulation around each trailing vortex.

Answer: *To keep the plane flying, the weight of the plane must balance the lift, so*

$$Mg = \rho V \Gamma (2L) \Rightarrow \Gamma = \frac{Mg}{2\rho V L}. \tag{6.1}$$

(c) Assume trailing vortices originating at both wings are parallel to each other and their vorticity is concentrated in thin tubes separated by $D = 2L$. Find the downdraft velocity produced by these trailing vortices at the mid point between them.

Answer: *Velocity at midpoint is two times the velocity at a distance of $D/2$ from a point vortex,*

$$U = \frac{\Gamma}{2\pi(D/2)} \times 2 = \frac{2\Gamma}{\pi D}. \tag{6.2}$$