The local energy inequality and the partial regularity results for the 3D incompressible Navier-Stokes equations.

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The 3D incompressible Navier-Stokes equations are

\[ u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{div } u = 0, \quad u(0) = u_0, \]

where (usually) \( u_0 \in L^2 \) or \( H^1 \) (on some domain, we will take \( \mathbb{T}^3 \) - the torus in \( \mathbb{R}^3 \)) and \( \text{div } u_0 = 0 \).

The essentials:

1) \( u_0 \in H^1 \Rightarrow \text{unique local strong solution} \) until at least \( T := M \| \nabla u_0 \|^{-4} \),

2) \( u_0 \in L^2 \Rightarrow \) a (nonunique) global \text{Leray-Hopf weak solution} (understood as a function \( u(x, t) \) such that \( u \in L^\infty((0, T); L^2) \cap L^2((0, T); H^1) \) and that \( u \) satisfies the weak form of the Navier-Stokes equations (tested against divergence free test functions)) satisfying the strong energy inequality

\[
\frac{1}{2} \| u(t) \|^2 + \int_s^t \| \nabla u \|^2 \leq \frac{1}{2} \| u(s) \|^2 \quad \text{for a.e. } s \geq 0 \text{ and every } t > s.
\]

Also, \( u \) is a weak solution \( \Rightarrow u \in L^{10/3}((0, T) \times \mathbb{T}^3) \).
Some other results:

1) $\|u_0\|_{H^1}$ small enough $\Rightarrow$ the global weak solution is strong,

2) $u_0 \in \dot{H}^{1/2}$ or $u_0 \in L^3 \Rightarrow$ local strong solution ($\dot{H}^{1/2}$, $L^3$ are the critical spaces for 3D NSE - i.e. the spaces with norms invariant under the rescaling $u_0(x) \mapsto \lambda u_0(\lambda x)$),

3) A weak solution $u$ is strong on $(s,t)$ if either

   (A) $u \in L^q((s,t), L^r(\mathbb{T}^3))$ for some $q$, $r$ satisfying $\frac{2}{q} + \frac{3}{r} \leq 1$ (the Serrin condition),

   (B) $\int_s^t ||\nabla u||_{L^\infty} < \infty$ or

   (C) $\int_s^t ||\text{curl } u||_{L^\infty} < \infty$ (the Beale-Kato-Majda condition).
Suppose \( u \) admits a blow-up \( \Rightarrow \) there exists a singular set

\[
S := \{(x, t) \in \mathbb{T}^3 \times (0, T) : u(x, t) \text{ is unbounded in any neighbourhood of } (x, t)\}.
\]

what is the 'size' of \( S \)?
Let $K \subset \mathbb{R}^n$.

1) Hausdorff dimension: for $\delta > 0$ let

$$
\mathcal{H}_{\delta}^s(K) := \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : K \subset \bigcup_{i=1}^{\infty} U_i, \text{diam } U_i \leq \delta \right\}.
$$

and

$$
\mathcal{H}^s(K) := \lim_{\delta \to 0} \mathcal{H}_{\delta}^s(K) \quad \leftarrow \text{this is } s\text{-dimensional Hausdorff measure}
$$

The **Hausdorff dimension** is

$$
d_H(K) := \inf \{ s \geq 0 : \mathcal{H}^s(K) = 0 \}.
$$

2) The **box-counting dimension** (also *fractal dimension* or *Minkowski dimension*) is

$$
d_B(K) := \limsup_{\varepsilon \to 0} \frac{\log N(K, \varepsilon)}{- \log \varepsilon},
$$

where $N(K, \varepsilon)$ is the minimum number of $\varepsilon$-balls required to cover $K$. 

