The Monge-Ampère equation and the two-dimensional Navier-Stokes equations
Summer Research Project dissertation

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Abstract

The dissertation focuses on the theory of the real Monge-Ampère equation and its application in the theory of the two-dimensional Navier-Stokes equations. We discuss the fundamental results regarding convex functions, the normal mapping and the concept of the Monge-Ampère measure. We then study the convergence of the Monge-Ampère measures corresponding to given convergent sequence of convex functions. We consider relevant inequalities, the maximum principle and the comparison principle of convex functions. We prove the existence and uniqueness of convex solutions to both homogeneous and inhomogeneous Dirichlet problems for the Monge-Ampère equation. We study a comparison principle for not necessarily convex functions and resulting bounds on such solutions to the Monge-Ampère equation. We then prove, for an even space dimension $n$, a nonexistence result of $W^{2,n}$ solutions of the Monge-Ampère equations with a nonpositive right-hand side and constant boundary data. This, in particular, gives a restriction on the class of functions for the pressure $p$ in the Navier-Stokes equations.
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1 Introduction

The Monge-Ampère equation $\det D^2 \phi = f$ is a fully nonlinear second order partial differential equation which arises in several areas, including the problem of prescribed Gauss curvature in Riemannian geometry and optimal transport. The subject of this dissertation is the application of the Monge-Ampère equation in the theory of the two-dimensional, incompressible Navier-Stokes equations. Namely, writing the velocity $u$ as $u = (\phi_y, -\phi_x)$ for some scalar function $\phi$ one can obtain a two-dimensional Monge-Ampère equation for $\phi$:

$$\det D^2 \phi = \frac{1}{2} \Delta p. \tag{1.1}$$

The study of this equation is an attempt to answer an open problems for the 2D Navier-Stokes equations, namely whether one can determine the velocity $u$ of the fluid from the pressure $p$. It is not known how to solve the resulting Monge-Ampère equation (1.1) as $\Delta p$ is not in general nonnegative and hence the equation is not elliptic. The purpose of the project is to study this equation with a possibly negative/sign-changing right-hand side.

1.1 Motivation

Let us consider the 2D incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0 \quad \tag{1.2}$$
$$\text{div } u = 0, \quad \tag{1.3}$$

where $\nu > 0$ is the kinematic viscosity. Following the calculation presented in Robinson [40], p. 6, we take the divergence of (1.2) and using the incompressibility condition (1.3), we obtain

$$\nabla \cdot [(u \cdot \nabla) u] + \Delta p = 0. \quad \tag{1.4}$$

Note that the condition (1.3) is equivalent to $\partial_x u_1 + \partial_y u_2 = 0$. This means that $u$ can be described by a velocity potential, i.e. one can write

$$u = (u_1, u_2) = (\partial_y \phi, -\partial_x \phi). \quad \tag{1.5}$$
1.1 Motivation

Substituting this into (1.4) one obtains

\[
- \Delta p = ((\phi_y)^2)_{xx} + ((\phi_x)^2)_{yy} - 2(-\phi_x \phi_y)_{xy}
= 2\phi_{xy}\phi_{xy} + 2\phi_y \phi_{xxx} + 2\phi_x \phi_{xyy} + 2\phi_{xy}\phi_{xy} - 2\phi_{xxy} \phi_{yy} - 2\phi_{xx} \phi_{yy}
- 2\phi_{xy} \phi_{xy} - 2\phi_x \phi_{xyy} = -2(\phi_{xx} \phi_{yy} - (\phi_{xy})^2).
\]

Hence one obtains the equation

\[
\phi_{xx} \phi_{yy} - (\phi_{xy})^2 = \frac{1}{2} \Delta p.
\] (1.6)

This means that the function \( \phi \) satisfies the Monge-Ampère equation, which was first observed by Larchevêque [29]. This equation is elliptic when the right-hand side is positive (recall that a nonlinear second-order equation \( F(x, \phi, \nabla \phi, D^2 \phi) = 0 \) is elliptic if the matrix \( \{ \frac{\partial F}{\partial r_{ij}} \} \) is positive definite, where \( F = F(x, z, p, r) \), and substitute \( F(x, z, p, r) = \det r \), see Gilbarg & Trudinger [24], p. 441, for the details). We note that the right hand side \( \frac{1}{2} \Delta p \) is not, in general, positive. This means that the general theory of the Monge-Ampère equation (see e.g. Gilbarg & Trudinger [24], pp. 467-482) will, in general, not apply in this case as the character of the equation may possibly change inside the domain.

It is natural to ask for which pressure functions \( p \) there exists a solution \( \phi \) of (1.6). This is, however, a difficult question. Nevertheless, it turns out that that we can eliminate some classes of functions for the pressure \( p \). For this let us consider \( u \in H^1_0(\Omega) \). This translates, using the substitution (1.5), to boundary conditions for \( \phi \) of the form

\[
\nabla \phi = 0 \quad \text{on } \partial \Omega.
\] (1.7)

In particular this means that \( \frac{\partial \phi}{\partial \nu} = 0 \) on \( \partial \Omega \), where \( \nu \) denotes the outward normal vector at the boundary \( \partial \Omega \). It is straightforward to show that there is no solution \( \phi \in C^2(\Omega) \) to the problem

\[
\begin{cases}
\phi_{xx} \phi_{yy} - (\phi_{xy})^2 = \frac{1}{2} \Delta p & \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\] (1.8)

if \( \Delta p > 0 \) in \( \Omega \). In fact, we can prove this directly, without introducing any theory of the Monge-Ampère equation. Writing

\[
\phi_{xx} \phi_{yy} \geq \phi_{xx} \phi_{yy} - \phi_{xy}^2 = \det D^2 \phi > 0
\]
we see that $\phi_{xx}, \phi_{yy}$ must have the same sign in $\Omega$ (by a continuity argument), i.e. either

$$\phi_{xx}, \phi_{yy} > 0 \text{ in } \Omega \quad \text{or} \quad \phi_{xx}, \phi_{yy} < 0 \text{ in } \Omega.$$ (1.9)

Hence, supposing that $\phi_{xx}, \phi_{yy} > 0$, we can use the divergence theorem to obtain

$$0 < \int_{\Omega} \Delta \phi \, dx \, dy = \int_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \, dS = 0,$$

a contradiction (and similarly for $\phi_{xx}, \phi_{yy} < 0$).

If we weaken the condition $\Delta p > 0$ to $\Delta p \geq 0$ in $\Omega$ then the nonexistence result of solution of (1.8) no longer holds. In fact, one can come up with the following counterexample (or rather an example of existence).

**Example 1.1.** Let $\Omega = (-2, 2)^2 \subset \mathbb{R}^2$ and let $\phi(x, y) := v(x)$, where $v : \mathbb{R} \to \mathbb{R}$ is any function such that $v \in C^2([-1, 1])$. It is easy to see that the boundary condition $\frac{\partial \phi}{\partial \nu} = 0$ is satisfied everywhere on $\partial \Omega$. Moreover

$$\det D^2 \phi = \det \begin{pmatrix} v'' & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{in } \Omega$$

(obviously, this example could be modified to the case of a domain $\Omega$ that is of smooth regularity).

However, there are no results (of which the author is aware) of nonexistence of a solution $\phi$ to the problem (1.8) if we consider the boundary condition (1.7) instead of the condition $\frac{\partial \phi}{\partial \nu} \big|_{\partial \Omega} = 0$.

Let us now focus on the case of $\Delta p \leq 0$ in $\Omega$. Considering constant boundary data $\phi \big|_{\partial \Omega} = C$, which comes from the boundary condition (1.7) provided sufficient regularity of $\partial \Omega$, one can also show a nonexistence result. Namely, there is no $\phi \in W^{2,2}(\Omega)$ (with $\Omega \subset \mathbb{R}^2$ open, bounded, strictly convex) that is a solution of the problem

$$\begin{cases} 
\phi_{xx} \phi_{yy} - (\phi_{xy})^2 = \frac{1}{2} \Delta p & \text{in } \Omega, \\
\phi = C & \text{on } \partial \Omega 
\end{cases}$$ (1.10)

with $\Delta p \leq 0$ in $\Omega$, $\Delta p \neq 0$ (see Corollary 5.3). This statement is by no means an elementary fact. As we will see, it requires an investigation of the theory of the Monge-Ampère equation. We will find out that the non-existence of the solution of the problem (1.10) can be proved essentially using a generalized comparison principle.
(see Theorem 5.1), which implies that any possible solution of this problem is bounded below by the unique convex solution of the respective Dirichlet boundary value problem and, similarly, bounded above by the unique concave solution (see Theorem 5.2). Since in the case of $\Delta p \leq 0$ both the convex and the concave solution will turn out to be constant functions, the nonexistence result will follow, see Corollary 5.3. This is the main result of the dissertation.

1.2 Literature review

The theory of the Monge-Ampère equation goes back to the work of A. V. Pogorelov [35], [36], [37] and A. D. Aleksandrov (see e.g. Kutateladze [28]), which focused mostly on geometry of convex surfaces and on the problem of finding a closed convex surface in $\mathbb{R}^n$ with prescribed Gaussian curvature. A more recent exposition, with the viewpoint of partial differential equations, can be found in Bakelman [5], Gutiérrez [25], Chapter 8 of Aubin [4] and Chapter 17 of Gilbarg & Trudinger [24]. Some important results regarding the Monge-Ampère equation include the notion of generalised solution (see Pogorelov [35], [36]), the notion of viscosity solution and its equivalence to the generalised solution (see Crandall, Ishii & Lions [16] and Caffarelli [10], pp. 137-139), Aleksandrov maximum principle (see Aleksandrov [2]), classification of Monge-Ampère equations (see Lychagin, Rubtsov & Chekalov [31]) and, most notably, Caffarelli’s regularity results, see Caffaralli [11] (see also Caffarelli [10], [12], [13], [14]). The Monge-Ampère equation has wide applications in geometry, particularly in the problem of prescribed Gaussian curvature (see Kutateladze [28], Pogorelov [36], [37], Urbas [42] and Oliker [32], [33]) and in optimal control, particularly in the mass transfer problems (see Evans [21], Evans & Gangbo [22], Dacorogna & Moser [18] and Benamou & Brenier [6]).

1.3 Structure of the dissertation

The structure of the report is as follows: Section 2 presents the concepts that are fundamental in the theory of Monge-Ampère equation. It includes a description of the properties of convex sets, convex functions and the concept of supporting hyperplanes, which is a generalisation of the concept of tangent hyperplanes. This leads to the idea of the normal mapping and consequently to the concept of Monge-Ampère measures, which is the main subject of the next section, Section 3. In that section, through the discussion of the convergence properties of Monge-Ampère measures, we learn the importance of uniform convergence on compact sets of convex functions (this
convergence is sufficient for the weak convergence of the corresponding Monge-Ampère measures). We further present the inequalities between Monge-Ampère measures of two convex functions, Aleksandrov maximum principle and comparison principle of convex functions. All these results will turn out essential in Section 4 where we show the existence of generalised solutions to homogeneous and inhomogeneous Dirichlet problems with continuous boundary data. In the same section we also prove the existence of at most two $C^2$ solutions of a 2D "strictly inhomogeneous" Dirichlet problem and we reflect on the similarities between Monge-Ampère equation and the Laplace equation. In Section 5 we prove a generalised comparison principle, which is a comparison principle for not necessarily convex functions. We then use this comparison principle to show that any solution of the Monge-Ampère equation, with not necessarily positive right-hand side, is bounded below and above by, respectively, the unique concave solution and the unique convex solution of a certain corresponding Dirichlet problem. Section 6 summarises the main results of the project and outlines future perspectives.

