MA4H7 Atmospheric Dynamics Support Handout 5 - Anelastic Approximation and Quasi-Geostrophic Equations

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1 Boussinesq Approximation

Variations of density ignored except when they give rise to a gravitational force (multiplied by g), that is ignored in inertia terms but not buoyancy terms.

2 Anelastic Approximation

Eliminates sound waves by assuming flow has velocities much smaller than the speed of sound. Assumes density as a decreasing function of height.

Example 1. *(Exam Question - Master Equation)*
(a) Explain the difference between the anelastic and Boussinesq approximations.
Answer: *See above.*

(b) For gravity waves in the linearised incompressible Navier-Stokes equations with mean shear flow $U(z)$ and ignoring rotation, diffusion and y-direction momentum, the master equation *is (found by putting general wave solutions $w = \hat{w}(z)e^{i(kx-\omega t)}$, etc. into the equations),*

$$\hat{w}'' + \frac{\bar{\rho}'}{\bar{\rho}}\hat{w}' + \left[\frac{N^2}{(c-U)^2} + \frac{U''}{c-U} + \frac{\rho_0'}{\rho_0}\frac{U'}{(c-U)} - k^2\right]\hat{w} = 0 \tag{1}$$

where $c = \omega/k$, $U(z)$ is the shear flow, $\bar{\rho}$ is average density and ρ_0 is from the linearisation $\rho = \rho_0(z) + \rho_1(x, z, t)$. The vertical velocity $w = \hat{w}e^{i(kx-\omega t)}$. Consider gravity waves with no mean shear and the average density is a function of height (anelastic approximation). What does the master equation reduce to?
Answer: *No mean shear gives $U = 0$. If mean density is a function of height then $\frac{\bar{\rho}'}{\bar{\rho}} = \frac{\rho_0'}{\rho_0}$ so the master equation (1) reduces to*

$$\hat{w}'' + \frac{\rho_0'}{\rho_0}\hat{w}' + \left(\frac{N^2}{c^2} - k^2\right)\hat{w} = 0. \tag{2}$$

(c) What is N and what does it represent a balance between?

Answer: *$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$ is the Brunt Väisälä or buoyancy frequency. It represents a balance between buoyancy and gravity, and is the frequency at which a vertically displaced air parcel will oscillate in a stable environment.*

Now assume no mean shear and allow for density as a function of height $\rho(z)$.

(d) Assume $\rho_0(z) = \rho_r e^{-\frac{z}{H}}$ for some reference density ρ_r. What is the dispersion relation for a wave $w(x, z, t) = w_0 e^{\frac{z}{aH}} e^{i(kx+mz-\omega t)}$ where (k, m) are horizontal and vertical wavenumbers, and ω the frequency.

Answer: *Substitute* $w(x,z,t) = w_0 e^{\frac{z}{aH}} e^{i(kx+mz-\omega t)}$ *into (2) noting that* $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$ *to get,*

$$\left[\left(\frac{1}{aH} + im\right)^2 + \frac{\rho_0'}{\rho_0}\left(\frac{1}{aH} + im\right) + \left(\frac{N^2}{c^2} - k^2\right)\right] w_0 = 0$$

$$\Rightarrow \left[\frac{1}{a^2H^2} + \frac{2im}{aH} - m^2 - \frac{1}{H}\left(\frac{1}{aH} + im\right) + k^2\left(\frac{N^2}{\omega^2} - 1\right)\right] w_0 = 0 \tag{3}$$

(e) Why is $a = 2$ *the proper choice?*
Answer: *If* $a = 2$ *the complex term in (3) vanishes. The complex part would cause the waves to decay as they rose since we would have*

$$w = w_0 e^{-Im[\omega]t} e^{\frac{z}{aH}} e^{i(kx+mz-Re[\omega]t)} \tag{4}$$

where $Re[\omega]$ *and* $Im[\omega]$ *are the real and imaginary parts of* ω*. It can also been seen that* $a = 2$ *by solving the second order ODE* (2)*.*

