The dimension of singular sets in the 3D incompressible Navier-Stokes equations and in a related 1D surface growth model

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Let $K \subset \mathbb{R}^n$. What is the dimension of $K$?

1) Hausdorff dimension

For $\delta > 0$ let

$$\mathcal{H}_\delta^s(K) := \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : K \subset \bigcup_{i=1}^{\infty} U_i, \text{diam } U_i \leq \delta \right\}.$$ 

and

$$\mathcal{H}^s(K) := \lim_{\delta \to 0} \mathcal{H}_\delta^s(K) \quad \leftarrow \text{this is } s\text{-dimensional Hausdorff measure}$$

The Hausdorff dimension is

$$d_H(K) := \inf \left\{ s \geq 0 : \mathcal{H}^s(K) = 0 \right\}.$$
2) Box-counting dimension

\[ d_B(K) := \limsup_{\varepsilon \to 0} \frac{\log M(K, \varepsilon)}{-\log \varepsilon}, \]

where \( M(K, \varepsilon) \) is the maximal number of disjoint \( \varepsilon \)-ball with centers in \( K \). In particular if \( d < d_B(K) \) then there exists a sequence \( \varepsilon_j \to 0 \) such that

\[ M(K, \varepsilon_j) > \varepsilon_j^{-d}. \]

3) Assouad dimension

\( K \) is called to be \((C, s)\)-homogeneous iff for any \( R \)-ball \( B_R \) one can cover \( K \cap B_R \) by at most \( C(R/\rho)^s \) many \( \rho \)-balls.

\[ d_A(K) := \inf \{ s : K \text{ is } (C, s)\text{-homogeneous for some } C \geq 1 \} \]
Examples:

1) If $K = [0, 1]^m$ then $d_H(K) = d_B(K) = d_A(K) = m$.

2) For any compact $K \subset \mathbb{R}^n$ we have

$$d_H(K) \leq d_B(K) \leq d_A(K),$$

3) If $K = \{0\} \cup \{1/m\}_{m=1}^\infty$ then $d_H(K) = 0$, $d_B(K) = 1/2$, $d_A(K) = 1$.

4) For any $\xi \in (0, 1)$ there exists a ”Cantor-like” set $C \subset \mathbb{R}$ such that $d_H(C') = d_B(C') = d_A(C') = \xi$. 
The 3D incompressible Navier-Stokes equations are

\[ u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{div} \, u = 0, \quad u(0) = u_0, \]

where (typically) \( u_0 \in L^2 \) or \( H^1 \) (on some domain, we will take \( \mathbb{T}^3 \) - the torus in \( \mathbb{R}^3 \)) and \( \text{div} \, u_0 = 0 \).

The essentials:

1) If \( u_0 \in H^1 \) then there exists a unique local smooth solution until at least \( T := M \| \nabla u_0 \|_{L^2}^{-4} \),

2) If \( u_0 \in L^2 \) then there exists a (nonunique) global weak solution (understood as a function \( u(x, t) \) such that \( u \in L^\infty((0, T); L^2) \cap L^2((0, T); H^1) \) and \( u \) satisfies the weak form of the Navier-Stokes equations (tested against divergence free test functions))

If \( u \) is a weak solution then \( u \in L^{10/3}((0, T) \times \mathbb{T}^3) \).
Moreover, for $u_0 \in L^2$ there exists a global weak solution $u$ such that the pair $(u, p)$ satisfies

a) $-\Delta p = \partial_{x_i} \partial_{x_j} (u_i u_j)$ for every $t \in (0, T)$,

b) $p \in L^{5/3}((0, T) \times \mathbb{T}^3)$,

c) the strong energy inequality

$$\frac{1}{2} \|u(t)\|_{L^2} + \int_s^t \|\nabla u\|^2 \leq \frac{1}{2} \|u(s)\|_{L^2}$$

for a.e. $s \geq 0$ and all $t > s$,

d) the local energy inequality

$$\int_{\mathbb{T}^3} |u(t)|^2 \phi \, dx + 2 \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \phi \, dx \, ds$$

$$\leq \int_0^t \int_{\mathbb{T}^3} |u|^2 (\phi_t + \Delta \phi) \, dx \, ds + \int_0^t \int_{\mathbb{T}^3} (|u|^2 + 2p) u \cdot \nabla \phi \, dx \, ds$$

for all $t \in (0, T)$ and for all $\phi \in C^\infty_c((0, T) \times \mathbb{T}^3)$ with $\phi \geq 0$.

This is called a suitable Leray-Hopf solution.
The parabolic cylinder $Q_r$:

$$Q_r(a, s) = \{ (x, t) : |x - a| < r, \ t \in (s - r^2, s) \}.$$
Theorem (First Local Regularity Theorem$^a$)

Let $(u, p)$ be a suitable Leray-Hopf solution of the Navier-Stokes equations. There exists an $\varepsilon_0 > 0$ such that whenever

$$
\frac{1}{r^2} \int_{Q_r} \left( |u|^3 + |p|^{3/2} \right) < \varepsilon_0
$$

then $u \in L^\infty(Q_{r/2})$.

Let $S$ denote the singular set of the $(u, p)$ (i.e. $(x, t) \in S \iff u(x, t)$ is unbounded in any neighbourhood of $(x, t)$).