The results presented in the dissertation are, in most part, based on Gutiérrez [25]. The dissertation includes a number of the author’s own contributions, which include the main result of the project, Section 5.2. Moreover, the presented proofs are, unless stated otherwise, constructed independently.

2 Basic Concepts

The main aim of this section is to understand the essential properties of convex functions and normal mapping, which we will use in the presentation of the Monge-Ampère theory later.

2.1 Convex sets

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\phi : \Omega \to \mathbb{R}$. In most of the analysis of the Monge-Ampère equation we will consider $\Omega$ convex (i.e. if $x, y \in \Omega$ then $\lambda x + (1 - \lambda)y \in \Omega$ for all $\lambda \in [0, 1]$) or strictly convex (i.e. if $x, y \in \overline{\Omega}$ then $\lambda x + (1 - \lambda)y \in \Omega$ for all $\lambda \in (0, 1)$).

One of the geometric importance of convexity is that we have supporting hyperplanes at each point on the boundary. In other words, we have the following elementary results.

**Lemma 2.1.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be an open, nonempty convex set and let $x_0 \in \partial \Omega$. 
(a) (Separation lemma) There exists an affine hyperplane \( \kappa \), defined by an equation
\[
a \cdot x + \alpha = 0,
\]
where \( a \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathbb{R} \), such that
\[
a \cdot x_0 + \alpha = 0 \quad \text{and} \quad \Omega \subset \{a \cdot x + \alpha > 0\}.
\]

(b) If \( \Omega \) is strictly convex then the supporting hyperplane \( \kappa \) is such that for each \( r > 0 \) there exists \( \eta > 0 \) such that
\[
\{x \in \overline{\Omega} : a \cdot x + \alpha \leq \eta\} \subset B(x_0, r)
\]
(see Fig. 1)

![Figure 1](image)

Clearly property (b) does not hold for general convex sets (take for instance \( \Omega := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\} \)).

**Proof.**

(a) This fact follows from the Geometric Hahn-Banach Theorem, see Leunberger [30], p. 133, for the proof and Ekeland & Témam [19], p. 5, for interesting applications. We note that this fact is true also in any infinite-dimensional topological vector space.

(b) Suppose otherwise that for all \( \eta > 0 \) we have \( \{x \in \overline{\Omega} : a \cdot x + \alpha \leq \eta\} \not\subset B(x_0, r) \). This means that there exist sequences \( \{\eta_k\}_{k \geq 1} \subset \mathbb{R}, \{x_k\}_{k \geq 1} \subset \mathbb{R}^n \) such that \( \eta_k \to 0^+ \) and \( x_k \in \overline{\Omega} \),
\[
a \cdot x_k + \alpha \leq \eta_k,
\]
and \( |x_k - x_0| \geq r \) for all \( k \). Moreover we can take \( \{x_k\}_{k \geq 1} \) to be such that \( |x_k - x_0| = r \) for all \( k \). In fact, for a given \( x_k \) we can define \( x_k := \frac{r}{|x_k - x_0|} x_k + \)}
2.2 Supporting hyperplanes

Denition 2.2. Given \( x_0 \in \Omega \), a supporting hyperplane to the function \( \phi \) at the point \( x_0 \) is an affine function \( l(x) = \phi(x_0) + p \cdot (x - x_0) \), \( p \in \mathbb{R}^n \), such that \( \phi(x) \geq l(x) \) for all \( x \in \Omega \) (see Fig. 2).

Figure 2: Supporting hyperplanes (note this is a 1D sketch of an \( n \)-dimensional situation).

We note that, even though the supporting hyperplane is defined for any function \( \phi \), its usual application is for \( \phi \) convex. That is because any convex function \( \phi: \Omega \to \mathbb{R} \) has at least one supporting hyperplane at each point \( x \in \Omega \) (see Lemma 2.7 (a)). Most of the theory of the Monge-Ampère equation presented in this report is concerned
with supporting hyperplanes and convex functions. However, considering hyperplanes supporting a function from above, i.e. affine functions \( l(x) := \phi(x_0) + p \cdot (x - x_0) \) such that \( \phi(x) \leq l(x) \) for all \( x \in \Omega \), we can obtain similar results for concave functions. From now on, we will be concerned only with supporting hyperplanes from below (and hence with convex functions) and we will keep in mind that the analysis of supporting hyperplanes from above (which corresponds to concave functions) is analogous.

We also note that for a convex function \( \phi \), a sufficient condition for \( l(x) = \phi(x_0) + p \cdot (x - x_0) \) to be a supporting hyperplane to \( u \) at point \( x_0 \in \Omega \) is that \( l(x) \) supports \( \phi \) over any open neighbourhood \( U \) of \( x_0 \).

**Lemma 2.3.** If \( \phi : \Omega \to \mathbb{R} \) is convex and \( \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) \) for all \( x \in U \), where \( U \subset \Omega \) is open and \( x_0 \in U \), then \( \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) \) for all \( x \in \Omega \).

**Proof.** Suppose otherwise that there exists \( x_1 \in \Omega \) such that \( \phi(x_1) < \phi(x_0) + p \cdot (x_1 - x_0) \). As \( U \) is open there exists \( \lambda \in (0, 1) \) such that \( x_\lambda := \lambda x_1 + (1 - \lambda) x_0 \in U \) and we get

\[
\phi(x_\lambda) \leq \lambda \phi(x_1) + (1 - \lambda) \phi(x_0) < \lambda(\phi(x_0) + p \cdot (x_1 - x_0)) + (1 - \lambda) \phi(x_0) = \phi(x_0) + p \cdot (x_\lambda - x_0) \leq \phi(x_\lambda),
\]
a contradiction. \( \square \)

**Definition 2.4.** The **normal mapping** of \( \phi \), or subdifferential of \( \phi \), is the set-valued function \( \partial \phi : \Omega \to \mathcal{P}(\mathbb{R}^n) \) defined by

\[
\partial \phi(x_0) := \{ p \in \mathbb{R}^n : \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) \ \forall x \in \Omega \}.
\]

Given \( E \subset \Omega \) we define \( \partial \phi(E) = \bigcup_{x \in E} \partial \phi(x) \).

We will need the following lemma.

**Lemma 2.5.** If \( \gamma > 0 \), then \( \partial(\gamma \phi)(E) = \gamma(\partial \phi(E)) \)

**Proof.**

\[
p \in \partial(\gamma \phi)(E) \iff \exists x_0 \in E \ \forall x \in \Omega \ \gamma \phi(x) \geq \gamma \phi(x_0) + p \cdot (x - x_0)
\]

\[
\iff \exists x_0 \in E \ \forall x \in \Omega \ \phi(x) \geq \phi(x_0) + \frac{1}{\gamma} p \cdot (x - x_0) \iff \frac{1}{\gamma} p \in \partial \phi(E) \square
\]
2.3 Some facts about convex functions

There are several properties of convex functions that make them the main subject of the analysis related to the Monge-Ampère equation. First of all, we have the following bound on the slope of the supporting hyperplane to a convex function.

**Lemma 2.6 (Bound on $|p|$).** Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set and $\phi$ a convex function in $\Omega$ such that $\phi \leq M$ on $\partial \Omega$ for some $M \in \mathbb{R}$. If $x \in \Omega$ and $p \in \partial \phi(x)$, then

$$|p| \leq \frac{M - \phi(x)}{\text{dist}(x, \partial \Omega)}$$

(see Fig. 3).

![Figure 3: Bound on $|p|$ (note this is a 1D sketch of an $n$-dimensional situation).](image)

**Proof** (we simplify the proof from Gutiérrez [25], p. 48). The claim follows trivially if $p = 0$. If $p \neq 0$ we have $\phi(y) \geq \phi(x) + p \cdot (y - x)$ for all $y \in \Omega$ and, by continuity of $\phi$, also for all $y \in \overline{\Omega}$. In particular, if $y_0 := x + \text{dist}(x, \partial \Omega) \frac{p}{|p|} \in \partial \Omega$ then $M \geq \phi(y_0) \geq \phi(x) + p \cdot (y_0 - x) = \phi(x) + \text{dist}(x, \partial \Omega)|p|$. 

We have the following properties of convex functions.

**Lemma 2.7.** Let $\phi : \Omega \to \mathbb{R}$ be convex. Then

(a) $\phi$ has a supporting hyperplane at every point $x \in \Omega$,

(b) $\phi$ is continuous,

(c) for $K \subset \Omega$ compact, $\partial \phi(K)$ is compact,

(d) for $K \subset \Omega$ compact, $\phi$ is uniformly Lipschitz in $K$,

(e) $\phi$ is differentiable a.e. in $\Omega$. 
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Proof.

(a) If \( \phi \) is convex then \( A := \{(x,z) \in \Omega \times \mathbb{R} : \phi(x) < z\} \) is convex. Indeed, if \((x_1,z_1),(x_2,z_2) \in A\) then \( \phi(x_1) < z_1, \phi(x_2) < z_2 \) and hence, for all \( \lambda \in [0,1] \),

\[
\phi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda \phi(x_1) + (1-\lambda)\phi(x_2) < \lambda z_1 + (1-\lambda)z_2,
\]

which means that \( \lambda(x_1,z_1) + (1-\lambda)(x_2,z_2) \in A \).

Now let \( x_0 \in \Omega \). From the separation lemma (Lemma 2.1, (a) ) we know that there exists a hyperplane \( \kappa \subset \mathbb{R}^{n+1} \) that separates \( A \) and \( (x_0, \phi(x_0)) \). In other words, if \( a \cdot x + az + \beta = 0, a \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R} \) with \( (a, \alpha) \neq 0 \) is the equation of \( \kappa \), then

\[
a \cdot x + \alpha z + \beta > 0 \quad \forall (x,z) \in A, \quad a \cdot x_0 + \alpha \phi(x_0) + \beta = 0. \tag{2.2}
\]

We will show that \( \alpha > 0 \). Taking any \( z_0 \) greater than \( \phi(x_0) \) we have \((x_0,z_0) \in A\) and hence

\[
a \cdot x_0 + \alpha z_0 + \beta > 0 = a \cdot x_0 + \alpha \phi(x_0) + \beta.
\]

Therefore \( \alpha(\phi(x_0) - z_0) < 0 \) and consequently \( \alpha > 0 \). Hence

\[
z > -\frac{1}{\alpha}a \cdot x - \frac{\beta}{\alpha} =: l(x) \quad \forall (x,z) \in A.
\]

Hence, as \((x,\phi(x)) \in \overline{A}\) for each \( x \in \Omega \), we obtain, by continuity, \( \phi(x) \geq l(x) \) for all \( x \in \Omega \). Noting that \( \phi(x_0) = l(x_0) \) we deduce that \( l(\cdot) \) is a supporting hyperplane of \( \phi \) at \( x_0 \).

