MA4H7 Atmospheric Dynamics Support Handout 7 - Thermal Wind

8th March 2017 Jorge Lindley email: J.V.M.Lindley@warwick.ac.uk

1 Baroclinicity

A *barotropic* fluid is one in which the pressure is a function only of the density $p(\rho)$. Otherwise the fluid is known as *baroclinic*, that is the fluid is stratified ($\partial\rho/\partial z \neq 0$).

Taking the curl of the compressible Euler equation gives the vorticity (ζ) equation

$$\frac{D\zeta}{Dt} = \frac{1}{\rho^2}\nabla\rho \times \nabla p. \tag{1.1}$$

In a baroclinic fluid, the term on the right hand side shows that vorticity can be produced due to a misalignment of surfaces of constant density (isopycnals) and pressure (isobars). In a barotropic fluid, these surfaces are parallel since pressure is a function of only density and the cross product of the gradients is identically zero, therefore a barotropic fluid cannot produce vorticity.

2 Thermal Wind

Suppose we have a mass of cool air at ground level that is wedged under a mass of warm air. The density then varies in vertical and horizontal directions. Assuming that the flow is steady, geostrophic and hydrostatic we have

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{2.1}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{2.2}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.3}$$

Taking the vertical z-derivative of (2.1) and (2.2) and eliminating the vertical pressure gradient with (2.3) we obtain the *thermal wind*

$$\frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f}\frac{\partial p}{\partial x} \tag{2.4}$$

$$\frac{\partial u}{\partial z} = \frac{g}{\rho_0 f}\frac{\partial p}{\partial y}. \tag{2.5}$$

When assuming that density is a linear function of temperature T with thermal expansion α, this can also be written as

$$\frac{\partial v}{\partial z} = \frac{g\alpha}{f}\frac{\partial T}{\partial x} \tag{2.6}$$

$$\frac{\partial u}{\partial z} = -\frac{g\alpha}{f}\frac{\partial T}{\partial y}. \tag{2.7}$$

This relation shows us that due to the Coriolis force, the system can be maintained in equilibrium without a tendency to level out the density variations. That is the rotation of the earth can keep the system away from a rest state without any continuous supply of energy. The result is a change in the horizontal velocities with height due to the horizontal temperature gradient, this is what is known as the *thermal wind*. One important example of this is the powerful jet stream, that increases in strength with height, which blows weather systems eastward at mid-latitudes in the Northern-Hemisphere.

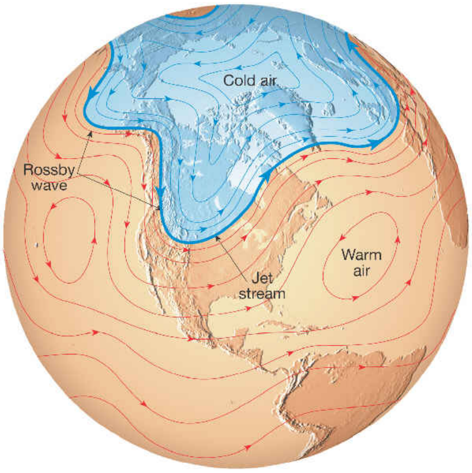

Figure 1: A westerly jet stream formed by a north-south temperature gradient.

3 Geostrophic Adjustment

In the atmosphere, it is often the case that fluid masses come into contact and have not yet had time to achieve thermal-wind balance. The process of a flow system being adjusted to satisfy this thermal-wind balance is known as *geostrophic adjustment*.

Example 1. *(Geostrophic adjustment of a slab of cold air) Consider two fluid masses recently brought into contact. One of the air masses of height H is cool with a density $\rho_1 = \rho_0 + \Delta\rho$, where ρ_0 is the density of the warm air around the slab of cold air. We assume there is no variation in the y-direction but allow for a velocity v in that direction. Using a reduced-gravity constant $g' = g(\rho_0 - \rho_1)/\rho_0$ the shallow water equations are*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x} \tag{3.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0 \tag{3.2}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0. \tag{3.3}$$

The initial conditions, as soon as the air masses come into contact, are $u = v = 0$, $h = H$ for $x < 0$, and $h = 0$ for $x > 0$. At the boundary we have $u, v \to 0$ and $h \to H$ as $x \to -\infty$, whereas the velocity component u at the front is given by the material derivative $u = dx/dt$ where $h = 0$ at some $x = a(t)$.

Solving this non-linear problem analytically is impossible. However we may observe that the following potential vorticity is conserved by the governing equations,

$$q = \frac{f + \zeta}{h} = \frac{f + \partial v/\partial x}{h}. \tag{3.4}$$

Initially $v = 0$ and $h = H$ so that the potential vorticity $q = f/H$. Since this is conserved, the fluid at the final state must also have $q = f/H$. This allows us to relate the initial state of the fluid to the final state without having to solve for intermediate evolution. Once the final steady state has been achieved, the time derivatives vanish and (3.3) *implies that $hu = c$ for some constant c. Since we have $h = 0$ at $x = a$, this constant must be zero, therefore $u = 0$ everywhere. The shallow water equations now become*

