11. Let \( \{S_n\} \) be a sequence such that \( S_n \to S \) as \( n \to \infty \). Define the Cesàro sum
\[
\sigma_n := \frac{S_1 + \ldots + S_n}{n}.
\]
Show that \( \sigma_n \to S \) as \( n \to \infty \). (i.e. Cesàro convergence extends the notion of “standard” convergence.)

(b) Let \( S_n := \sum_{k=1}^{n} (-1)^{k+1} \). Show that \( \sigma_n \to 1/2 \) as \( n \to \infty \).

12. Let \( \{a_k\} \) be a sequence such that the series \( \sum_{k=1}^{\infty} a_k \) is convergent. For \( r \in (0,1) \) consider the Abel sum \( \sum_{k=1}^{\infty} a_k r^k \).

(a) Show that \( \sum_{k=1}^{\infty} a_k r^k \) is convergent for every \( r \in (0,1) \).

(b) Show that
\[
\sum_{k=1}^{\infty} a_k r^k \to \sum_{k=1}^{\infty} a_k \quad \text{as} \quad r \to 1^-.
\]
(i.e. Abel summability extends the notion of “standard” summability.)

(c) Let \( a_k := (-1)^{k+1}k \). Show that
\[
\sum_{k=1}^{\infty} a_k r^k \to 1/4 \quad \text{as} \quad r \to 1^-.
\]

13. Let \( \{a_k\} \) be a sequence and \( S_n := \sum_{k=1}^{n} a_k \). Suppose that \( S_n \) is Cesàro summable to \( S \), i.e.
\[
S_1 + \ldots + S_n \to S \quad \text{as} \quad n \to \infty.
\]
Show that \( \sum_{k=1}^{\infty} a_k r^k \) converges for every \( r \in (0,1) \) and that
\[
\sum_{k=1}^{\infty} a_k r^k \to S \quad \text{as} \quad r \to 1^-.
\]
(i.e. Cesàro summability implies Abel summability.)

14. Let \( f \in L^1(\mathbb{R}) \) and let \( \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} \, dx \) be its Fourier transform. Let \( f_L \) be the restriction of \( f \) to \([-L/2, L/2)\) and let \( \hat{f}_L(n), n \in \mathbb{Z} \), denote the Fourier modes of \( f_L \), that is
\[
\hat{f}_L(n) := \frac{1}{L} \int_{-L/2}^{L/2} f(x)e^{-2\pi inx/L} \, dx.
\]
For \( \xi \in \mathbb{R} \) define
\[
g_L(\xi) := L \hat{f}_L(n) \quad \text{if} \quad \xi \in \left[ \frac{n}{L}, \frac{n + 1}{L} \right).
\]
(a) Show that for each $\xi \in \mathbb{R}$

$$g_L(\xi) \to \hat{f}(\xi) \quad \text{as} \quad L \to \infty.$$  

(i.e. Fourier modes approximate the Fourier transform $\hat{f}$ for large $L$.)

(b) Suppose additionally that $f \in C^1(\mathbb{R})$ and that, for some $F \in L^1(\mathbb{R})$, $|g_L(\xi)| \leq F(\xi)$ for all $L, \xi$. Consider the Fourier series of $f_L$,

$$Sf_L(x) := \sum_{n=-\infty}^{+\infty} \hat{f}_L(n)e^{2\pi inx/L}.$$  

Show that for each $x \in \mathbb{R}$

$$Sf_L(x) \to \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi \quad \text{as} \quad L \to \infty.$$  

(i.e. Fourier series approximates the Fourier inversion formula for large $L$.)

Deduce that in particular the Fourier inversion formula $f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi$ is satisfied for such $f$.

15. Let $H$ be a Banach space with norm $\| \cdot \|$ and let $\mathcal{S}$ be a dense subset of $H$. Let $T: \mathcal{S} \to \mathcal{S}$ be a linear operator such that $\|Tf\| = \|f\|$ (or $\|Tf\| \leq \|f\|$) for all $f \in \mathcal{S}$. Show that there exists a unique extension $\tilde{T}$ of $T$ to the whole of $H$ such that $\|\tilde{T}f\| = \|f\|$ (or, respectively, $\|\tilde{T}f\| \leq \|f\|$) for all $f \in H$.

(So that, if $H = L^2$, $\mathcal{S}$ is the Schwartz space and $T = \hat{\cdot}$ there is a unique extension $\mathcal{F}$ of $\hat{\cdot}$ to the whole of $L^2$.)

