Title: How to construct a solution to the Navier–Stokes inequality that blows up.

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Abstract: The 3D incompressible Navier–Stokes equations are the central model of fluid mechanics and, despite a significant effort since the celebrated work of Leray (1934), the fundamental question of existence of global in time strong solutions remains unsolved. Although we do not know whether singularities occur, we are able to estimate the Hausdorff dimension $d_H(S)$ of the possible set $S$ of singular points, that is the dimension of the set of points $(x,t)$ at which the velocity field $u$ blows up. This is possible thanks to the partial regularity theory developed by Caffarelli, Kohn & Nirenberg (1982). This theory is concerned with velocity fields satisfying the weak form of the Navier–Stokes inequality,

$$u \cdot (u_t - \Delta u + (u \cdot \nabla)u + \nabla p) \leq 0,$$

and it provides sufficient conditions for boundedness of $|u|$ in small cylinders in space-time, which implies the bound $d_H(S) \leq 1$. It turns out that this result is sharp, which is demonstrated by the work of Scheffer (1985 & 1987), who constructed weak solutions to the Navier–Stokes inequality with internal blow-ups. The blow-up in these solutions occurs on a set $S = (S', T_0)$, where $S' \subset \mathbb{R}^3$ either consist of one point or is a uniform Cantor set with $d_H(S') = \xi$ for any preassigned $\xi \in (0, 1)$. In the talk we will discuss some of the details of these constructions and we will see that they provide unique insights into the structure of the Navier–Stokes equations and suggest that the question of the existence of global in time strong solutions might have a negative answer.

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