(b) Let \( x_0 \in \Omega \), we will show that \( \phi \) is continuous at \( x_0 \). Without loss of generality we can assume that \( x_0 = 0 \) and that \( Q := [-1,1]^n \subset \Omega \) (the general case follows by translation and scaling). For \( x \in Q \) we have \( \frac{x+1}{2} \in [-1,1] \) and, by convexity,

\[
\phi(x) = \phi\left(\frac{x_1+1}{2}(-1,x_2,\ldots,x_n) + \frac{1-x_1}{2}(1,x_2,\ldots,x_n)\right)
\leq \frac{x_1+1}{2} \phi(-1,x_2,\ldots,x_n) + \frac{1-x_1}{2} \phi(1,x_2,\ldots,x_n)
\leq \frac{x_1+1}{2} \sup_{y \in \partial Q} \phi(y) + \frac{1-x_1}{2} \sup_{y \in \partial Q} \phi(y) = \sup_{y \in \partial Q} \phi(y).
\]

Hence \( \phi \) is bounded by its values on \( \partial Q \). Repeating this reasoning for each of the sides of \( \partial Q \) (i.e. considering a function \( \psi_m^{\pm 1} : [-1,1]^{n-1} \), defined by
For \( y \in \partial Q \) and \( \lambda \in [0, 1] \) write
\[
\phi(\lambda y) \leq (1 - \lambda)\phi(0) + \lambda\phi(y) = \phi(0) + \lambda(\phi(y) - \phi(0)),
\]
which is equivalent to
\[
\phi(\lambda y) - \phi(0) \leq \lambda(\phi(0) - \phi(-y)).
\]

Also, for all \( t \in [0, \frac{1}{2}] \) we have
\[
\phi(0) = \phi\left((1 - t)\frac{t}{1 - t}y + t(-y)\right) \leq (1 - t)\phi\left(\frac{t}{1 - t}y\right) + t\phi(-y),
\]
which, after dividing by \( 1 - t \) and setting \( \lambda := \frac{t}{1 - t} \in [0, 1] \), gives
\[
(1 + \lambda)\phi(0) \leq \phi(\lambda y) + \lambda\phi(-y)
\]
or, equivalently,
\[
-\lambda(\phi(-y) - \phi(0)) \leq \phi(\lambda y) - \phi(0).
\]
This, together with (2.3) gives as the following lower and upper bounds on the difference \( \phi(\lambda y) - \phi(0) \):
\[
-\lambda(\phi(-y) - \phi(0)) \leq \phi(\lambda y) - \phi(0) \leq \lambda(\phi(0) - \phi(-y)).
\]
Now for \( \lambda \in [0, 1] \) let \( Q_\lambda := \{ \lambda y : y \in Q \} \). For \( z \in \partial Q_\lambda \) (2.4) implies that
\[
|\phi(z) - \phi(0)| \leq \lambda \left| \sup_{y \in \partial Q} \phi(y) - \phi(0) \right| \leq \lambda(C - \phi(0)).
\]
On the other hand, for any \( z \in Q_\lambda \), there exists \( \lambda_1 \leq \lambda \) such that \( z \in \partial Q_{\lambda_1} \) and
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hence

$$|\phi(z) - \phi(0)| \leq \lambda_1(C - \phi(0)) \leq \lambda(C - \phi(0))$$ \quad \forall z \in Q_\lambda

Taking \(\lambda \to 0^+\) gives continuity of \(\phi\) at 0.

(c) (we modify the proof from Gutiérrez [25], p. 2) Let \(\{p_k\}_{k \geq 1} \subset \partial \phi(K)\) be any sequence. For each \(k\) there exists \(x_k \in K\) such that \(p_k \in \partial \phi(x_k)\), that is

$$\phi(x) \geq \phi(x_k) + p_k \cdot (x - x_k)$$ \quad for all \(x \in \Omega\).

Since \(K\) is compact, there exists a subsequence, which we will also denote \(x_k\), such that \(x_k \xrightarrow{k \to \infty} x_0\) for some \(x_0 \in K\). Since the set \(K_\delta := \{x \in \Omega : \text{dist}(x, K) \leq \delta\}\) is compact and contained in \(\Omega\) for sufficiently small \(\delta\) and since \(x_k \in \text{Int}K_\delta\) for all \(k\), we have, by Lemma 2.6,

$$|p_k| \leq \frac{1}{\delta} \left( \max_{K_\delta} \phi - \min_K \phi \right),$$

where \((\max_{K_\delta} \phi - \min_K \phi) < \infty\) by continuity of \(\phi\) (see (b) ). Hence the sequence \(\{p_k\}_{k \geq 1}\) is bounded. Consequently, there exists a convergent subsequence \(\{p_{k_m}\}_{m \geq 1}\) such that \(p_{k_m} \xrightarrow{m \to \infty} p_0\) for some \(p_0 \in \mathbb{R}^n\) with \(|p_0| \leq \frac{1}{\delta} (\max_{K_\delta} \phi - \min_K \phi)\). We will show that \(p_0 \in \partial \phi(K)\). We note that

$$\phi(x) \geq \phi(x_{k_m}) + p_{k_m} \cdot (x - x_{k_m}) \quad \forall x \in \Omega, \forall m.$$

Hence, taking the limit \(m \to \infty\) and using continuity of \(\phi\) we get \(\phi(x) \geq \phi(x_0) + p_0 \cdot (x - x_0)\) for all \(x \in \Omega\). Hence \(p_0 \in \partial \phi(x_0) \subset \partial \phi(K)\).

(d) (we paraphrase the proof from Gutiérrez [25], p. 3) From (a) \(\phi\) has a supporting hyperplane at any \(x \in \Omega\), hence, for each \(x \in K\) we have \(p_x \in \partial \phi(x)\) such that \(\phi(y) \geq \phi(x) + p_x \cdot (y - x)\) for all \(y \in \Omega\). Let \(C := \sup_{p \in \partial \phi(K)} |p|\). From (c) we have \(C < \infty\) and therefore we may write

$$\phi(y) - \phi(x) \geq -|p_x||y - x| \geq -C|y - x| \quad \forall x, y \in K.$$

This and reversing the roles of \(x\) and \(y\) gives \(|\phi(y) - \phi(x)| \geq C|y - x|\) for all \(x, y \in K\).

(e) (following Lemma 1.1.7 in Gutiérrez [25]) The function \(\phi\) is locally Lipschitz (by (d) ) and hence it is differentiable a.e. in \(\Omega\) by Rademacher’s Theorem (see
Corollary 2.8. If $\phi \in C(\overline{\Omega})$ is convex and $\sup_{\overline{\Omega}} |\phi| \leq M$ then for every $K \subset \Omega$ compact
\[ |\phi(x) - \phi(y)| \leq \frac{2M}{\text{dist}(K, \partial \Omega)} |x - y| \quad \forall x, y \in K. \]

Proof. Follows directly from Lemma 2.6 and the proof of point (d) of Lemma 2.7.

2.4 Properties of the normal mapping

The Legendre transform is a useful tool in studying the normal mapping.

Definition 2.9. The Legendre transform of the function $\phi : \Omega \to \mathbb{R}$ is the function $\phi^* : \mathbb{R}^n \to \mathbb{R}$ defined by
\[ \phi^*(p) := \sup_{x \in \Omega} (x \cdot p - \phi(x)) \]
(see Fig. 4).

![Figure 4: Interpretation of the Legendre transform (note this is a 1D sketch of an $n$-dimensional situation).](image)

We have the following two simple facts.

Fact 2.10. Let $\phi^*$ be the Legendre transform of $\phi$. Then

(a) $\phi^*$ is convex in $\mathbb{R}^n$,

(b) if $\Omega$ is bounded and $\phi$ is bounded in $\Omega$, then $\phi^*$ is finite (at each $p \in \mathbb{R}^n$).

Proof.

(a) Let $p_1, p_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. We have, for all $x \in \Omega$,
\[ x \cdot (\lambda p_1 + (1 - \lambda) p_2) - \phi(x) = \lambda (x \cdot p_1 - \phi(x)) + (1 - \lambda) (x \cdot p_2 - \phi(x)) \leq \lambda \phi^*(p_1) + (1 - \lambda) \phi^*(p_2). \]
Taking $\sup_{x \in \Omega}$ finishes the proof.
(b) We have $\phi^*(p) \leq \sup_{x \in \Omega} |x| |p| + \sup_{x \in \Omega} |\phi(x)| \leq C_1 |p| + C_2$ for each $p \in \mathbb{R}^n$. □

We can use Legendre transform to prove the following result, which is important in studying convergence properties of Monge-Ampère measures (see Lemma 3.6 (b)) and in the proof of existence of solution of the homogeneous Dirichlet problem (Theorem 4.2)

**Lemma 2.11.** If $\Omega$ is open and $\phi$ is a continuous function in $\Omega$, then the set of points in $\mathbb{R}^n$ that belong to the image by the normal mapping of more than one point of $\Omega$ has Lebesgue measure zero. That is, the set

$$S := \{ p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \text{ and } p \in \partial \phi(x) \cap \partial \phi(y) \}$$

has measure zero. This also means that the set of supporting hyperplanes that touch the graph of $\phi$ at more than one point has measure zero.

**Proof** (we paraphrase the proof from Gutiérrez [25], p. 4). We may assume that $\Omega$ is bounded because otherwise we can write $\Omega = \bigcup_{k \geq 1} \Omega_k$, where $\Omega_k$ is open, $\overline{\Omega_k}$ is compact and $\Omega_k \subseteq \Omega_{k+1}$ for all $k$ (one can take for instance $\Omega_k := \Omega \cap B(0, k)$). If $p \in S$, then there exist $x, y \in \Omega$, $x \neq y$ with $\phi(z) \geq \phi(x) + p \cdot (z - x)$, $\phi(z) \geq \phi(y) + p \cdot (z - y)$ for all $z \in \Omega$. Since $\{\Omega_k\}_{k \geq 1}$ is an increasing family of sets, $x, y \in \Omega_{k_0}$ for some $k_0 \geq 1$ and obviously the previous inequalities hold true for $z \in \Omega_{k_0}$. That is, if

$$S_{k_0} := \left\{ p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \text{ and } p \in \partial \phi|_{\Omega_{k_0}}(x) \cap \partial \phi|_{\Omega_{k_0}}(y) \right\}$$

then we have $p \in S_{k_0}$. This means that $S \subseteq \bigcup_{k} S_k$ and then we show that each $S_k$ has measure zero. Hence we may assume boundedness of $\Omega$.

Let $\phi^*$ be the Legendre transform of $\phi$. By Fact 2.10 $\phi^*$ is convex and hence differentiable almost everywhere (see Lemma 2.7 (e) ). We will show that if $p \in S$ then $u^*$ is not differentiable at $p$, which implies $|S| = 0$. Indeed, if $p \in S$, then $p \in \partial \phi(x_1) \cap \partial \phi(x_2)$ with $x_1 \neq x_2$. This means that $\phi(z) \geq \phi(x_i) + p \cdot (z - x_i)$ for all $z \in \Omega$, $i = 1, 2$ or, equivalently,

$$\phi^*(p) = x_i \cdot p - \phi(x_i), \quad i = 1, 2.$$

On the other hand $\phi^*(z) \geq x_i \cdot z - \phi(x_i)$ for all $z \in \Omega$ and so $\phi^*(z) \geq \phi^*(p) + x_i \cdot (z - p)$ for all $z$ and $i = 1, 2$. Hence, if $\phi^*$ were differentiable at $p$ we would have $\nabla \phi^*(p) = x_i$, $i = 1, 2$, a contradiction since $x_1 \neq x_2$. □
3 The Monge-Ampère measure

3.1 Introduction

We define the Monge-Ampère measure to be the Lebesgue measure of the normal mapping. We have the following lemma.

