MA3D1 Fluid Dynamics Support Class 6 - Inertial Waves, Boundary Layers

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1   Inertial Waves in a Rotating Fluid

This is the derivation of the dispersion relation for inertial waves from a 2014 problem.

Suppose an inviscid, incompressible fluid is rotating uniformly with angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$. The linearised rotating equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p, \ \ \nabla \cdot \boldsymbol{u} = 0. \tag{1.1}$$

We find the pressure equation by taking the divergence and curl (looking at the z component) of (1.1),

$$\nabla \cdot \left[\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \right] \Rightarrow 2\Omega \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -\frac{1}{\rho}\Delta p. \tag{1.2}$$

$$\nabla \times \left[\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \right] \Rightarrow \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = 2\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{1.3}$$

We also take the partial derivative with respect to z of the z component of

$$\frac{\partial}{\partial z} \left[\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \right] \Rightarrow \frac{\partial^2}{\partial z^2} p = -\rho \frac{\partial}{\partial z}\frac{\partial}{\partial t} w. \tag{1.4}$$

Then putting these results together we get the pressure equation, we begin by multiplying (1.4) by $4\Omega^2$ and using incompressibility

$$4\Omega^2 \frac{\partial^2}{\partial z^2} p = 4\Omega^2 \rho \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 2\Omega \rho \frac{\partial^2}{\partial t^2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -\frac{\partial^2}{\partial t^2}\Delta p \tag{1.5}$$

$$\Rightarrow \left[\frac{\partial^2}{\partial t^2}\Delta + 4\Omega^2 \frac{\partial^2}{\partial z^2} \right] p = 0. \tag{1.6}$$

We look for solutions $p = \hat{p}e^{i(kx+ly+mz-\omega t)}$, putting this into (1.6) gives the dispersion relation,

$$\omega^2 = \frac{4\Omega^2 m^2}{k^2 + l^2 + m^2}. \tag{1.7}$$

The phase and group velocities are

$$c_{ph} = \frac{\omega \boldsymbol{k}}{|\boldsymbol{k}|^2}, \ \ c_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) = \frac{2\Omega}{|\boldsymbol{k}|^3}(-km, -lm, (k^2 + l^2)) \tag{1.8}$$

Taking the dot product shows orthogonality,

$$(k, l, m) \cdot (-km, -lm, (k^2 + l^2)) = -k^2 m - l^2 m + m(k^2 + l^2) = 0.$$

Example 1. *Inertial waves in rotating fluids*
This problem concerns the dynamics of inertial waves, i.e. small perturbations to the state of uniform rotation.

1. *Write down an expression for the velocity field corresponding to uniform rotation. Find the vorticity corresponding to this flow.*

2. *Consider a small perturbation u to the state of uniform rotation (with an angular velocity $\boldsymbol{\Omega}$) which has the form of a harmonic wave*

$$\boldsymbol{u} = \boldsymbol{A}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + \boldsymbol{A}^*e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{1.9}$$

   where $\boldsymbol{k}$ is the wave vector, ω is the frequency, $\boldsymbol{A}$ is the complex amplitude and $\boldsymbol{A}^*$ is the complex conjugate of $\boldsymbol{A}$. How small does the amplitude $\boldsymbol{A}$ need to be for the nonlinear (with respect to $\boldsymbol{A}$) terms to be much smaller than the linear ones in the fluid equations?

3. *Consider the linearized equations of motion and derive the dispersion relation $\omega = \omega(\boldsymbol{k})$. (**Hint:** Take the curl of the Euler equation and use the identity $\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = -(\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u}$.)*

4. What is the polarisation of the inertial waves?

5. Are these waves dispersive or non-dispersive? Isotropic or anisotropic? Explain why. Find the group velocity for the inertial waves and comment on its relative direction with respect to the wave number.

2 Boundary Layers

In the inviscid ($\nu = 0$) case we have the free-slip boundary condition, that is the velocity normal to the boundary is zero, $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ but fluid is free to move along the boundary.