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Wojciech Ożarński
Partial regularity of 3D NSE
Longville in the Dale, 24th May 2016 5 / 18
Notes:

1) If \( K = [0, 1]^m \) then \( d_H(K) = d_B(K) = m \),

2) If \( K \) is countable then \( d_H(K) = 0 \),

3) For any compact \( K \subset \mathbb{R}^n \)

\[
d_H(K) \leq d_B(K),
\]

4) In particular if \( K = \{0\} \cup \{1/m\}_{m=1}^{\infty} \) then \( d_H(K) = 0 \), but \( d_B(K) = 1/2 \).
For any $u_0 \in L^2$ and any $T > 0$ there exists a global Leray-Hopf weak solution $u$ such that the pair $(u, p)$ satisfies

a) $-\Delta p = \partial_{x_i} \partial_{x_j} (u_i u_j)$ for every $t \in (0, T)$,

b) $p \in L^{5/3}((0, T) \times \mathbb{T}^3)$,

c) the local energy inequality

$$
\int_{\mathbb{T}^3} |u(t)|^2 \phi(t) - \int_{\mathbb{T}^3} |u(s)|^2 \phi(s) + 2 \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \phi
$$

$$
\leq \int_0^t \int_{\mathbb{T}^3} |u|^2 (\phi_t + \Delta \phi) + \int_0^t \int_{\mathbb{T}^3} (|u|^2 + 2p) u \cdot \nabla \phi
$$

for all $s, t \in (0, T)$, $s < t$ and for all $\phi \in C^\infty_c((0, T) \times \mathbb{T}^3)$ with $\phi \geq 0$.

This is called a **suitable Leray-Hopf solution**.
The parabolic cylinder $Q_r$:

$$Q_r(a, s) = \{(x, t) : |x - a| < r, \ t \in (s - r^2, s)\}.$$
Theorem (First Local Regularity Theorem)

Let \((u, p)\) be a suitable Leray-Hopf solution of the Navier-Stokes equations. There exists an \(\epsilon_0 > 0\) such that whenever

\[
\frac{1}{r^2} \int_{Q_r} \left( |u|^3 + |p|^{3/2} \right) < \epsilon_0
\]

then \(u \in L^\infty(Q_{r/2})\).

Let \(S\) denote the singular set of the \((u, p)\).

Corollary (First Partial Regularity Theorem)

For any compact \(K \subset (0, T) \times \mathbb{T}^3\) we have \(d_H(S \cap K) \leq d_B(S \cap K) \leq 5/3\).

(recall \(d_B(S \cap K) := \limsup_{\epsilon \to 0} - \log N(S \cap K, \epsilon) / \log \epsilon\))

Sketch of the proof (without the pressure $p$)

Let $(a, s) \in Q_{1/2}(0, 0)$, $r_n := 2^{-n}$ and

$$(A_n) \quad \frac{1}{r_n^2} \int_{Q_{r_n}(a,s)} |u|^3 \leq \varepsilon_0^{2/3} r_n^3,$$

$$(B_n) \quad \frac{1}{r_n \sup_{s-r_n^2 < t < s}} \int_{B_{r_n}(a)} |u(t)|^2 + \frac{1}{r_n} \int_{Q_{r_n}(a,s)} |\nabla u|^2 \leq C_B \varepsilon_0^{2/3} r_n^2.$$

Induction:

(i) $(A_1)$ by the assumption,

(ii) $(B_n) \Rightarrow (A_n)$ by the interpolation inequality

$$\frac{1}{r^2} \int_{Q_r(a,s)} |u|^3 \leq C \left[ \frac{1}{r} \sup_{s-r^2 < t < s} \int_{B_r(a)} |u(t)|^2 + \frac{1}{r} \int_{Q_r(a,s)} |\nabla u|^2 \right]^{3/2},$$

(iii) $(A_1), \ldots, (A_{n-1}) \Rightarrow (B_n)$:
Sketch of the proof (without the pressure $p$)