(f) Find the dispersion relation $\omega(k,m)$ *with* $a = 2$*.*
Answer: *Use* $a = 2$ *in (3) to get*

$$\omega^2 = \frac{k^2N^2}{k^2 + m^2 + \frac{1}{4H^2}} \tag{5}$$

(g) How would you apply the Boussinesq approximation, what would the dispersion relation be?
Answer: *We assume density variations with height are small so* $\rho_0'/\rho_0 = -1/H$ *is small, meaning that* H *is relatively large, in particular in comparison to wavelengths* $H \gg \lambda = \frac{2\pi}{(k^2+m^2)^{1/2}}$ *so (5) becomes*

$$\omega^2 = \frac{N^2k^2}{k^2 + m^2}.$$

3 Quasi-Geostrophic Equations

These equations ignore some of the more complex dynamics whilst still capturing the basic atmospheric motions, allowing the study of large motions using simplified equations. These equations assume close to geostrophic velocities, a maintained state of stratification and use the β-plane approximation for the Coriolis parameter $f \approx f_0 + \beta y$. For a streamfunction ψ, and potential vorticity q, the 2D version of these equations are

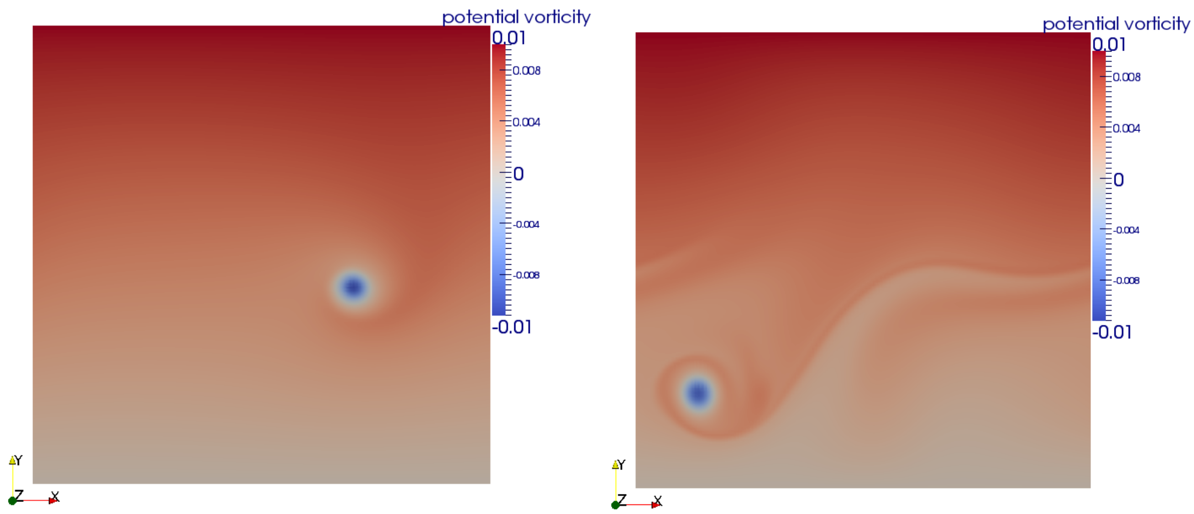
$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{6}$$

$$q = \Delta\psi - \frac{f_0^2}{gH\mathcal{B}}\psi + \beta y, \tag{7}$$

where the *Burger's number* $\mathcal{B} = (NH/\Omega L)^2$ measures stratification against rotation effects. The Jacobian operator describes advection of potential vorticity by the gesotrophic velocity $u_g = -\partial_y\psi, v_g = \partial_x\psi$, it is defined as

$$J(\psi, q) = \frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x} = \left[(u_g, v_g)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right] q. \tag{8}$$

Note that although these equations are 2D, the stratification N is non-zero, therefore stratification effects will be observed. These equations can be used to study planetary Rossby waves, the formation/motion of the jet stream, and the motion of large high/low pressure cells in the atmosphere.