In reality there is a thin *boundary layer* where velocity undergoes a smooth but rapid adjustment to zero. This is because at the boundary the normal and tangential components of velocity must equal that of the boundary. If the boundary is at rest then $\boldsymbol{u} = 0$. The condition on the tangential component of velocity is known as the *no-slip* condition.
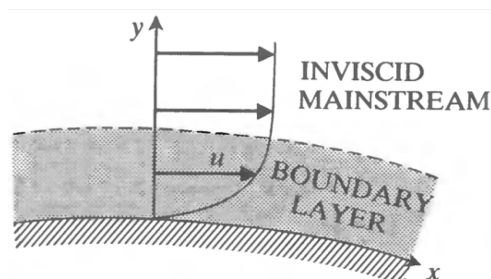

Figure 1: Boundary layers form near the boundary of a body in a flow, here viscous effects are very important. The mainstream can be considered as inviscid for low viscosity flows.

Velocity gradients in boundary layer much larger than in the main part of the flow, so viscous effect become significant no matter how small ν is.

In (Newtonian[1]) viscous fluids, shear stress τ is proportional to the velocity gradient $\frac{\partial u}{\partial y}$,

$$\tau = \mu \frac{\partial u}{\partial y}, \tag{2.1}$$

where μ is called the coefficient of viscosity. We define the *kinematic viscosity* as $\nu = \frac{\mu}{\rho}$.

Boundary layers may separate from the boundary causing dramatic differences in the flow from that of inviscid theory. Wakes behind objects will form causing drag.
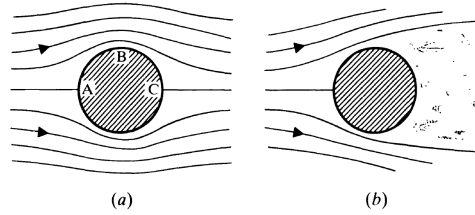

Figure 2: Flow past a cylinder for (a) inviscid fluid, (b) fluid of small viscosity.

Referring to Figure 2, inviscid theory says the pressure p has a local maximum at A, a minimum at B and a local maximum at C. So between B and C there is a substantial increase in pressure along the boundary in the direction of the flow. This severe adverse pressure gradient along the boundary causes the boundary layer to separate. We can see how this adverse pressure gradient drives a reversed flow in Figure 3 on the right hand side. This pushes the boundary layer (typically of thickness $O(R^{-\frac{1}{2}})$) into the mainstream. Here $R = UL/\nu$ is the *Reynolds number*, where U is a typical flow speed and L is a characteristic length scale. This is a ratio of inertia/viscosity, a high Reynolds number means viscous effects are negligible. Note that near the boundary U becomes very small so viscous effects become important even for small ν. At high R reattachment can occur, this is where the boundary layer reattaches to the surface so the boundary layer doesn't fully separate until further towards the rear end of the object reducing the size of the wake, hence reducing drag. This reattachment principle is behind the design of aerofoils, the dimples on golf balls and the fur on tennis balls.
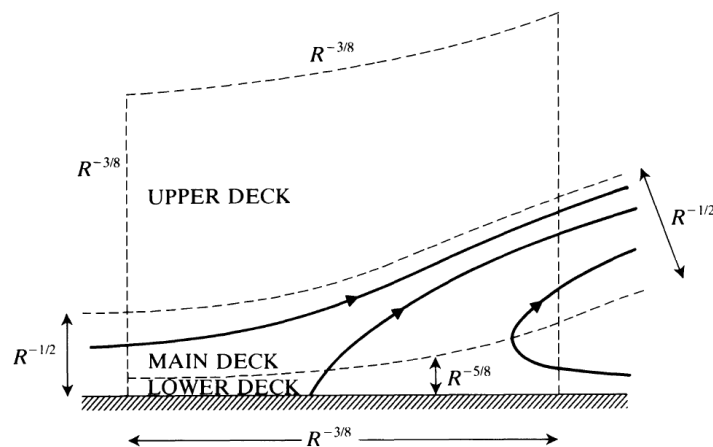

Figure 3: Separation point of a boundary layer on a rigid boundary.

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[1]A fluid is said to be Newtonian if the viscous stresses that arise from its flow, at every point, are proportional to the local strain rate - the rate of change of its deformation over time. This course will not consider non-Newtonian fluids.