**Lemma 3.1.** For $\phi \in C(\Omega)$ let

$$S := \{ E \subset \Omega : \partial \phi(E) \text{ is Lebesgue measurable} \}$$

Then $S$ is a Borel $\sigma$-algebra and the set function $M\phi : S \to \mathbb{R}$ defined by

$$M\phi(E) := |\partial \phi(E)|,$$

where $\mathbb{R} := \mathbb{R} \cup \{\infty\}$, is a measure, that is finite on compact sets. This measure is called the **Monge-Ampère measure associated with the function $\phi$.**

The proof of the above lemma is rather technical and hence it is omitted. It can be found in Gutiérrez [25], p. 5.

The reason for the name of this measure is clear from the following lemma.

**Lemma 3.2.** If $\phi \in C^2(\overline{\Omega})$ then $M\phi$ is absolutely continuous with respect to the Lebesgue measure and

$$M\phi(E) = \int_E \det D^2\phi \, dx$$

for all Borel sets $E \subset \Omega$.

The proof of this lemma, which uses Sard’s Theorem, can be found in Gutiérrez [25], p. 4.

**Example 3.3** (Examples of Monge-Ampère measures). Let $x_0 \in \Omega$.

(a) If $\phi \in C^2(\overline{\Omega})$ then, by Lemma 3.2,

$$M\phi(E) = |\nabla \phi(E)|$$

for all Borel subsets $E \subset \Omega$.

In particular, if $\phi(x) := q \cdot x + \alpha$, where $q \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, then $M\phi = 0$. 

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3.2 Convergence properties

3.2.1 THE MONGE-AMPÈRE MEASURE

(b) If \( \phi(x) := \delta|x - x_0| \) then

\[
\partial \phi(x) = \begin{cases} 
B(0, \delta) & \text{for } x = x_0, \\
\frac{|x - x_0|}{|x - x_0|} & \text{for } x \neq x_0.
\end{cases}
\]

Hence for each Borel subset \( E \subset \mathbb{R}^n \)

\[
M_\phi(E) = \begin{cases} 
B(0, \delta) & \text{if } \{x_0\} \subset E, \\
|\nabla \phi(E)| & \text{if } \{x_0\} \not\subset E.
\end{cases}
\]

(c) If \( \phi(x) := \delta|x - x_0|^2 \) then, \( D^2\phi = 2\delta I \) and so by Lemma 3.2,

\[
M_\phi(E) = \int_E \det D^2\phi \, dx = (2\delta)^n|E|.
\]  \( \text{(3.1)} \)

(d) If \( \gamma > 0 \), then \( M(\gamma\phi) = \gamma^n M\phi \) for every \( \phi \) convex (cf. Lemma 2.5).

3.2 Convergence properties

We want to prove certain convergence properties of Monge-Ampère measures.

Definition 3.4. Let \( \mu, \mu_j \) be Borel measures defined on a \( \sigma \)-algebra of Borel subsets of an open set \( \Omega \subset \mathbb{R}^n \) and let \( \mu(\Omega), \mu_j(\Omega) \leq M \) for some \( M > 0 \). We say that \( \mu_j \) converge weakly to \( \mu \) (denoted \( \mu_j \rightharpoonup \mu \)) if

\[
\int_{\Omega} f(x) d\mu_j(x) \xrightarrow{j \to \infty} \int_{\Omega} f(x) d\mu(x) \quad \forall f \in C_0(\Omega).
\]

We will show that uniform convergence on compact subsets of convex functions is sufficient for the weak convergence of the respective Monge-Ampère measures, which will later turn out to be essential in showing the existence of solutions to an inhomogeneous Dirichlet problem of Monge-Ampère equation (Section 4.3). The convergence properties can also be used in proving inequalities regarding Monge-Ampère measures (Section 3.3). Namely, they enable one to use approximation arguments in showing the superadditivity of Monge-Ampère measure in the case of convex functions that are not \( C^2 \) (see Lemma 3.10).

In the statement of convergence lemma (Lemma 3.6), which we borrow from Gutiérrez [25], pp. 6-7, we will apply the formulation of the Fatou lemma which differs from its usual occurrence in analysis (i.e. if \( f_j \in L^1 \), \( f_j \geq 0 \) and \( \liminf_{j \to \infty} \int f_j < \infty \)
then \( \int \liminf_{j \to \infty} f_j \leq \liminf_{j \to \infty} \int f_j \) (see Alt, [3], p. 96) and which is not explained in detail in Gutiérrez [25]. For this reason we first introduce this formulation. Recall the definitions

\[
\limsup_{k \to \infty} A_k := \bigcap_{k \geq 0} \bigcup_{m \geq k} A_m, \quad \liminf_{k \to \infty} A_k := \bigcup_{k \geq 0} \bigcap_{m \geq k} A_m.
\]

In other words, \( \limsup_{k \to \infty} A_k \) consists of elements that belong to \( \textit{ininitely many} \) \( A_k \) and \( \liminf_{k \to \infty} A_k \) consists of elements that belong to \( \textit{all but finitely many} \) \( A_k \). Obviously, \( \liminf_{k \to \infty} A_k \subset \limsup_{k \to \infty} A_k \). We have the following result.

**Lemma 3.5** (Fatou lemma). Let \( |\cdot| \) be a measure defined on a \( \sigma \)-algebra \( \Sigma \). If \( A_k \in \Sigma \) then \( \liminf_{k \to \infty} A_k \in \Sigma \) and

\[
\left| \liminf_{k \to \infty} A_k \right| \leq \liminf_{k \to \infty} |A_k|. \quad (3.2)
\]

In particular if \( A \subset \liminf_{k \to \infty} A_k \) then \( |A| \leq \liminf_{k \to \infty} |A_k| \).

Similarly, \( \limsup_{k \to \infty} A_k \in \Sigma \),

\[
\limsup_{k \to \infty} |A_k| \leq \limsup_{k \to \infty} |A_k|
\]

and if \( \limsup_{k \to \infty} A_k \subset A \) then \( \limsup_{k \to \infty} |A_k| \leq A \).

**Proof** (inspired by the proof of the "usual" Fatou lemma in Alt, [3], pp. 96-97). We have \( \liminf_{k \to \infty} A_k = \bigcup_{k \geq 0} \bigcap_{m \geq k} A_m \), hence \( \liminf_{k \to \infty} A_k \in \Sigma \) as a countable intersection and sum of sets \( A_m \in \Sigma \). Let

\[
G_k := \bigcap_{m \geq k} A_m.
\]

Then \( \liminf_{k \to \infty} A_k = \bigcup_k G_k \), \( G_k \) is an increasing family of sets (i.e. \( G_k \subset G_{k+1} \)) and \( G_k \subset A_j \) for all \( j \geq k \). Hence also \( |G_k| \leq |A_j| \) for all \( j \geq k \) and taking \( \liminf_{j \to \infty} \) we get

\[
|G_k| \leq \liminf_{j \to \infty} |A_j| \quad \forall k \quad (3.3)
\]

Letting \( k \to \infty \) and using the monotonicity \( G_k \subset G_{k+1} \) we get

\[
|\liminf_{j \to \infty} A_j| = \left| \bigcup_k G_k \right| = \lim_{k \to \infty} |G_k| \leq \liminf_{j \to \infty} |A_j|.
\]

For the \( \limsup_{k \to \infty} A_k \) replace "\( \liminf \)" and "\( \bigcup \)" with "\( \limsup \)" and "\( \bigcap \)" respectively.
\[ \limsup \bigcup_k \bigcap_i \]" decreasing respectively. \[ \limsup_{k \to \infty} \partial \phi_k(K) \subset \liminf_{k \to \infty} \partial \phi_k(V_k), \]

where the inequality holds for almost every point of the set on the left-hand side, and by the Fatou lemma

\[ M\phi(U) \leq \liminf_{k \to \infty} M\phi_k(V_k). \]

Proof (we modify the proof of Lemma 1.2.2 from Gutièrrez [25])

(a) If \( p \in \limsup_{k \to \infty} \partial \phi_k(K) \), then for each \( n \) there exist \( k_n \) and \( x_{k_n} \in K \) such that \( p \in \partial \phi_{k_n}(x_{k_n}) \). By selecting a subsequence \( x_j \) of \( x_{k_n} \), we may assume that \( x_j \to x_0 \in K \). On the other hand,

\[ \phi_j(x) \geq \phi_j(x_j) + p \cdot (x - x_j), \quad \forall x \in \Omega, \]

and by letting \( j \to \infty \) we get, by the uniform convergence of \( \phi_j \) on compact sets,

\[ \phi(x) \geq \phi(x_0) + p \cdot (x - x_0), \quad \forall x \in \Omega, \]

that is \( p \in \partial \phi(x_0) \subset \partial \phi(K) \).
(b) Let \( S := \{ p : p \in \partial \phi(x_1) \cap \partial \phi(x_2) \text{ for some } x_1, x_2 \in \Omega, x_1 \neq x_2 \} \). By Lemma 2.11, \(|S| = 0\). Consider \( \partial \phi(U) \setminus S \). If \( p \in \partial \phi(U) \setminus S \), then there exists a unique \( x_0 \in U \) such that \( p \in \partial \phi(x_0) \). Hence

\[
\phi(x) > \phi(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega, x \neq x_0.
\]

Otherwise, if \( \phi(x_1) = \phi(x_0) + p \cdot (x_1 - x_0) \) for some \( x_1 \in \Omega, x_1 \neq x_0 \) then \( \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) = \phi(x_1) + p \cdot (x - x_1) \) for all \( x \in \Omega \). Letting \( \lambda \in (0,1) \) be such that \( x_\lambda := \lambda x_0 + (1 - \lambda)x_1 \in U \) (such a \( \lambda \) exists because \( U \) is open). Since \( p \in \partial \phi(x_0) \) and since \( \phi \) is convex, we have for all \( x \in \Omega \)

\[
\phi(x) = \lambda \phi(x) + (1 - \lambda)\phi(x) \\
\geq \lambda(\phi(x_0) + p \cdot (x - x_0)) + (1 - \lambda)(\phi(x_1) + p \cdot (x - x_1)) \\
= \lambda \phi(x_0) + (1 - \lambda)\phi(x_1) + p \cdot (x - x_\lambda) \geq \phi(x_\lambda) + p \cdot (x - x_\lambda),
\]

that is, \( p \in \partial \phi(x_\lambda) \), which contradicts the uniqueness of \( x_0 \).