(iii) $(A_1), \ldots, (A_{n-1}) \Rightarrow (B_n)$: by the local energy inequality

$$
\int_{B_1} |u(t)|^2 \phi(t) + 2 \int_{Q_1} |\nabla u|^2 \phi \leq \int_{Q_1} |u|^2 (\phi_t + \Delta \phi) + \int_{Q_1} |u|^2 u \cdot \nabla \phi,
$$

splitting up

$$
Q_1(a, s) = Q_{r_n}(a, s) \cup \bigcup_{k=1}^{n} Q_{r_{k-1}}(a, s) \setminus Q_{r_k}(a, s),
$$

a choice of test functions $\phi_n$, namely the $C_0^\infty(\mathbb{R}^4)$ functions such that

(i) $(\text{supp } \phi_n) \cap Q_1 \subset Q_{1/3}$,

(ii) $|\partial_t \phi_n + \Delta \phi_n| \leq C r_n^2$ for all $t \leq 0$,

(iii) $\frac{1}{C r_n} \leq \phi_n \leq \frac{C}{r_n}$ and $|\nabla \phi_n| \leq \frac{C}{r_n^2}$ in $Q_{r_n}$,

(iv) $|\nabla \phi_n| \leq C r_n^2 r_k^{-4}$ on $Q_{r_{k-1}} \setminus Q_{r_k}$ for all $k \in \{2, \ldots, n\}$,

and $(A_1), \ldots, (A_{n-1})$. 
Sketch of the proof (without the pressure $p$)

Finish:

$$(B_n)'s \Rightarrow \frac{1}{r_n^3} \int_{B_{r_n}(a)} |u(s)|^2 \leq C_B \varepsilon_0^{2/3} \text{ for all } (a, s) \in Q_{1/2}(0, 0), \text{ for all } n.$$ 

\[
\left( \begin{array}{c}
\text{Lebesgue} \\
\text{differentiation} \\
\text{theorem}
\end{array} \right) \Rightarrow |u(a, s)|^2 \leq C_B \varepsilon_0^{2/3} \text{ for a.e. } (a, s) \in Q_{1/2}(0, 0) \quad \square
\]
Recent improvements of the First Partial Regularity Theorem

Let \((u,p)\) be a suitable weak solution on some time interval \((0,T)\) and let \(S\) denote its singular set.

**Theorem**

(Kukavica, (2009)\(^a\)) \(d_B(S) \leq \frac{135}{82}(\approx 1.65)\),

(Kukavica & Pei (2012)\(^b\)) \(d_B(S) \leq \frac{45}{29}(\approx 1.55)\),

(Koh & Yang (March 2016)\(^c\)) \(d_B(S) \leq \frac{95}{63}(\approx 1.51)\),

(Wang & Wu (April 2016)\(^d\)) \(d_B(S) \leq \frac{180}{131}(\approx 1.37)\).


\(^b\) I. Kukavica, Y. Pei *An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system*, Nonlinearity 25(9):2775-2783 (2012)


Theorem (Second Local Regularity Theorem\textsuperscript{a})

Let \((u, p)\) be a suitable Leray-Hopf solution of the Navier-Stokes equations on some cylinder \(Q_R\). There exists an \(\varepsilon_1 > 0\) such that whenever

\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 < \varepsilon_1
\]

then \(u \in L^\infty(Q_\rho)\) for some \(\rho \in (0, R)\).

Corollary (Second Partial Regularity Theorem)

We have

\[
d_H(S) \leq 1.
\]

Theorem (Scheffer (1985))

There exist functions \( u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3 \) and \( p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R} \) such that

1) \( S \neq \emptyset \),

2) there exists a compact set \( K \subset \mathbb{R}^3 \) such that \( \operatorname{supp} u(t) \subset K \) for all \( t \),

3) \( u(x, t) \) is a \( C^\infty \) function for each \( t \geq 0 \),

4) there exists \( M > 0 \) such that \( \| u(t) \| \leq M \) for all \( t \geq 0 \),

5) \( \nabla u \in L^2((0, \infty) \times \mathbb{R}^3), u \in L^3((0, \infty) \times \mathbb{R}^3), |u| |p| \in L^1((0, \infty) \times \mathbb{R}^3) \),