Example 2. *Boundary layer produced by a pure strain flow*
Consider a boundary layer at an infinite at plate at $y = 0$ produced by the flow which is in the inviscid exterior (i.e. outside of the boundary layer) has the form of a pure strain flow,

$$\boldsymbol{u}^{ext} = (u^{ext}, v^{ext}) = (\alpha x, -\alpha y). \tag{2.2}$$

1. *Find the pressure field $p^{ext}(x, y)$ in the inviscid exterior flow using Bernoulli's theorem.*

2. *The boundary layer equations are (c.f. problem 5.2.1):*

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\partial_{yy}u, \tag{2.3}$$

$$\partial_x u + \partial_y v = 0, \tag{2.4}$$

where $p \equiv p(x) = p^{ext}(x, 0)$. Find the first term on the right-hand side of equation (2.3). From the x-dependence of this term, deduce that every term in equation (2.3) must scale linearly with x.

3. *Thus deduce that u scales linearly with x and v is independent of x, i.e. the boundary layer thickness δ is x-independent. Use a dimensional argument to find δ in terms of ν and α.*

4. Express v in terms of a non-dimensional function f of non-dimensional variable $\eta = y/\delta$:

$$v = v^* f(\eta). \tag{2.5}$$

Using a dimensional argument find v^* in terms of ν and α.

5. Find u in terms of f using the incompressibility condition (2.4).

6. Substitute u and v into the equation (2.3) and find the ordinary differential equation for $f(\eta)$.

7. Use the no-slip boundary conditions on the plate and the conditions of matching to the exterior inviscid flow to find the boundary conditions for $f(\eta)$.

8. You might like to solve the found equation for $f(\eta)$ with the obtained initial conditions using e.g. Matlab's ODE45 programme and thereby find the profile of the boundary layer.

3 Point Vortices

Example 3. *Point vortex in a channel* *A two-dimensional incompressible inviscid fluid is occupying a channel bounded by two straight walls, $x = 0$ and $x = h$. Consider a flow in such a channel which is generated by a point vortex of circulation $\Gamma > 0$ initially located at the point $(x, y) = (a, 0)$, see figure 4.*
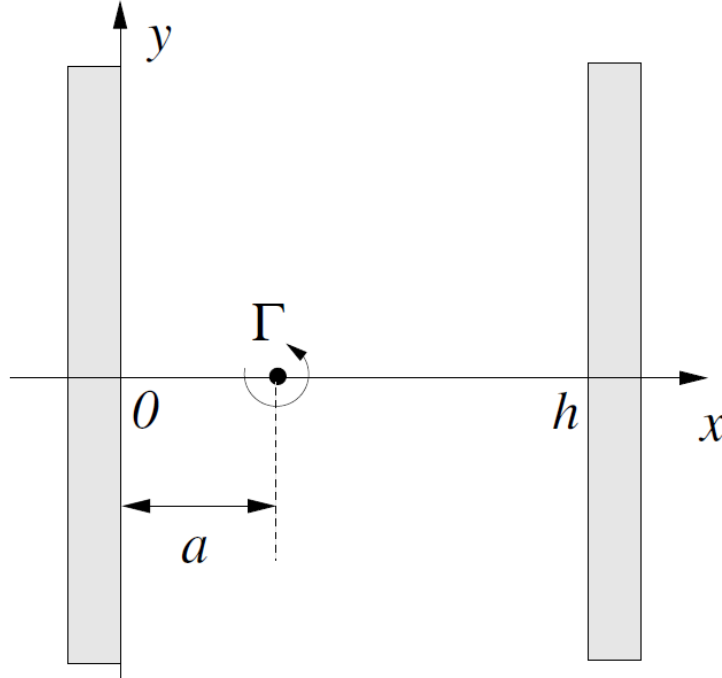

Figure 4: Scattering of a point vortex dipole at an isolated vortex.

