MA4H7 Atmospheric Dynamics Support Handout 8 - Instability

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1 Two-Layer QGE and Baroclinic Instability

Baroclinic instabilities are caused by the presence of a horizontal temperature gradient in a rapidly rotating, strongly stratified fluid like the atmosphere. This instability can be studied using a two-layer quasigeostrophic model with layer thicknesses $H_1 = H_2 = H/2$,

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0, \tag{1.1}$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0, \tag{1.2}$$

where the potential vorticities in the layers are

$$q_1 = \Delta\psi_1 + \beta y - \frac{f_0^2}{g' H_1}(\psi_1 - \psi_2), \tag{1.3}$$

$$q_2 = \Delta\psi_2 + \beta y + \frac{f_0^2}{g' H_2}(\psi_1 - \psi_2). \tag{1.4}$$

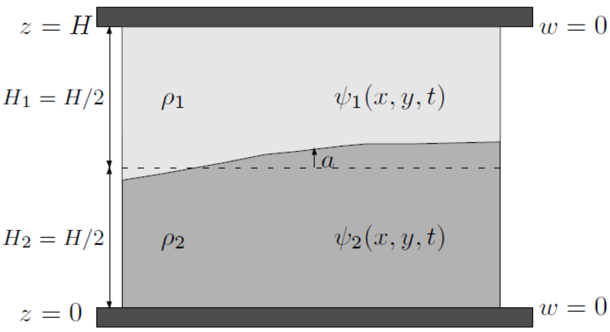

Figure 1: Representation of the vertical stratification by two layers of uniform density in a quasigeostrophic model. The vertical displacement $a = (f_0/g')(\psi_2 - \psi_1)$. Figure in Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics*, Academic Press, 2009.

Note that the last terms in potential vorticity equations are equivalent to a finite difference approximation of the term $\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)$ in the full three-dimensional quasigeostrophic equations. Linearising these equations with an average flow $\overline{\psi}_1 = -Uy$ and $\overline{\psi}_2 = Uy$ gives

$$\frac{\partial q_1'}{\partial t} + U\frac{\partial q_1'}{\partial x} + v_1'\left[\beta + \frac{U}{R^2}\right] = 0, \tag{1.5}$$

$$\frac{\partial q_2'}{\partial t} - U\frac{\partial q_2'}{\partial x} + v_2'\left[\beta - \frac{U}{R^2}\right] = 0 \tag{1.6}$$

where $R = \sqrt{g'H}/2f_0$ and

$$q_1' = \Delta\psi_1' - \frac{f_0^2}{g' H_1}(\psi_1' - \psi_2'), \tag{1.7}$$

$$q_2' = \Delta\psi_2' + \frac{f_0^2}{g' H_2}(\psi_1' - \psi_2'). \tag{1.8}$$

Now we assume the fluctuating component to be a wave of the form $\psi'_j = \Psi_j \exp\left(i(kx + ly - \omega t)\right)$ for layers $j = 1, 2$. Using this in the above equations we get

$$(\omega - kU)\left[K^2\Psi_1 + \frac{1}{2R^2}(\Psi_1 - \Psi_2)\right] + k\left[\beta + \frac{U}{R^2}\right]\Psi_1 = 0, \tag{1.9}$$

$$(\omega + kU)\left[K^2\Psi_2 - \frac{1}{2R^2}(\Psi_1 - \Psi_2)\right] + k\left[\beta - \frac{U}{R^2}\right]\Psi_2 = 0. \tag{1.10}$$

Using $C_x = \omega/k$ and defining the barotropic and baroclinic components of the Fourier coefficients,

$$\Psi_{\text{tr}} = \frac{1}{2}(\Psi_1 + \Psi_2) \text{ and } \Psi_{\text{cl}} = \frac{1}{2}(\Psi_1 - \Psi_2) \tag{1.11}$$

our equations now become

$$[C_x K^2 + \beta]\Psi_{\text{tr}} - UK^2\Psi_{\text{cl}} = 0 \tag{1.12}$$

$$-U(K^2 - R^{-2})\Psi_{\text{tr}} + [C_x(K^2 + R^{-2}) + \beta]\Psi_{\text{cl}} = 0. \tag{1.13}$$

A pure barotropic wave occurs when $U = 0$ and $\Psi_{\text{cl}} = 0$ then $C_x = -\beta/K^2$ which is the same wavespeed derived for planetary Rossby waves using the Charney equation (rotation but no stratification, ie barotropic).

A pure baroclinic wave occurs when $U = 0$ and $\Psi_{\text{tr}} = 0$ then $C_x = -\beta/(K^2 + R^{-2})$ which is the same wavespeed derived for planetary Rossby waves using the single-layer 2D quasigeostrophic equations (rotation and stratification, ie. baroclinic).