Corollary (First Partial Regularity Theorem)

For any compact $K \subset (0, T) \times \mathbb{T}^3$ we have $d_B(S \cap K) \leq 5/3$.

(recall $d_B(S \cap K) := \limsup_{\varepsilon \to 0} -\log M(S \cap K, \varepsilon) / \log \varepsilon$)

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Partial regularity results, Navier-Stokes equations

**Theorem (Second Local Regularity Theorem\(^a\))**

Let \((u, p)\) be a suitable Leray-Hopf solution of the Navier-Stokes equations on some cylinder \(Q_R\). There exists an \(\varepsilon_1 > 0\) such that whenever

\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 < \varepsilon_1
\]

then \(u \in L^\infty(Q_\rho)\) for some \(\rho \in (0, R)\).

**Corollary (Second Partial Regularity Theorem)**

We have

\[d_H(S) \leq 1.\]

The 1D surface growth equation is

\[
    u_t + u_{xxxx} + (|u_x|^2)_{xx} = 0, \quad u(0) = u_0.
\]

The essentials:

1) If \( u_0 \in H^1(0, 2\pi) \) then there exists a unique local smooth solution until some time \( T > 0 \),

2) If \( u_0 \in L^2(0, 2\pi) \) then there exists a (nonunique) global weak solution (understood as a function \( u(x, t) \) such that \( u \in L^\infty((0, T); L^2) \cap L^2((0, T); H^2) \) and \( u \) satisfies the weak form of the equation).

Moreover, there exists a global weak solution satisfying the local energy inequality

\[
    \frac{1}{2} \int_0^{2\pi} |u(t)|^2 \phi \, dx + \int_0^t \int_0^{2\pi} u_{xx}^2 \phi \, dx \, ds
    \leq \int_0^t \int_0^{2\pi} \left( \frac{1}{2}(\phi_t + \phi_{xxxx})|u|^2 + 2u_x^2 \phi_{xx} - \frac{5}{3}u_x^3 \phi_x - u_x^2 u \phi_{xx} \right) \, dx \, ds.
\]
"First local regularity", surface growth model

Let $Q(x, t, r) := (t - r^4, t) \times (x - r, x + r)$.

Theorem (J. Robinson, D. Blömker, 2016$^a$)

There exists an $\varepsilon > 0$ such that whenever

$$\frac{1}{r^2} \int_{Q(x, t, r)} |u_x|^3 < \varepsilon$$

then $u$ is Hölder continuous in some neighbourhood of $(x, t)$.

Let $S$ denote the singular set of a weak solution $u$ of the surface growth model, that satisfies the local energy inequality.

Corollary

For any compact $K \subset (0, T) \times (0, 2\pi)$ we have $d_B(S \cap K) \leq 5/3$.

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Theorem (J. Robinson, D. Blömker, 2016\textsuperscript{a})

There exists a $\delta > 0$ such that whenever

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q(x,t,r)} u_{xx}^2 < \delta$$

then $u$ is regular at $(x,t)$.

Corollary

We have

$$d_H(S) \leq 1.$$  

Theorem (V. Scheffer, 1987\textsuperscript{a})

Let $\xi \in (0, 1)$. There exists a Cantor set $S \subset \mathbb{R}^3 \times \{1\}$ and functions $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ such that

1) $d_H(S) = \xi$,  
2) there exists a compact set $K \subset \mathbb{R}^3$ such that $\text{supp } u(t) \subset K$ for all $t$,  
3) $u(x, t)$ is a $C^\infty$ function for each $t$,  
4) there exists $M > 0$ such that $\|u(t)\|_{L^2} \leq M$ for all $t$,  
5) $\nabla u \in L^2((0, \infty) \times \mathbb{R}^3)$, $u \in L^3((0, \infty) \times \mathbb{R}^3)$, $|u||p| \in L^1((0, \infty) \times \mathbb{R}^3)$,  
6) the local energy inequality

\[
\int_{\mathbb{R}^3} |u(t)|^2 \phi \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx \, ds 
\leq \int_0^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta \phi) \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} (|u|^2 + 2p)u \cdot \nabla \phi \, dx \, ds
\]

holds for all $t \geq 0$ and for all $\phi \in C_c^\infty((0, \infty) \times \mathbb{R}^3)$ with $\phi \geq 0$,  
7) the singular set of $u$ contains $S$.

\textsuperscript{a} V. Scheffer, Nearly One Dimensional Singularities of Solutions to the Navier-Stokes Inequality, Comm. Math. Phys. 110, 525-551 (1987)
Directions:

1) Construct an example of a solution to the surface growth inequality (or a weak solution to the equation) with a given \( d_H(S) \in (0, 1) \),

2) Construct a shorter proof of the local regularity results for the surface growth model and try showing boundedness in the half-cylinder (modifying the approach of Caffarelli, Kohn, Nirenberg for Navier-Stokes equations),

3) Construct an example of a solution to the Navier-Stokes inequality with \( d_H(S) \leq 1 \) and \( d_B(S) > 1 \) (add concentration of blow-up times),

4) Establish any bounds on \( d_A \) of the singular set \( S \) of a solution to the Navier-Stokes equations.