16. Let

$$f(x) := \begin{cases} 
1 & \text{for } x \in (-1/2, 1/2), \\
0 & \text{otherwise},
\end{cases} \quad g(x) := \begin{cases} 
1 - |x| & \text{for } x \in (-1, 1), \\
0 & \text{otherwise}.
\end{cases}$$

Calculate $\hat{f}(\xi)$ and hence show that

$$\hat{g}(\xi) = \left( \frac{\sin \pi \xi}{\pi \xi} \right)^2.$$  

17. Let $f \in L^1(\mathbb{R})$ and suppose that the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi$$

holds for all $x \in \mathbb{R}$. Suppose additionally that $\hat{f}(\xi) = 0$ for $|\xi| \geq M/2$ for some $M > 0$ (i.e. $f$ is a band-limited signal).

(a) Show that $f$ and $\hat{f}$ are continuous.
(b) Show that
\[
f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{M}\right) \frac{\sin(\pi(Mx-n))}{\pi(Mx-n)} \quad \text{for all } x \neq n/M, n \in \mathbb{Z}.
\]
(i.e. \( f \) is determined by its values at the points \( n/M, n \in \mathbb{Z} \).)

18. Let \( f \in L^1(\mathbb{R}) \) and suppose that the Fourier inversion formula holds in \( \mathbb{R} \). Show that if for some \( C > 0, \alpha \in (0,1) \)
\[
|\hat{f}(\xi)| \leq \frac{C}{|\xi|^{1+\alpha}} \quad \text{for all } \xi \in \mathbb{R},
\]
then \( f \in C^{0,\alpha}(\mathbb{R}) \), i.e. there exists \( M > 0 \) such that
\[
|f(x+h) - f(x)| \leq M|h|^\alpha \quad \text{for all } x, h \in \mathbb{R}.
\]
(i.e. the decay of \( \hat{f} \) is related to the continuity properties of \( f \).)

19. Prove that for all real \( f \in \mathcal{S} \) with \( \|f\|_{L^2} = 1 \) and for all \( x_0, \xi_0 \in \mathbb{R} \)
\[
\left( \int_{-\infty}^{+\infty} (x-x_0)^2|f(x)|^2\,dx \right) \left( \int_{-\infty}^{+\infty} (\xi-\xi_0)^2|\hat{f}(\xi)|^2\,d\xi \right) \geq \frac{1}{16\pi^2}.
\]
This is the Heisenberg uncertainty principle.

(In quantum mechanics the location of a particle on a line is described by a probability distribution, i.e. its location is a random variable \( X \). Suppose that \( X \) has a distribution given by a density \( |f|^2 \), i.e. the probability that the particle is located in the interval \((a,b)\) is \( \int_a^b |f|^2 \). Then one can check that the momentum of the particle is given by density \( |\hat{f}|^2 \), that is the momentum \( M \) is a random variable which has density \( |\hat{f}|^2 \). Then, taking \( x_0 := \mathbb{E}X, \xi_0 := \mathbb{E}M \), the Heisenberg uncertainty principle simply states that
\[
\text{Var}X \text{Var}M \geq 1/16\pi^2,
\]
i.e. there is an uncertainty in determining the location and the momentum of a particle.)

Prove that the equality holds if and only if \( f(x) = \pm \left( \frac{a}{2} \right)^{1/4} e^{-a|x|^2/2} \) for some \( a > 0 \).

(Lo(3 that the only particles with the optimal uncertainty are the ones described by a Gaussian distribution.)

20. Show that if \( \lambda \in \{1, -1, i, -i\} \) then \( \lambda \) is an eigenvalue of the Fourier transform, i.e.
\[
\hat{f} = \lambda f
\]
for some $f \in \mathcal{F}$. You can use the fact that

$$
\begin{align*}
    f_1(x) &:= xe^{-\pi x^2}, \\
f_2(x) &:= x^2 e^{-\pi x^2}, \\
f_3(x) &:= x^3 e^{-\pi x^2},
\end{align*}
$$

then

$$
\begin{align*}
    \hat{f}_1(\xi) &= -i\xi e^{-\pi \xi^2}, \\
    \hat{f}_2(\xi) &= \frac{1-2\xi^2}{2\pi} e^{-\pi \xi^2}, \\
    \hat{f}_3(\xi) &= \frac{\xi(2\xi^2-3)}{2\pi} e^{-\pi \xi^2}.
\end{align*}
$$

Show that these four $\lambda$'s are the only eigenvalues of the Fourier transform.