When $U \neq 0$, the barotropic and baroclinic components are coupled. Note that (1.12) and (1.13) form a system of line equations in Ψ_{tr} and Ψ_{cl} and can be written in matrix form

$$\underbrace{\begin{bmatrix} [C_x K^2 + \beta] & -UK^2 \\ -U(K^2 - R^{-2}) & [C_x(K^2 + R^{-2}) + \beta] \end{bmatrix}}_{A} \begin{bmatrix} \Psi_{\text{tr}} \\ \Psi_{\text{cl}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.14}$$

If the matrix A is invertible then we have trivial solutions $\Psi_{\text{tr}} = 0$ and $\Psi_{\text{cl}} = 0$. Therefore the non-trivial solutions are when the matrix A is not invertible and so the determinant $|A| = 0$. That is

$$[C_x K^2 + \beta][C_x(K^2 + R^{-2}) + \beta] - U^2K^2(K^2 - R^{-2}) = 0. \tag{1.15}$$

To get the wavespeed C_x here we must calculate the discriminant $\mathcal{P}$ of the quadratic equation (1.15) for C_x. Doing this we get

$$C_x = -\frac{\beta(2K^2 + R^{-2}) \pm \sqrt{\mathcal{P}}}{2K^2(K^2 + R^{-2})} \tag{1.16}$$

where

$$\mathcal{P} = \beta^2 R^{-4} + 4U^2K^4(K^{-4} - R^{-4}). \tag{1.17}$$

The solution is stable when $\mathcal{P} > 0$. Otherwise $\mathcal{P} < 0$ and the wavespeed has an imaginary, growing component, ie unstable. It can be shown that the wave is stable for

$$U \leq \beta R^2. \tag{1.18}$$

Recall that the layers had average flows of speed U in opposite directions, therefore the greater the vertical shear, the more likely that it will breach the threshold of instability.

2 Jet Stream

The motion of vortices in the jet stream develop planetary Rossby waves which can be explained in terms of baroclinic instability. Figures 2 and 3 show how a vertical vortex column can be subjected to baroclinic instability causing it to oscillate in the north-south direction.

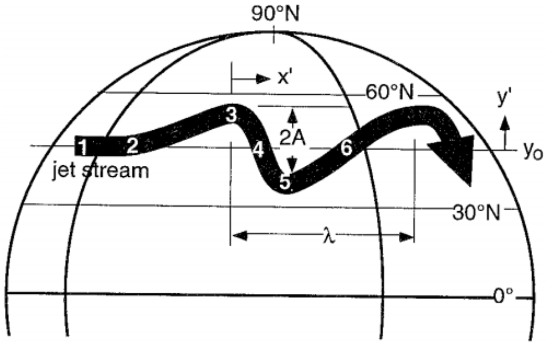

Figure 2: Initially zonal flow at point 1, if disturbed at point 2, will develop north-south meanders called Rossby waves. Roland Stull, *Meteorology for Scientists and Engineers.*

The atmosphere is thinner near the poles since the air is cooler and heavier, so the thickness $H = h_0 - h'y$ decreases towards the poles. This can also be represented by two fluid layers using a sloping density surface as shown in Figure 3 with the stratosphere acting like a rigid lid to the troposphere since it is so strongly stratified and stable. The Coriolis parameter $f = f_0 + \beta y$ increases towards the poles. Given that the relative vorticity $\zeta = 0$ at point 1, we require that the potential vorticity

$$q = \frac{f + \zeta}{H} = \frac{f}{H} \tag{2.1}$$

be conserved. Suppose at point 2 the flow is perturbed towards the north, the air is now moving to greater latitudes where f increases and H decreases. Therefore to conserve potential vorticity q, the relative vorticity ζ must decrease to the point of becoming negative at point 3 and turning anti-cyclonic (clockwise), causing the jet to point south-east.

Now moving south, the jet experiences a decrease in f and an increase in H, therefore to conserve q the vorticity ζ increases. At point 4 the vorticity has increased so much that it is now positive and the jet turns cyclonic (anti-clockwise) heading back north-east. The initially stable jet at point 1 has become unstable. We can see this Rossby wave requires variation of the Coriolis parameter f and thickness H (due to stratification) with latitude to create the baroclinic instability.

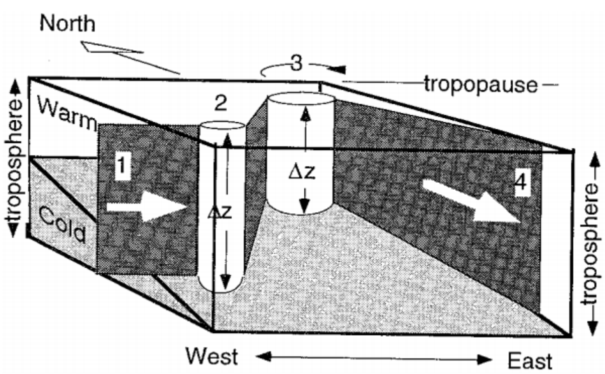

Figure 3: Dark grey ribbon represents jet stream axis, white columns indicate absolute vorticity. Roland Stull, *Meteorology for Scientists and Engineers.*