Now, since \( \overline{V}_1 \subset \Omega \) is compact and \( \phi_k \to \phi \) uniformly on \( \overline{V}_1 \), we have from (3.6) that for any \( \delta > 0 \) there exists \( k_\delta \) such that \( \phi_k(x) \geq \phi_k(x_0) + p \cdot (x - x_0) - \delta \) for all \( x \in \overline{V}_1 \) and \( k \geq k_\delta \). Fix any \( \delta > 0 \) and hence also \( k_\delta \). Let

\[
\delta_k := \min_{x \in \overline{V}_k} \{ \phi_k(x) - \phi_k(x_0) - p \cdot (x - x_0) + \delta \}.
\]

This minimum is attained at some \( x_k \in \overline{V}_k \), i.e.

\[
\delta_k = \phi_k(x_k) - \phi_k(x_0) - p \cdot (x_k - x_0) + \delta.
\]

Hence we have

\[
\phi_k(x) \geq \phi_k(x_0) + p \cdot (x - x_0) - \delta + \delta_k \overset{(3.8)}{=} \phi_k(x_k) + p \cdot (x - x_k) \quad \forall x \in \overline{V}_k
\]

and, in particular,

\[
\phi_k(x) \geq \phi_k(x_k) + p \cdot (x - x_k) \quad \forall x \in U
\]

We will show that \( p \) is the slope of a supporting hyperplane to \( \phi_k \) at the point \( x_k \) for all but finitely many \( k \). Note that, by Lemma 2.3 (for an affine function to support a convex function at a given point over \( \Omega \) it is sufficient to support it over
an open neighbourhood of this point), we only need to show that \( x_k \in V_k \) for all but finitely many \( k \). Suppose otherwise that \( x_k \in \partial V_k \) for infinitely many \( k \), i.e. suppose that there exists a subsequence \( \{x_{k_m}\}_{m \geq 1} \subset V_1 \) such that \( x_{k_m} \in \partial V_{k_m} \) for all \( m \geq 1 \). As \( V_1 \) is compact, there exists a subsequence, which we will also denote \( \{x_{k_m}\}_{m \geq 1} \), such that \( x_{k_m} \xrightarrow{m \to \infty} y \) for some \( y \in V_1 \). Moreover \( y \neq x_0 \) as \( |x_{k_m} - x_0| \geq \text{dist}(x_0, \partial V_{k_m}) \geq \text{dist}(x_0, \partial U) > 0 \) for all \( m \). By the uniform convergence of \( \phi_k \) to a continuous function \( \phi \) we get \( \phi_{k_m}(x_{k_m}) \xrightarrow{m \to \infty} \phi(y) \) (indeed, one can write \( |\phi_{k_m}(x_{k_m}) - \phi(y)| \leq |\phi_{k_m}(x_{k_m}) - \phi(x_{k_m})| + |\phi(x_{k_m}) - \phi(y)| \leq ||\phi_{k_m} - \phi||_{L^{\infty}(\Omega)} + |\phi(x_{k_m}) - \phi(y)| \xrightarrow{m \to \infty} 0 \)). Hence, taking \( \lim_{m \to \infty} \) in (3.10) and letting \( x := x_0 \) we get

\[
\phi(x_0) \geq \phi(y) + p \cdot (x_0 - y) \quad (3.6)
\]

a contradiction. Therefore indeed \( x_k \in V_k \) for all but finitely many \( k \) and consequently \( p \in \partial \phi_k(x_k) \subset \partial \phi_k(V_k) \) for all but finitely many \( k \). This means that \( p \in \liminf_{k \to \infty} \partial \phi_k(V_k) \). \( \square \)

One of the consequences of this lemma is the weak convergence of Monge-Ampère measures. Namely, we have the following result (formulated without proof in Gutiérrez [25], p. 8).

**Lemma 3.7.** If \( \phi_k \) are convex functions in \( \Omega \) such that \( \phi_k \to \phi \) uniformly on compact subsets of \( \Omega \) and \( M\phi(\Omega), M\phi_k(\Omega) \leq L \) for some constant \( L > 0 \), then the associated Monge-Ampère measures \( M\phi_k \) tend to \( M\phi \) weakly, that is

\[
\int_{\Omega} f(x) \, dM\phi_k(x) \to \int_{\Omega} f(x) \, dM\phi(x)
\]

for every \( f \) continuous with compact support in \( \Omega \) (cf. Definition 3.4).

**Proof** (inspired by a suggestion of one of the supervisors). We will assume that \( f \geq 0 \) (the general case follows by considering the positive and negative parts \( f^+, f^- \) and writing \( f = f^+ - f^- \)).

Fix \( \epsilon > 0 \) and let \( F := \text{max}_f \) and \( \Omega' := \{ x \in \Omega : f(x) > 0 \} \equiv \text{Int}(\text{supp} \, f) \). Since \( f \) is integrable with respect to \( M\phi \), there exist \( N > 0, a_i \in (0, L], U_i \subset \Omega' \) Borel sets, \( i = 1, \ldots, N \), such that \( U_i \cap U_j = \emptyset \) for \( i \neq j \), \( \sum_{i=1}^{N} a_i \chi_{U_i} \leq f \) (where \( \chi_{E} \) denotes the characteristic function of a borel set \( E \)) and

\[
\sum_{i=1}^{N} a_i M\phi(U_i) \leq \int_{\Omega'} f \, dM\phi \leq \sum_{i=1}^{N} a_i M\phi(U_i) + \epsilon. \quad (3.11)
\]
Moreover, we will show that, by continuity of \( f \), we can take \( U_i \) to be open. For this reason let

\[
\mathcal{T} := \{ \tau \in \mathbb{R} : M\phi(\{x \in \Omega : f = \tau\}) = 0 \}.
\]

Note that \((-\infty, 0] \cup (F, +\infty) \subset \mathcal{T} \). We have \((0, F] \setminus \mathcal{T} = \bigcup_{n \geq 1} \mathcal{T}_n \), where

\[
\mathcal{T}_n := \{ \tau \in (0, F] : M\phi(\{x \in \Omega : f = \tau\}) \geq 1/n \}.
\]

It is clear that \( \# \mathcal{T}_n \leq Ln \) (otherwise \( L < \sum_{\tau \in \mathcal{T}_n} 1/n \leq \sum_{\tau \in \mathcal{T}_n} M\phi(\{f = \tau\}) = M\phi(\bigcup_{\tau \in \mathcal{T}_n} \{f = \tau\}) \leq M\phi(\Omega') \leq L \), a contradiction). Consequently, \([0, F] \setminus \mathcal{T} \) is at most countable. We will use this fact to see that \( \mathcal{T} \) is dense in \((0, F] \). In fact, if \( \mathcal{T} \) is not dense in \((0, F] \), then there exists an open interval \((a, b) \subset (0, F] \) such that \((a, b) \cap \mathcal{T} = \emptyset \), i.e. \((a, b) \subset (0, F] \setminus \mathcal{T} \), which is a contradiction since \((a, b) \) is uncountable. Hence \( \mathcal{T} \) is indeed dense in \((0, F] \).

Therefore, we can find \( N \geq 1 \) and a sequence \( \{\tau_i\}_{i=0,\ldots,N} \subset \mathcal{T} \) such that \( 0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < F < \tau_N \) and \( \tau_i - \tau_{i-1} \leq \frac{\epsilon}{L} \) for all \( i = 1, \ldots, N \) (for instance take any \( N \geq 4FL/\epsilon \) and \( \{\tau_i\}_{i=0,\ldots,N} \subset \mathcal{T} \) such that \( \tau_0 := 0 \), \( \tau_N := F \left( 1 + \frac{1}{4N} \right) \) and \(|\tau_i - \tau_{i-1}| \leq \frac{\epsilon}{2N} \) for \( i = 1, \ldots, N-1 \).

We now define

\[
U_i := \{ x \in \Omega' : \tau_{i-1} < f(x) < \tau_i \} \quad \text{for} \quad i = 1, \ldots, N.
\]

We note that the sets \( U_i \), \( i = 1, \ldots, N \) are open (by continuity of \( f \)) and mutually disjoint and that \( \sum_{i=1}^N \tau_{i-1} \chi_{U_i} \leq f \). We also have

\[
M\phi\left(\Omega' \setminus \bigcup_{i=1}^N U_i\right) = \sum_{i=1}^N M\phi(\{f = \tau_i\}) = 0 \quad (3.12)
\]

and therefore

\[
\int_{\Omega'} f \, dM\phi - \sum_{i=1}^N \tau_{i-1} M\phi(U_i) = \sum_{i=1}^N \int_{U_i} (f - \tau_{i-1}) \, dM\phi \leq \sum_{i=1}^N \int_{U_i} (\tau_i - \tau_{i-1}) \, dM\phi \leq \frac{\epsilon}{L} \sum_{i=1}^N M\phi(U_i) \leq \frac{\epsilon}{L} M\phi(\Omega') \leq \epsilon,
\]

which means that (3.11) holds (with \( a_i := \tau_{i-1} \) for all \( i = 1, \ldots, N \), as claimed.

For each \( i \) let \( \{V^k_i\}_{k \geq 1} \) be a family of open sets such that \( V^k_i \subset \Omega, \overline{U_i} \subset V^k_i \), \( V^{k+1}_i \subset V^k_i \) and \( M\phi_k(V^k_i) \leq M\phi_k(U_i) + \frac{\epsilon}{2^{3N}} \) for all \( k \geq 1 \) (such a family exists
by compactness of supp f and by the regularity of Borel measures \( M\phi_k, k \geq 1 \), see Parthasarathy [34], p. 27). Note that, by point (b) of the above lemma, we have
\[ M\phi(U_i) \leq \lim \inf_{k \to \infty} M\phi_k(V_i^k) \] for each \( i = 1, \ldots, N \) and hence
\[
\int \Omega f \, dM\phi = \int \Omega' f \, dM\phi \leq \sum_{i=1}^{N} \tau_{i-1} M\phi(U_i) + \epsilon \leq \sum_{i=1}^{N} \tau_{i-1} \lim \inf_{k \to \infty} M\phi_k(V_i^k) + \epsilon \\
\leq \lim \inf_{k \to \infty} \sum_{i=1}^{N} \tau_{i-1} \left( M\phi_k(U_i) + \frac{\epsilon}{2^i \tau_{N}} \right) + \epsilon \leq \lim \inf_{k \to \infty} \sum_{i=1}^{N} \tau_{i-1} M\phi_k(U_i) + 2\epsilon \\
= \lim \inf_{k \to \infty} \left( \sum_{i=1}^{N} \tau_{i-1} M\phi_k(U_i) + 0 \cdot M\phi_k \left( \Omega' \setminus \bigcup_{i=1,\ldots,N} U_i \right) \right) + 2\epsilon \\
\leq \lim \inf_{k \to \infty} \int \Omega' f \, dM\phi_k + 2\epsilon = \lim \inf_{k \to \infty} \int \Omega f \, dM\phi_k + 2\epsilon. \tag{3.13}
\]