21.
(a) Show that

$$
\int_{\mathbb{R}^d} \hat{f}(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \hat{g}(x) dx \quad \text{whenever } f, g \in L^1(\mathbb{R}^d)
$$

and

$$
f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \quad \text{for almost every } x \in \mathbb{R}^d
$$

whenever $f, \hat{f} \in L^1(\mathbb{R}^d)$. Deduce that $f$ can be modified on a set of measure zero such that the resulting $f$ is continuous.

(This is an extension of the result for Schwartz functions from the lecture.)

(b) Suppose that the space dimension $d = 1$ and that $f, \hat{f}$ are of moderate decrease, i.e. there exists $C > 0$ such that

$$
|f(x)|, |\hat{f}(x)| \leq \frac{C}{1 + x^2} \quad \text{for all } x \in \mathbb{R}.
$$

Show that then both $f, \hat{f}$ are almost Lipschitz in a sense that for each $\alpha \in (0, 1)$ there exists $C_\alpha > 0$ such that

$$
|f(x) - f(y)| \leq C_\alpha |x - y|^{\alpha}, \quad |\hat{f}(x) - \hat{f}(y)| \leq C_\alpha |x - y|^{\alpha} \quad \text{for all } x, y \in \mathbb{R}.
$$

22.
Suppose that $f, \hat{f}$ are of moderate decrease.

(a) Show that

$$
\sum_{n \in \mathbb{Z}} f(x + n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i x n} \quad \text{for all } x \in \mathbb{R},
$$

which, letting $x := 0$, reduces to

$$
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).
$$

(This is the Poisson summation formula.)

(b) Show that

$$
\sum_{n \in \mathbb{Z}} e^{-2\pi \varepsilon |n|} = \frac{\varepsilon}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\varepsilon^2 + n^2} \quad \text{for all } \varepsilon > 0.
$$

(c) Deduce that

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$
23.
(a) Let \( F_N(y) \) denote the \( N \)-th Fejér kernel on the circle, that is
\[
F_N(y) := \frac{D_0(y) + \ldots + D_{N-1}(y)}{N} \quad \text{for } y \in [-\pi, \pi],
\]
where \( D_n(y) := \sum_{k=-n}^{n} e^{iky} \) denotes the \( n \)-th Dirichlet kernel.
Show that
\[
F_N(y) = \sum_{n=-N}^{N} \left( 1 - \left| \frac{n}{N} \right| \right) e^{iny} \quad \text{for all } y \in [-\pi, \pi].
\]
(b) Let \( F_N(y) \) denote the \( N \)-th Fejér kernel on the line, that is
\[
F_N(y) := \frac{1}{N} \sin^2 \left( \frac{\pi Ny}{\pi y} \right).
\]
Use the Poisson summation formula to obtain
\[
F_N(2\pi x) = \sum_{n \in \mathbb{Z}} F_N(x + n) \quad \text{for all } x.
\]
(i.e. the Fejér kernel on the circle is a periodization of the Fejér kernel on the line.)

24.
(a) Consider the Cauchy problem for the heat equation on the unit interval \((0, 1)\) with periodic boundary conditions, that is the problem
\[
\begin{align*}
&u_t(x, t) = u_{xx}(x, t) \quad \text{for } (x, t) \in [0, 1] \times (0, \infty), \\
&u(0, t) = u(1, t) \quad \text{for } t \geq 0, \\
&u(x, 0) = f(x) \quad \text{for } x \in [0, 1],
\end{align*}
\]
where \( f \) is a given function such that \( f(0) = f(1) \) and \( f \in L^1((0, 1)) \). Use the method of separation of variables to deduce that a function
\[
\begin{equation}
\tag{1}
u(x, t) := \sum_{n \in \mathbb{Z}} f_n e^{-4\pi^2 n^2 t} e^{2\pi inx}
\end{equation}
\]
is a solution of the above problem, where \( f_n \) is the \( n \)-th Fourier mode of \( f \) (i.e. \( f_n := \int_0^1 f(x) e^{-2\pi inx} dx \)).
(b) Show that (1) is equivalent to
\[
u(x, t) = (f * H_t)(x),
\]
where \( f * g(x) := \int_0^1 f(y) g(x-y) dy \) is the convolution on the torus (for 1-periodic functions \( f, \) \( g \)) and
\[
H_t(x) := \sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi inx}
\]
is the heat kernel on the unit interval.