(a) t=30 (b) t=200

Figure 1: Simulation of a vortex in 2D quasi-geostrophic equations. (a) An initially axisymmetric anti-cyclonic (high pressure) vortex propagating South-West due to β-plane rotation effects. (b) Rossby waves are seen to the East in the wake of the vortex.

For a characteristic velocity scale U and length scale L, we can estimate a scaling ratio between the β-term and vorticity advection in the potential vorticity equation (6),

$$\mathcal{R} = \frac{\beta v}{\boldsymbol{u} \cdot \nabla \zeta} \sim \frac{\beta U}{(U/L)(U/L)} = \frac{\beta L^2}{U}, \tag{9}$$

where vorticity $\zeta = \Delta \psi$. The β influence is strong when $\mathcal{R}$ is large. When this is true the initially axisymmetric streamfunction ψ propagates westward and changes shape by Rossby wave dispersion. To understand this propagation, note that when we have a primarily axisymmetric vortical flow $\psi(x, y) \approx \Psi(r)$, the β-term creates a forcing term in (6) since

$$(\boldsymbol{u} \cdot \nabla)\beta y = \beta \frac{\partial \psi}{\partial x} \approx \beta \frac{\partial}{\partial x} \Psi(r) = \beta \cos(\theta) \frac{d\Psi}{dr}. \tag{10}$$

The $\cos(\theta)$ represents a dipole structure, which generates an advective flow. Initially the dipole centres are separated in the zonal direction (East-West), but after time this is rotated by azimuthal (angular) advection from Ψ towards a more meridional (North-South) separation. This results in an approximately westward advection. The dipole centre separation is not completely meridional, so the vortex propagation is not exactly westward. The North-South motion is dependent on the sign of $d\Psi/dr$; if it is positive (low pressure) then the motion will be to the North and if it is negative (high pressure) then motion will be to the South as in Figure 1 (assuming Northern Hemisphere).

4 Inertial Waves in a Rotating Fluid

Suppose an inviscid, incompressible fluid is rotating uniformly with angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$. The linearised rotating equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho} \nabla p, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{11}$$

We find the pressure equation by taking the divergence and curl (looking at the z component) of (11),

$$\nabla \cdot \left[\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho} \nabla p \right] \Rightarrow 2\Omega \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -\frac{1}{\rho} \Delta p. \tag{12}$$

$$\nabla \times \left[\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p\right] \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{13}$$

We also take the partial derivative with respect to z of the z component of (11)

$$\frac{\partial}{\partial z}\left[\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\right] \Rightarrow \frac{\partial^2}{\partial z^2}p = -\rho\frac{\partial}{\partial z}\frac{\partial}{\partial t}w. \tag{14}$$

Then putting these results together we get the pressure equation, we begin by multiplying (14) by $4\Omega^2$ and using incompressibility

$$4\Omega^2\frac{\partial^2}{\partial z^2}p = 4\Omega^2\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 2\Omega\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\frac{\partial^2}{\partial t^2}\Delta p \tag{15}$$

$$\Rightarrow \left[\frac{\partial^2}{\partial t^2}\Delta + 4\Omega^2\frac{\partial^2}{\partial z^2}\right]p = 0. \tag{16}$$

We look for solutions $p = \hat{p}e^{i(kx+ly+mz-\omega t)}$, putting this into (16) gives the dispersion relation,

$$\omega^2 = \frac{4\Omega^2 m^2}{k^2 + l^2 + m^2}. \tag{17}$$

The phase and group velocities are

$$c_{ph} = \frac{\omega\boldsymbol{k}}{|\boldsymbol{k}|^2}, \;\; c_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) = \frac{2\Omega}{|\boldsymbol{k}|^4}(-km^2, -lm^2, m(k^2 + l^2)) \tag{18}$$

Taking the dot product shows orthogonality,

$$(k, l, m) \cdot (-km^2, -lm^2, m(k^2 + l^2)) = -k^2m^2 - l^2m^2 + m^2(k^2 + l^2) = 0.$$