We will now show that
\[ \int \Omega f \, dM\phi \geq \lim \sup_{k \to \infty} \int \Omega f \, dM\phi_k - \epsilon. \]
For each \( i = 1, \ldots, N \), let
\[ W_i := \{ x \in \overline{\Omega} : \tau_{i-1} \leq f(x) \leq \tau_i \}. \]
Note that each \( W_i, i = 1, \ldots, N \), is a compact subset of \( \Omega \) and that \( \overline{\Omega} = \bigcup_{i=2,\ldots,N}(W_i \setminus W_{i-1}) \cup W_1 \). We have \( f \leq \sum_{i=1}^{N} \tau_i \chi_{W_i} \) and, since \( M\phi(\{ x \in \Omega' : f = \tau_i \}) = 0 \) for all \( i = 1, \ldots, N \) we have
\[
\sum_{i=1}^{N} \tau_i M\phi(W_i) - \int \Omega f \, dM\phi = \sum_{i=1}^{N} \int_{W_i} (\tau_i - f) \, dM\phi \leq \sum_{i=1}^{N} \int_{W_i} (\tau_i - \tau_{i-1}) \, dM\phi \\
\leq \frac{\epsilon}{L} \sum_{i=1}^{N} M\phi(W_i) = \frac{\epsilon}{L} M\phi(\overline{\Omega}) \leq \epsilon.
\]
Now, using point (a) of the above lemma, we can write
\[
\int \Omega f \, dM\phi \geq \sum_{i=1}^{N} \tau_i M\phi(W_i) - \epsilon \geq \sum_{i=1}^{N} \tau_i \lim \sup_{k \to \infty} M\phi_k(W_i) - \epsilon \\
\geq \lim \sup_{k \to \infty} \sum_{i=1}^{N} \tau_i M\phi_k(W_i) - \epsilon \\
\geq \lim \sup_{k \to \infty} \left( \tau_1 M\phi_k(W_1) + \sum_{i=2}^{N} \tau_i M\phi_k(W_i \setminus W_{i-1}) \right) - \epsilon \\
\geq \lim \sup_{k \to \infty} \int \Omega' f \, dM\phi_k - \epsilon = \lim \sup_{k \to \infty} \int \Omega f \, dM\phi_k - \epsilon.
\]
From here and from (3.13), taking the limit $\epsilon \to 0^+$, we get $\lim_{k \to \infty} \int_\Omega f \, dM\phi_k = \int_\Omega f \, dM\phi$. \hfill $\square$

Another proof of this lemma, which uses a roundabout approach of a certain space of currents, can be found in Rauch & Taylor [39], pp. 353-354.

We also state other consequences of Lemma 3.6.

**Corollary 3.8.** If $\phi_k, \psi_k \in C(\overline{\Omega})$ are convex in $\Omega$ such that $\phi_k \to \phi$, $\psi_k \to \psi$ uniformly on compact subsets of $\Omega$ and

$$M\phi_k \leq M\psi_k \quad \text{in} \quad \Omega \quad \forall \ k$$ (3.14)

then

$$M\phi \leq M\psi \quad \text{in} \quad \Omega.$$ (3.15)

**Proof.** It is enough to prove (3.15) for $K \subset \Omega$ with $K$ compact (by regularity of Borel measures, see Parthasarathy [34], p. 27). Let $U_m := \{x \in \Omega : \text{dist}(x, K) < \delta/m\}$ for $\delta > 0$ such that $U_1 \subset \Omega$. We have $K = \bigcap_{m \geq 1} \overline{U_m}$. For a fixed $m$ consider a family of open sets $\{U^k_m\}_{k \geq 1}$ such that $U^k_m \subset \Omega$, $U^{k+1}_m \subset U^k_m$ and $\overline{U_m} \subset U^k_m$ for all $k$ (take for instance $U^k_m := \{x \in \Omega : \text{dist}(x, \overline{U_m}) < 1/k\}$). Then, by Lemma 3.6,

$$M\phi(K) \leq M\phi(U_m) \leq \liminf_{k \to \infty} M\phi_k(U^k_m) \leq \liminf_{k \to \infty} M\phi_k(\overline{U^k_m}) \leq \limsup_{k \to \infty} M\phi_k(\overline{U^k_m}) \leq \limsup_{k \to \infty} M\psi_k(\overline{U^k_m}) \leq M\psi(\overline{U_m}).$$

Therefore, taking $m \to \infty$, we obtain

$$M\phi(U) \leq \lim_{m \to \infty} M\psi(\overline{U_m}) = M\psi\left(\bigcap_{m \geq 1} \overline{U_m}\right) = M\psi(K),$$

which finishes the proof. \hfill $\square$

**Corollary 3.9.** Similarly, if $\phi_k, \psi_k, w_k \in C(\overline{\Omega})$ are convex in $\Omega$ such that $\phi_k \to \phi$, $\psi_k \to \psi$, $\varphi_k \to w$ uniformly on compact subsets of $\Omega$ and $M\phi_k \leq M\psi_k + M\varphi_k$ in $\Omega$ then

$$M\phi \leq M\psi + M\varphi \quad \text{in} \quad \Omega.$$ (3.16)

**Proof.** Similar to the proof of Corollary 3.8. \hfill $\square$
3.3 Inequalities

The last Corollary can be applied in showing the superadditivity of Monge-Ampère measure for $C(\overline{\Omega})$ functions.

**Lemma 3.10** (Superadditivity of Monge-Ampère measure). Let $\phi, \psi \in C(\overline{\Omega})$ be convex with $\Omega \subset \mathbb{R}^n$ convex. Then

$$M(\phi + \psi)(E) \geq M\phi(E) + M\psi(E) \quad \forall \text{ Borel sets } E \subset \Omega. \quad (3.17)$$

**Proof.** First suppose that $\psi, \phi \in C^2(\overline{\Omega})$. Then (3.17) is equivalent to

$$\det (D^2\phi + D^2\psi) \geq \det D^2\phi + \det D^2\psi \quad \text{in } \Omega. \quad (3.18)$$

As $\phi, \psi$ are convex functions, each of $D^2\phi$, $D^2\psi$, $D^2(\phi + \psi)$ is a positive definite matrix in $\Omega$. Therefore (3.18) follows as $\det (A + B) \geq \det A + \det B$ for any $A, B \in \mathbb{R}^{n \times n}$ positive definite (in fact a stronger inequality holds: $(\det (A + B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}$, see Horn & Johnson [27], pp. 510–511, or Bhatia [9], p. 114, for further generalisations).

Now let $\psi, \phi \in C(\overline{\Omega})$ and fix $K \subset \Omega$ compact. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2} \text{dist}(K, \partial \Omega)$,

$$\Omega_\epsilon := \{x \in \Omega : \text{dist} (x, \partial \Omega) > \epsilon\}$$

(note that $\Omega_\epsilon$ is convex) and let $\eta_\epsilon \in C^\infty_0(B(0, \epsilon))$ be a nonnegative function such that $\int_{B(0, \epsilon)} \eta_\epsilon dx = 1$ (for instance take a standard mollifier $\eta_\epsilon := C \exp \left( \left( \frac{|x|^2}{\epsilon^2} - 1 \right)^{-1} \right)$, where $C > 0$ is chosen such that $\int_{B(0, \epsilon)} \eta_\epsilon dx = 1$). Let

$$\psi_\epsilon := (\psi \ast \eta_\epsilon)(x) \equiv \int_{B(0, \epsilon)} \psi(x - y)\eta_\epsilon(y)dy \quad \forall x \in \Omega_\epsilon$$

be a mollification of a function $\psi$ and similarly define $\phi_\epsilon := \phi \ast \eta_\epsilon$. We have $\psi_\epsilon, \phi_\epsilon \in$
3.3 Inequalities

$C^\infty(\Omega_\epsilon)$ (as $\eta_\epsilon \in C^\infty_0(B(0,\epsilon))$) and $\psi_\epsilon, \phi_\epsilon$ are convex. Indeed

\[
\psi_\epsilon(\lambda x_1 + (1 - \lambda)x_2) = \int_{B(0,\epsilon)} \psi(\lambda x_1 + (1 - \lambda)x_2 - y)\eta_\epsilon(y)dy
\]

\[
= \int_{B(0,\epsilon)} \psi(\lambda(x_1 - y) + (1 - \lambda)(x_2 - y))\eta_\epsilon(y)dy
\]

\[
\leq \lambda \int_{B(0,\epsilon)} \psi(x_1 - y)\eta_\epsilon(y)dy + (1 - \lambda) \int_{B(0,\epsilon)} \psi(x_2 - y)\eta_\epsilon(y)dy
\]

\[
= \lambda \psi_\epsilon(x_1) + (1 - \lambda) \psi_\epsilon(x_2) \quad \forall \lambda \in [0,1], x_1, x_2 \in \Omega_\epsilon.
\]

From the above argument we have that

\[
M(\phi_\epsilon + \psi_\epsilon)(K) \geq M\phi_\epsilon(K) + M\psi_\epsilon(K)
\]

(3.19)

Noting that $\psi_\epsilon \xrightarrow{\epsilon \to 0^+} \psi$ and $\phi_\epsilon \xrightarrow{\epsilon \to 0^+} \phi$ uniformly on compact subsets of $\Omega_\epsilon$ we may use Corollary 3.9 to take the limit $\epsilon \to 0^+$ in (3.19) and get $M(\phi + \psi)(K) \geq M\phi(K) + M\psi(K)$.

One of the important implications of this lemma is that adding an affine function to a convex function does not change its Monge-Ampère measure.

**Corollary 3.11.** Let $\psi \in C(\overline{\Omega})$ be convex and let $q \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then the function $\phi(x) := \psi(x) + l(x)$, where $l(x) := q \cdot x + \alpha$ satisfies $M\phi(E) = M\psi(E)$ for all Borel sets $E \subset \Omega$.

**Proof.** Recall that $Ml = 0$ by Example 3.3 (a). Hence

\[
M\psi = M\psi + Ml \leq M(\psi + l) = M\phi = M\phi + M(-l) \leq M(\phi - l) = M\psi.
\]

Another important implication of Lemma 3.10 is that adding a quadratic functional to a convex function strictly increases its Monge-Ampère measure. As we will see later, this trick makes it straightforward to prove a comparison principle for the solutions of the Monge-Ampère equation (see Theorem 3.15). In other words, we have the following result.

**Corollary 3.12.** Let $\psi \in C(\overline{\Omega})$ be convex and let $x_0 \in \Omega$. For $\delta > 0$ let

\[
\psi_\delta(x) := \psi(x) + \delta|x - x_0|^2.
\]

Then

\[
M\psi_\delta(E) \geq M\psi(E) + (2\delta)^n|E| \quad \forall \text{ Borel set } E \subset \Omega,
\]

(3.20)
where $|E|$ denotes the Lebesgue measure of a Borel set $E$.

Proof.

\[
M_{\psi}(E) \geq M_{\psi}(E) + M(\delta \cdot -x_0^2)(E) \geq M_{\psi}(E) + (2\delta)^n|E|
\]

\[\square\]

### 3.4 Aleksandrov maximum principle

Here we state the Aleksandrov maximum principle, which is a classical result in the theory of the Monge-Ampère equation and will be useful in constructing the solution to the inhomogeneous Dirichlet problem for the Monge-Ampère equation (i.e. problem (4.6)).