$$-fv = -g'\frac{dh}{dx}. \tag{3.5}$$

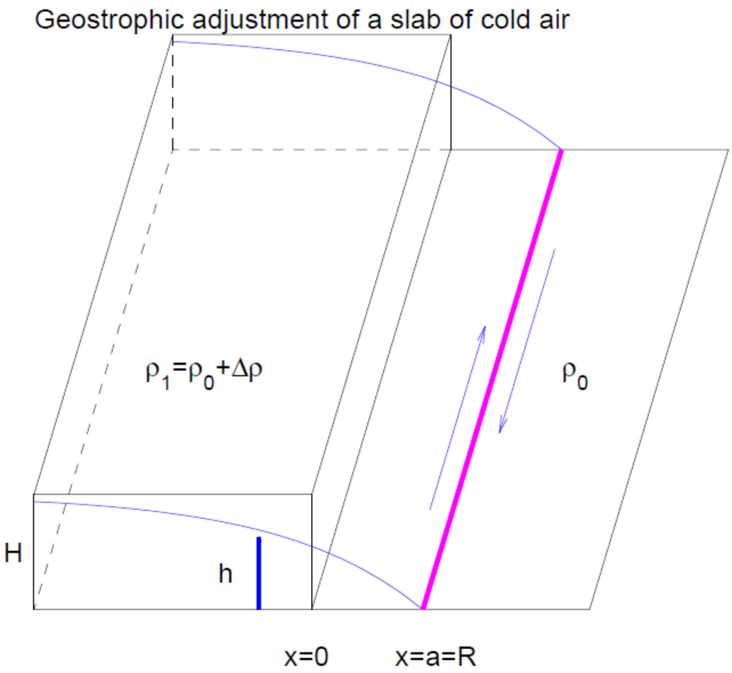

Figure 2

This is a geostrophic balance between the velocity and the pressure gradient set by the sloping interface. Conservation of potential vorticity provides a second equation,

$$\frac{\partial v}{\partial x} + f = \frac{f}{H} h. \tag{3.6}$$

Eliminating h from (3.5) and (3.6) we get the second order differential equation

$$\frac{\partial^2 v}{\partial x^2} = \frac{f^2}{g' H} v. \tag{3.7}$$

Solving this with the boundary condition $h = 0$ at $x = a$ and ignoring the exponential solution that grows as $x \to -\infty$ we have

$$h = H \left[1 - \exp\left(\frac{x-a}{R} \right) \right] \tag{3.8}$$

$$v = -\sqrt{g' H} \exp\left(\frac{x-a}{R} \right), \tag{3.9}$$

where $R = \sqrt{g' H}/f$. To find the value of a we must use conservation of mass, that is the displacement of cold air on the left of $x = 0$ must be replaced by warm air on the right,

$$\int_{-\infty}^{0} (H - h)\, dx = \int_0^a h\, dx \tag{3.10}$$

which gives $a = R$.

4 Thermal Wind Vectors

When the thermal wind pushes the geostrophic wind clockwise with height then the wind is *veering*, this is associated with warm air advection and dynamics lifting. This brings calm weather.

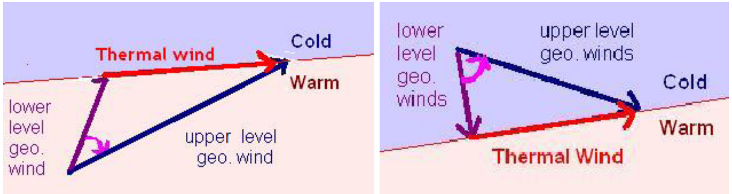

Figure 3

When the thermal wind pushes the geostrophic wind anti-clockwise with height then the wind is *backing*, this is associated with cold air advection and dynamic sinking. This brings stormy weather.

Example 2. *Referring to the map in Figure 4 with solid contours for isobars of the mean sea level pressure (MSLP) and dashed lines for isopleths of the thickness (measured in 10s of meters, or decameters) between the 500 and 1000 hPa levels. Indicate the direction of the thermal wind across the British Isles by adding denoted lines to this map.*

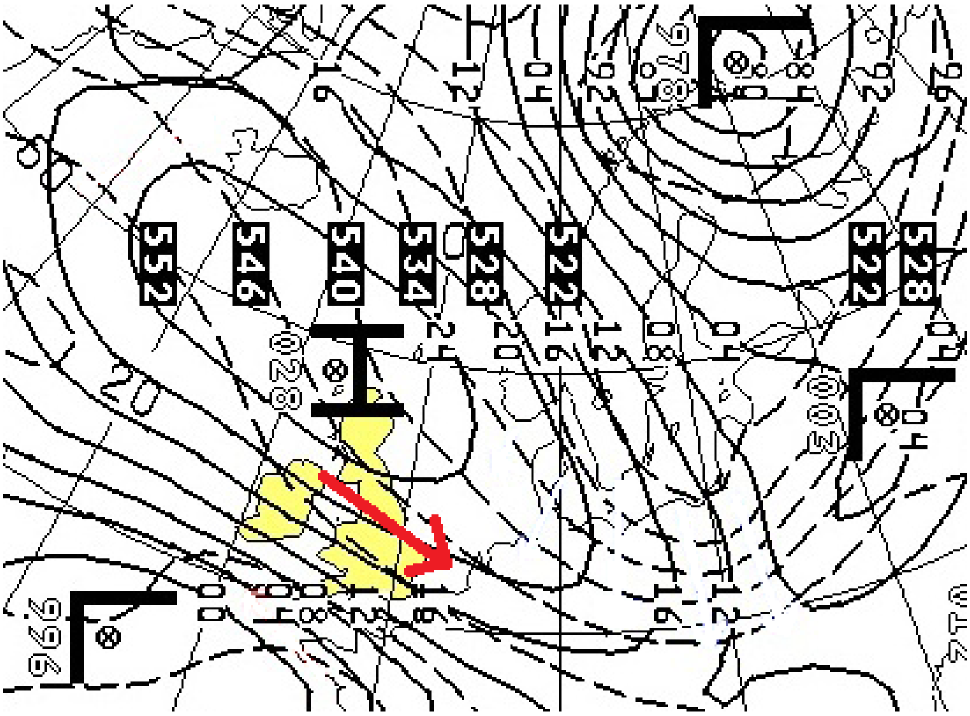

Figure 4: The thermal wind goes from northwest to southeast.