1. *Formulate the free-slip boundary conditions for this problem.*

   Answer: *Let the velocity field be $u = (u(x, y, t), v(x, y, t))$. The free-slip boundary conditions for this problem are:*

   $$u(x, 0, t) = u(x, h, t) = 0; \quad v(x, 0, t) \ \& \ v(x, h, t) \ \text{arbitrary}.$$

2. *Explain why the flow generated by such a point vortex in the channel with the free-slip boundary conditions is the same (for $0 < x < h$) as the flow generated by an infinite chain of the image vortices on an unbounded two-dimensional plane which are initially located as shown in figure 5.*

   Answer: *From symmetry, the flow generated by an infinite chain of the image vortices on an unbounded two-dimensional plane which are located as shown in figure 5 will have $u(x, 0, t) = u(x, h, t) = 0$, as required by the free-slip boundary conditions of the original problem. Thus, the rest of the flow for $0 < x < h$ will also be the same, because it contains the same point vortex inside this interval and it satisfies the fluid equations (basically the Laplace equation for the velocity potential of an irrotational incompressible flow) outside of the point vortex.*

3. *Explain why the point vortex will move parallel to the walls with a constant speed.*

   Answer: *The point vortex and its images will remain on the same line for any $t > 0$. Thus, they will not produce any x-component of the velocity at each other's positions. Because the x-positions of the vortices do not change, they produce the same y-component of the velocity at each other's positions for any $t > 0$. Therefore, the point vortex (and its images) will move parallel to the walls with a constant speed.*
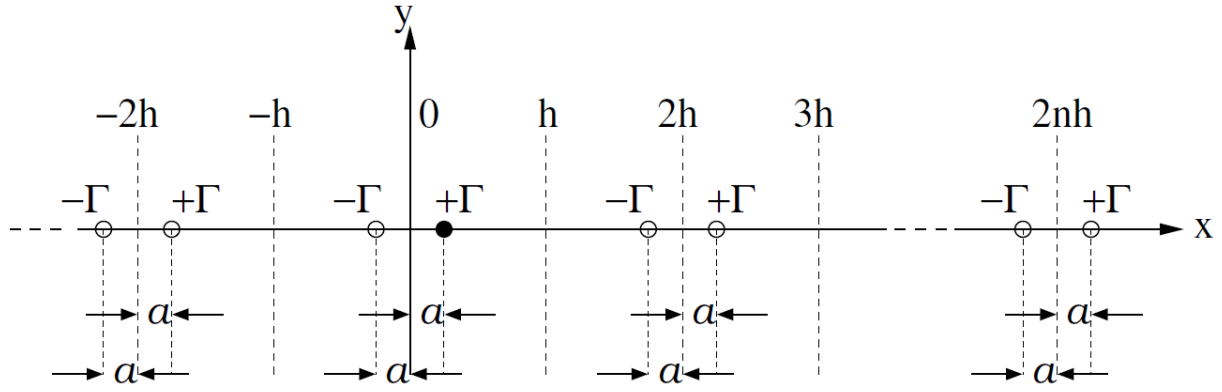
Figure 5: Infinite chain of image vortices.

4. *Find the velocity of the point vortex if it is located in the centre of the channel (i.e. $a = h/2$).*

   Answer: *From the symmetry, the velocity of the point vortex located in the centre of the channel (when $a = h/2$) is zero.*

5. *By summing the contributions from all of the image vortices, find the velocity of the point vortex for the case $a = h/4$.* ***Hint:*** *to evaluate the infinite sum involved use the following identity,*

$$\sum_{k=1}^{\infty} \frac{1}{16k^2 - 1} = \frac{1}{2} - \frac{\pi}{8}.$$

   Answer: *Here we have to refer to figure 5. By summing the contributions from all of the image vortices for the case $a = h/4$, we have for the vortex velocity V :*

$$V = \frac{\Gamma}{2\pi h} \sum_{k=-\infty}^{k=+\infty} \left[\frac{1}{2k - 1/2} - \frac{1}{2k + 1/2} \right] = -\frac{\Gamma}{\pi h} + \frac{4\Gamma}{\pi h} \sum_{k=1}^{k=+\infty} \frac{1}{16k^2 - 1} = \frac{\Gamma}{\pi h}(1 - \pi/2).$$

6. *Find the velocity of the point vortex for the case $a = 3h/4$.*

   Answer: *From the symmetry, the velocity of the point vortex for the case $a = 3h/4$ is just the negative of the velocity found for the case $a = h/4$,*

$$V = -\frac{\Gamma}{\pi h}(1 - \pi/2).$$