**Lemma 3.13 (Aleksandrov maximum principle).** If $\Omega \subset \mathbb{R}^n$ is a bounded, open and convex set with diameter $D$, then for all $\phi \in C(\overline{\Omega})$ convex with $\phi = 0$ on $\partial \Omega$

\[|\phi(x)|^n \leq C D^{n-1} \text{dist}(x, \partial \Omega) |\partial \phi(\Omega)| \quad \forall x \in \Omega,
\]

where $C$ is a constant depending only on the dimension $n$.

The proof of this lemma can be found in Gutiérrez [25], p. 12.

### 3.5 Comparison principle

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

**Lemma 3.14.** Let $\phi, \psi \in C(\overline{\Omega})$. If $\phi = \psi$ on $\partial \Omega$ and $\psi \geq \phi$ in $\Omega$, then

\[\partial \psi(\Omega) \subset \partial \phi(\Omega)
\]

(see Fig. 5).

![Figure 5: (based on Fig. 1.2. from Gutiérrez [25], p. 11)](image-url)
3.5 Comparison principle

**Proof** (we simplify the proof from Gutiérrez [25], pp. 10-11). If \( p \in \partial \psi(\Omega) \) then there exists \( x_0 \in \Omega \) such that

\[
\psi(x) \geq \psi(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega.
\]  
(3.21)

Let

\[
a := \sup_{x \in \Omega} (\psi(x_0) + p \cdot (x - x_0) - \phi(x))
\]

(see Fig. 5). Since \( \psi(x_0) \geq \phi(x_0) \), we have \( a \geq 0 \). Moreover, since \( \Omega \) is bounded and \( \phi \) is continuous, there exists \( x_1 \in \Omega \) such that \( a = \psi(x_0) + p \cdot (x_1 - x_0) - \phi(x_1) \) and so

\[
\phi(x) \geq \psi(x_0) + p \cdot (x - x_0) - a = \phi(x_1) + p \cdot (x - x_1) \quad \forall x \in \Omega,
\]

where the inequality holds by definition of \( a \). Clearly, \( \phi(x_1) + p \cdot (x - x_1) \) is a supporting hyperplane to the function \( \phi \) at \( x_1 \). This means that \( p \in \partial \phi(\Omega) \) if \( x_1 \in \Omega \). Otherwise, if \( x_1 \in \partial \Omega \), then \( \psi(x_1) = \phi(x_1) \) and by taking \( x := x_1 \) in (3.21), we have

\[
\phi(x_1) = \psi(x_1) \geq \psi(x_0) + p \cdot (x_1 - x_0) = \phi(x_1) + a.
\]

Hence \( a = 0 \) and therefore \( \psi(x_0) = \phi(x_0) \). Also \( \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) \) for all \( x \in \Omega \) and hence \( p \in \partial \phi(x_0) \subset \partial \phi(\Omega) \).

We can now state the comparison principle for convex (concave) functions.

**Theorem 3.15** (Comparison principle). Let \( \phi, \psi \in C(\Omega) \) be convex functions such that

\[
M \phi \leq M \psi \quad \text{on } \Omega.
\]  
(3.22)

Then

\[
\min_{x \in \Omega} (\phi - \psi)(x) = \min_{x \in \partial \Omega} (\phi - \psi)(x).
\]  
(3.23)

**Proof** (from Gutiérrez [25], pp. 16-17). Suppose (3.23) is not true and we have

\[
a := \min_{x \in \Omega} (\phi - \psi)(x) < \min_{x \in \partial \Omega} (\phi - \psi)(x) =: b.
\]

Then there exists \( x_0 \in \Omega \) such that \( a = \phi(x_0) - \psi(x_0) \). Let \( \delta > 0 \) be such that \( \delta (\text{diam } \Omega)^2 < \frac{b-a}{2} \) and let

\[
\varphi(x) = \psi(x) + \delta |x - x_0|^2 + \frac{b + a}{2}.
\]
Consider the set \( G := \{ x \in \Omega : \phi(x) < \varphi(x) \} \). We have \( x_0 \in G \). Also, \( G \cap \partial \Omega = \emptyset \). In order to see it, we note that for every \( x \in \partial \Omega \) we have \( \psi(x) \leq \phi(x) - b \) (by definition of \( b \)) and hence

\[
\varphi(x) \leq \phi(x) + \delta |x - x_0|^2 - \frac{b - a}{2} \leq \phi(x) + \delta (\text{diam } \Omega)^2 - \frac{b - a}{2} < \phi(x),
\]

which means that \( \varphi(x) < \phi(x) \) for \( x \in \partial \Omega \). Hence \( G \cap \partial \Omega = \emptyset \) and consequently \( \partial G = \{ x \in \Omega : \varphi(x) = \phi(x) \} \). By Lemma 3.14 we obtain \( \partial \varphi(G) \subset \partial \phi(G) \). Noting that \( \partial \varphi = \partial(\psi + \delta |\cdot - x_0|^2) \) (adding a constant does not change the normal mapping) and using Corollary 3.12 we obtain

\[
M_\varphi(G) \geq M_\phi(G) = |\partial(\psi + \delta |\cdot - x_0|^2)(G)| = M_\psi(G) + (2\delta^2)^n |G|,
\]

which contradicts (3.22).

\[ \square \]

### 4 Existence theorems

First, let us introduce the generalised notion of solution to the Monge-Ampère equation.

#### 4.1 Generalised solutions of the Monge-Ampère equation

Let \( \Omega \subset \mathbb{R}^n \) be open and convex.

**Definition 4.1.** Let \( \nu \) be a Borel measure defined in \( \Omega \). The convex function \( \phi \in C(\Omega) \) is a **generalised solution** of the Monge-Ampère equation

\[
\det D^2 \phi = \nu \tag{4.1}
\]

if the Monge-Ampère measure \( M \phi \) associated with \( \phi \) equals \( \nu \).

If \( \phi \in C^2(\overline{\Omega}) \) and the Borel measure \( \nu \) is absolutely continuous with respect to the Lebesgue measure with density \( f \in C(\Omega) \), \( f \geq 0 \) then \( \phi \) is a generalised solution of (4.1) iff \( \det D^2 \phi = f \) in \( \Omega \).
4.2 Homogeneous Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be bounded and strictly convex and let us consider a homogeneous Dirichlet problem for the Monge-Ampère equation

\[
\begin{aligned}
\det D^2 \phi &= 0 & \text{in } \Omega, \\
\phi &= g & \text{on } \partial \Omega,
\end{aligned}
\]

where $g : \partial \Omega \to \mathbb{R}$ is continuous. Let
\[
\mathcal{F} := \{a(x) : a \text{ is an affine function and } a \leq g \text{ on } \partial \Omega\}.
\]

We will show that problem (4.2) has a unique convex generalized solution (see Definition 4.1). What is more, the solution $\phi$ is given by

\[
\phi(x) := \sup_{a \in \mathcal{F}} a(x).
\]

Let us give a geometric interpretation of this characterisation of the solution $\phi$. Consider a set $G := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \partial \Omega \text{ and } z = g(x)\}$, i.e. a plot of $g$ in $\mathbb{R}^n \times \mathbb{R}$. Then one can think of $\phi$ as of the "lower boundary" of the set $\text{conv} G$ (where $\text{conv}$ denote the convex hull of a set), i.e. $\phi(x) = \min_{z : (x, z) \in G} z$. Also note that $\mathcal{F}$ is the set of hyperplanes supporting $G$ from below (i.e. below with respect to the $x_{n+1}$ direction).

Similarly one can show that there exists a unique concave solution to (4.2) (in a generalized sense) and the concave solution is the "upper boundary" of $\text{conv} G$.

We make these comments precise in the following theorem.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^n$ be bounded and strictly convex, and $g : \partial \Omega \to \mathbb{R}$ be a continuous function. There exists a unique convex function $\phi \in C(\Omega)$ that is a generalized solution of the problem (4.2).

Similarly, there exists a unique concave $C(\Omega)$ solution to (4.2).

**Proof** (we paraphrase and simplify the proof from Gutiérrez [25], pp. 17-19). First, we note that uniqueness of solution follows directly from the comparison principle (Theorem 3.15). For the existence we will show that

(a) $\mathcal{F} \neq \emptyset$ and $\phi$ is convex with $\phi(x) \leq g(x)$ for $x \in \partial \Omega$,

(b) $\phi = g$ on $\partial \Omega$,
4.2 Homogeneous Dirichlet problem

(c) \( \phi \in C(\overline{\Omega}) \),

(d) \( \partial \phi(\Omega) \subset \{ p \in \mathbb{R}^n : \text{there exists } x, y \in \Omega, x \neq y \text{ and } p \in \partial \phi(x) \cap \partial \phi(y) \} \).

Then, by Lemma 2.11, we will have \( M\phi(\Omega) = |\partial \phi(\Omega)| = 0 \), which will finish the proof. We have

(a) As \( g \) is continuous we have \( a(x) := \min_{y \in \partial \Omega} g(y) \in F \). Hence \( F \neq \emptyset \) and \( \phi \) is well defined. We have, for \( x, y \in \Omega, \lambda \in [0, 1] \),

\[
\phi(\lambda x + (1 - \lambda)y) = \sup_{a \in F} a(\lambda x + (1 - \lambda)y) \leq \lambda \sup_{a \in F} a(x) + (1 - \lambda) \sup_{a \in F} a(y) = \lambda \phi(x) + (1 - \lambda)\phi(y),
\]

i.e. \( \phi \) is convex. Also, \( \phi(x) \leq g(x) \) for \( x \in \partial \Omega \) as each \( a(x) \in F \) has this property.

(b) Let \( \xi \in \partial \Omega \). By continuity of \( g \), given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |g(x) - g(\xi)| \leq \epsilon \) for \( |x - \xi| < \delta, x \in \partial \Omega \). Let \( l(x) = 0 \) be the equation of the supporting hyperplane to \( \Omega \) at the point \( \xi \) (see Lemma 2.1 (a) ). Since \( \Omega \) is strictly convex, there exists \( \eta > 0 \) such that \( S = \{ x \in \overline{\Omega} : l(x) \leq \eta \} \subset B(\xi, \delta) \) (see Lemma 2.7 (b) ). Let

\[
M := \min\{ g(x) : x \in \partial \Omega, l(x) \geq \eta \}
\]

and consider

\[
a_\epsilon(x) := g(x) - \epsilon - A_\epsilon l(x), \quad (4.4)
\]

where

\[
A_\epsilon := \max \left\{ \frac{g(\xi) - \epsilon - M}{\eta}, 0 \right\}.
\]

We have \( l(\xi) = 0 \) and hence \( a_\epsilon(\xi) = g(\xi) - \epsilon \). We will show that \( a_\epsilon \in F \). Indeed, if \( x \in \partial \Omega \cap S \), then \( x \in \partial \Omega \cap B(\xi, \delta) \) and \( g(x) \geq g(\xi) - \epsilon \geq g(\xi) - \epsilon - A_\epsilon l(x) = a_\epsilon(x) \). If \( x \in \partial \Omega \cap S^c \), then \( l(x) > \eta \) and by the definition of \( M \) and the choice of \( A_\epsilon \) we have \( g(x) \geq M = a_\epsilon(x) + M - g(\xi) + \epsilon + A_\epsilon l(x) \geq a_\epsilon(x) + M - g(\xi) + \epsilon + A_\epsilon \eta \geq a_\epsilon(x) \).