6) the local energy inequality

\[
\int_{\mathbb{R}^3} |u(t)|^2 \phi(t) - \int_{\mathbb{R}^3} |u(s)|^2 \phi(s) + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \\
\leq \int_s^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta \phi) + \int_s^t \int_{\mathbb{R}^3} (|u|^2 + 2p)u \cdot \nabla \phi
\]

holds for all \( 0 \leq s < t \) and for all \( \phi \in C^\infty_c((0, \infty) \times \mathbb{R}^3) \) with \( \phi \geq 0 \).

Scheffer’s construction, sketch

Define $u^{(0)}$ on $(0, t_1)$ by

1) 

$$u^{(0)}(x_1, x_2, 0, 0) := u[v, f](x_1, x_2, 0) := (v_1, v_2, (f^2 - |v|^2)^{1/2}),$$

2) 

$$u^{(0)}(R_c(x_1, x_2, 0), 0) := R_c(u^{(0)}(x_1, x_2, 0, 0)),$$

where $R_c(x_1, x_2, x_3) := (x_1, c_1x_2 - c_2x_3, c_1x_3 + c_2x_2)$ with $c_1^2 + c_2^2 = 1$ denotes the rotation around the $Ox_1$ axis, $v = (v_1, v_2): \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the PDE

$$x_2 \text{div } v + v_2 = 0$$

and $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f > |v|$ everywhere,

3) 

$$|u^{(0)}(x, 0)| = f_1(x), \quad |u^{(0)}(x, t_1)| = f_2(x)$$

for all $x$, where $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ satisfy

$$f_2(\tau x + (a_1, a_2, 0)) \geq \tau^{-1} f_1(x)$$

for all $x$ for some fixed $\tau \in (0, 1)$, and $a := (a_1, a_2, 0) \in \mathbb{R}^3$. 
We construct $u^{(1)}$ on the time interval $(t_1, t_2)$ by the rescaling

$$u^{(1)}(x, t) := \tau^{-1} u^{(0)}(\tau^{-1} (x - a), \tau^{-2} (t - t_1)).$$

This way

$$|u^{(1)}(x, t_1)| = |\tau^{-1} u^{(0)}(\tau^{-1} (x - a), 0)|
= \tau^{-1} f_1(\tau^{-1} (x - a))
\leq f_2(x)
= |u^{(0)}(x, t_1)|.$$

Hence, by local energy inequality, we can combine $u^{(0)}$ and $u^{(1)}$ (one after another).
Directions:

1) Understand the solutions constructed by Scheffer and try to construct an example of a solution to the Navier-Stokes inequality with \( d_H(S) \leq 1 \) and \( d_B(S) > 1 \) (add concentration of blow-up times),

2) Consider the 1D surface growth model

\[
\frac{\partial u}{\partial t} + u_{xxxx} + (|u_x|^2)_{xx} = 0, \quad u(0) = u_0
\]

for \( x \in (0, 2\pi) \), \( t \geq 0 \). This model shares many striking similarities with the 3D incompressible Navier-Stokes equations, including the existence and uniqueness results.

The local energy inequality:

\[
\frac{1}{2} \int_0^{2\pi} |u(t)|^2 \phi(t) + \int_0^t \int_0^{2\pi} u^2_{xx} \phi \\
\leq \int_0^t \int_0^{2\pi} \left( \frac{1}{2} (\phi_t + \phi_{xxxx}) |u|^2 + 2u_x^2 \phi_{xx} - \frac{5}{3} u_x^3 \phi_x - u_x^2 u \phi_{xx} \right).
\]

2a) Prove the partial regularity theory for the surface growth model in the similar way as in the case of 3D NSE,

2b) Construct an example of a solution to the surface growth inequality (or a weak solution to the equation) with a given \( d_H(S) \in (0, 1) \).