(c) Now consider the Cauchy problem for the heat equation on the line, that is the problem
\[
\begin{cases}
  u_t(x,t) = u_{xx}(x,t) & \text{for } (x,t) \in \mathbb{R} \times (0, \infty), \\
  u(x,0) = g(x) & \text{for } x \in \mathbb{R},
\end{cases}
\]
where \( g \in L^1(\mathbb{R}) \) is a given function. Repeat the steps from the lecture to obtain that
\[
u(x,t) := (g * \mathcal{H}_t)(x)
\]
is a solution to the above problem, where
\[
\mathcal{H}_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}
\]
is the heat kernel on the line.

(d) Use the Poisson summation formula to deduce that
\[
\mathcal{H}_t(x) = \sum_{n \in \mathbb{Z}} \mathcal{H}_t(x + n) \quad \text{for all } x.
\]
(i.e. the heat kernel on the unit interval \( \mathcal{H}_t \) is a periodization of the heat kernel on the line \( \mathcal{H}_t \).

25.

Consider a Cauchy problem for the wave equation in \( \mathbb{R}^3 \):
\[
\begin{cases}
  u_{tt}(x,t) = \Delta u(x,t) & \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\
  u(x,0) = f(x) & \text{for } x \in \mathbb{R}^3, \\
  u_t(x,0) = g(x) & \text{for } x \in \mathbb{R}^3,
\end{cases}
\]
where \( f,g \in \mathcal{S}(\mathbb{R}^3) \) and \( \Delta \) denotes the Laplacian, i.e. \( \Delta u := u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} \).

(a) Take the Fourier transform of the wave equation to obtain that a solution to the above problem satisfies
\[
\hat{u}(\xi,t) = \hat{f}(\xi) \cos(2\pi |\xi| t) + \hat{g}(\xi) \frac{\sin(2\pi |\xi| t)}{2\pi |\xi|} \quad \text{for all } \xi \neq 0.
\]

(b) Show that
\[
u(x,t) := \int_{\mathbb{R}^3} \left( \hat{f}(\xi) \cos(2\pi |\xi| t) + \hat{g}(\xi) \frac{\sin(2\pi |\xi| t)}{2\pi |\xi|} \right) e^{2\pi i x \cdot \xi} d\xi
\]
is a classical solution to (2).

(c) Let \( \alpha \in \mathbb{R} \), \( a, b \in \mathbb{C} \). Show that
\[
|a \cos \alpha + b \sin \alpha|^2 + |a \sin \alpha + b \cos \alpha|^2 = |a|^2 + |b|^2.
\]

(d) Define the energy of the solution \( u \) to the wave equation by
\[
E(t) := \int_{\mathbb{R}^3} (|u_t(x,t)|^2 + |\nabla u(x,t)|^2) \, dx.
\]
Use the Plancherel property of the Fourier transform to show that the solution of (2) given by (3) above conserves the energy, i.e. that

\[ E(t) = E(0) \quad \text{for all } t \in \mathbb{R}. \]

26.

(a) Prove that

\[ \frac{1}{4\pi} \int_{\partial B(0,1)} e^{-2\pi i \xi \cdot y} dS(y) = \frac{\sin(2\pi |\xi|)}{2\pi |\xi|} \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\}. \]

(b) For \( f \in \mathcal{S}(\mathbb{R}^3) \) define the spherical mean \( M_t f(x) \) of radius \( t > 0 \) of \( f \) by

\[ M_t f(x) := \frac{1}{4\pi} \int_{\partial B(0,1)} f(x + ty) dS(y) \quad \text{for } x \in \mathbb{R}^3. \]

Show that

\[ \widehat{M_t f}(\xi) = \frac{i}{2\pi |\xi| t} \frac{\sin(2\pi |\xi| t)}{2\pi |\xi|} \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}, t > 0. \]

(c) Hence show that for \( f, g \in \mathcal{S}(\mathbb{R}^3) \) the formula (3) is equivalent to

\[ u(x,t) = \frac{\partial}{\partial t} (tM_t f(x)) + tM_t g(x). \]

Deduce that if \( g \equiv 0 \) then \( u(x,t) \) depends only on the behaviour of \( f \) in the immediate neighbourhood of the sphere \( \partial B(x,t) \). In particular, the speed of propagation of a wave is finite (and equals 1). Moreover, if \( \text{supp } f \subset B(0,R) \) for some \( R > 0 \) then there exists \( T > 0 \) such that \( u(x,t) = 0 \) for \( x \in B(0,R), t > T \).

(This is the so-called Huygens principle, i.e. in 3-dimensional space (or, more generally, in any odd dimension space) the initial condition \( u(x,0) = f(x) \) can be thought of the “initial sound”. Then the Huygens principle says that at time \( t > 0 \) the observer located in the origin only “hears” the initial sound that is located \( t \) away from the origin.)

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