Therefore \( a_\epsilon \in F \) for each \( \epsilon > 0 \). In particular \( \phi(\xi) \geq a_\epsilon(\xi) = g(\xi) - \epsilon \) for every \( \epsilon > 0 \). Taking \( \epsilon \to 0^+ \) we get \( \phi(\xi) \geq g(\xi) \).
(c) Since \( \phi \) is convex in \( \Omega \), \( \phi \) is continuous in \( \Omega \) (see Lemma 2.7 (b)). Now let \( \xi \in \partial \Omega \) and let \( \{x_n\}_{n \geq 1} \subset \Omega \) be a sequence converging to \( \xi \). We will show that \( \phi(x_n) \to g(\xi) \). Let \( a_\epsilon \) be defined by (4.4). As \( a_\epsilon \in F \) for each \( \epsilon > 0 \), we have \( \phi(x) \geq a_\epsilon(x) \) for all \( x \in \overline{\Omega} \). In particular \( \phi(x_n) \geq a_\epsilon(x_n) \) and so \( \liminf \phi(x_n) \geq \liminf a_\epsilon(x_n) = \liminf (g(\xi) - \epsilon - A_\epsilon l(x_n)) = g(\xi) - \epsilon \) for all \( \epsilon > 0 \). Hence \( \liminf \phi(x_n) \geq g(\xi) \). We now prove that \( \limsup \phi(x_n) \leq g(\xi) \).

Let \( h \) be the solution of the problem
\[
\begin{align*}
\Delta h &= 0 \quad \text{in } \Omega, \\
h &\in C(\overline{\Omega}) \text{ with } h = g \quad \text{on } \partial \Omega
\end{align*}
\]
(such an \( h \) exists as the boundary \( \partial \Omega \) of \( \Omega \) is sufficiently regular (i.e. as a convex set it has a barrier at each point), see e.g. Theorem 2.14 in Gilbarg & Trudinger [24]). On the other hand we have \( \Delta a = 0 \) in \( \Omega \) (trivially) and \( a \leq g \) on \( \partial \Omega \) for any \( a \in F \). Then \( a - h \) is also harmonic and, as \( a|_{\partial \Omega} \leq g = h|_{\partial \Omega} \), by the maximum principle of harmonic functions we obtain \( a \leq h \) in \( \Omega \). Taking supremum over \( a \in F \) we obtain \( \phi(x) \leq h(x) \) for \( x \in \Omega \). In particular, \( \phi(x_n) \leq h(x_n) \) and therefore \( \limsup \phi(x_n) \leq \limsup h(x_n) = g(\xi) \).

(d) If \( p \in \partial \phi(\Omega) \) then there exists \( x_0 \in \Omega \) such that
\[
\phi(x) \geq \phi(x_0) + p \cdot (x - x_0) =: a(x) \quad \forall x \in \Omega.
\] (4.5)

Since \( \phi = g \) on \( \partial \Omega \), we have \( g(x) \geq a(x) \) for all \( x \in \partial \Omega \). There exists \( \xi \in \partial \Omega \) such that \( g(\xi) = a(\xi) \). Otherwise, there exists some \( \epsilon > 0 \) such that \( g(x) \geq a(x) + \epsilon \) for all \( x \in \partial \Omega \) and then, by definition of \( \phi \), \( \phi(x) \geq a(x) + \epsilon \) for all \( x \in \Omega \). In particular \( \phi(x_0) \geq a(x_0) + \epsilon = \phi(x_0) + \epsilon \), a contradiction. Since \( \Omega \) is convex, the open segment \( I \) joining \( x_0 \) and \( \xi \) is contained in \( \Omega \). We will show that \( a(x) \) is a supporting hyperplane to \( \phi \) at any \( z \in I \). By (4.5) we only need to show that \( \phi(z) = a(z) \) for all \( z \in I \). We have \( \phi(x_0) = a(x_0) \), \( \phi(\xi) = g(\xi) = a(\xi) \) and if \( z = tx_0 + (1-t)\xi, \ t \in [0, 1] \) is any point in \( I \), we have, by convexity of \( u \),
\[
\phi(z) \leq t\phi(x_0) + (1-t)\phi(\xi) = ta(x_0) + (1-t)a(\xi) = a(z) \leq \phi(z),
\]
i.e. \( \phi(z) = a(z) \) for all \( z \in I \), as needed. This means that \( p \in \partial \phi(z) \) for all \( z \in I \), which finishes the proof. \( \square \)
4.3 Inhomogeneous Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and a strictly convex set, $\mu$ a Borel measure in $\Omega$ with $\mu(\Omega) < \infty$, and $g \in C(\partial \Omega)$. We will focus on the inhomogeneous Dirichlet problem

\[
\begin{aligned}
\text{det} \, D^2 \phi &= \mu \quad \text{in } \Omega, \\
\phi &= g \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(4.6)

Let

\[ F(\mu, g) := \{ v \in C(\overline{\Omega}) : v \text{ is convex, } M v \geq \mu, v = g \text{ on } \partial \Omega \}. \]

We first note that if $\psi \in F(\mu, g)$, then $\psi \leq \phi_0$, where $\phi_0 \in C(\overline{\Omega})$ is the convex solution to the problem (4.2). Indeed, we have $0 = M \phi_0 \leq \mu \leq M \psi$ in $\Omega$ and by the comparison principle (Theorem 3.15) we have that $\psi \leq \phi_0$ in $\Omega$. Therefore, in particular, all functions in $F(\mu, g)$ are uniformly bounded above.

We will show that if $\mu$ is a finite combination of delta masses, i.e. if $\mu = \sum_{i=1}^{N} a_i \delta_{x_i}$ for $N \geq 1$, and $x_i \in \Omega$, $a_i > 0$ for $i = 1, \ldots, N$, then the solution $\Phi$ to the problem (4.6) is given by

\[ \Phi(x) := \sup_{\psi \in F(\mu, g)} \psi(x). \]  

(4.7)

In general, i.e. for any Borel measure $\mu$, we are going to use an approximation argument to claim existence of a solution. For this reason we need the following two lemmas.

**Lemma 4.3.** Let $\mu$ be a Borel measure on $\Omega$ with $\mu(\Omega) < \infty$. Then there exists a sequence $\{\mu_j\}_{j \geq 1}$ of measures such that

\[ \mu_j = \sum_{i=1}^{N_j} a_{ji} \delta_{x_{ji}}, \]

\[ \sup_j \mu_j(\Omega) =: A < \infty \text{ and } \mu_j \xrightarrow{j \to \infty} \mu \text{ (see Definition 3.4 for the definition of the weak convergence)}. \]

We omit the proof.

**Lemma 4.4.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and strictly convex domain, $\mu_j, \mu$ be Borel measures in $\Omega$, $\phi_j \in C(\overline{\Omega})$ be convex and $g \in C(\partial \Omega)$ such that

1. $\phi_j = g$ on $\partial \Omega$,
2. $M \phi_j = \mu_j$ in $\Omega$,
4.3 Inhomogeneous Dirichlet problem

3. \( \mu_j \rightharpoonup \mu \) in \( \Omega \), and

4. \( \mu_j(\Omega) \leq A \) for all \( j \).

Then \( \{\phi_j\} \) contains a subsequence \( \{\phi_{j_k}\} \), such that \( \phi_{j_k} \xrightarrow{k \to \infty} \phi \) uniformly on compact subsets of \( \Omega \), where \( \phi \in C(\overline{\Omega}) \) is convex and \( M\phi = \mu \) in \( \Omega \), \( \phi = g \) on \( \partial \Omega \).

Proof (we paraphrase the proof from Gutiérrez [25], pp. 20-21). We have \( \phi_j \in \mathcal{F}(\mu_j, g) \) and therefore \( \phi_j \) are uniformly bounded above by \( \phi_0 \) (see the explanation above the previous lemma). We will show that \( \phi_j \) are also uniformly bounded below in \( \Omega \). Similarly as in (4.4), let \( \xi \in \partial \Omega \), \( \epsilon > 0 \) and \( a_\epsilon(x) := g(\xi) - \epsilon - A_\epsilon \ l(x) \), where \( l(x) = 0 \) is the equation of a supporting hyperplane of \( \Omega \) at \( \xi \) and constant \( A_\epsilon \) is defined as in (4.4). Recall that \( a_\epsilon(x) \leq g(x) \) for \( x \in \partial \Omega \), \( l(\xi) = 0 \), \( l(x) > 0 \) for \( x \in \Omega \) and \( A_\epsilon \geq 0 \). Set \( \psi_j(x) := \phi_j(x) - a_\epsilon(x) \). Then the \( \psi_j \) are convex in \( \Omega \) and \( \psi_j(x) = g(x) - a_\epsilon(x) \geq 0 \) for \( x \in \partial \Omega \). If \( \psi_j(x) \geq 0 \) for all \( x \in \Omega \) then \( \phi_j \) is bounded below in \( \Omega \) (by boundedness of \( \Omega \)). If at some point \( x \in \Omega \) we have \( \psi_j(x) < 0 \), then by the Aleksandrov maximum principle (Lemma 3.13) applied to \( \psi_j \) on the set \( G := \{x \in \Omega : \psi_j(x) \leq 0\} \) \( (G \) is convex by convexity of \( \psi \); indeed, if \( x_1, x_2 \in G \) then \( \psi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \psi(x_1) + (1 - \lambda)\psi(x_2) \leq 0 \) for all \( \lambda \in [0, 1] \)), we obtain

\[
(-\psi_j(x))^n \leq C \text{dist}(x, \partial \Omega)\text{diam}(G)^{n-1} M\psi_j(\Omega) \leq C \text{dist}(x, \partial \Omega)\text{diam}(\Omega)^{n-1} A,
\]

and consequently \( \psi_j(x) \geq -C \text{dist}(x, \partial \Omega)^{1/n} \), that is

\[
\phi_j(x) \geq g(\xi) - \epsilon - A_\epsilon \ l(x) - C \text{dist}(x, \partial \Omega)^{1/n},
\]

which proves that \( \phi_j \) are uniformly bounded below in \( \Omega \).

Since \( \phi_j \leq \phi_0 \) for all \( j \) and \( \text{dist}(x, \partial \Omega) \leq |x - \xi| \) we get from (4.8) that

\[
\phi_0(x) \geq \phi_j(x) \geq g(\xi) - \epsilon - A_\epsilon \ l(x) - C|x - \xi|^{1/n},
\]

and \( \phi_j(x) \to g(\xi) \) as \( x \to \xi \) uniformly in \( j \).

Therefore, by Corollary 2.8, we get that \( \phi_j \) are locally uniformly Lipschitz in \( \Omega \) and hence, by Arzela-Ascoli theorem, there exists a subsequence \( \{\phi_{j_k}\} \), and a convex function \( \phi \) in \( \Omega \) such that \( \phi_j \to \phi \) uniformly on compact subsets of \( \Omega \). We also have from (4.9) that \( \phi \in C(\overline{\Omega}) \) and that \( \phi = g \) on \( \partial \Omega \). The weak convergence \( M\phi_{j_k} \to M\phi \) follows from from Lemma 3.7 and hence \( M\phi = \mu \) by the uniqueness of weak limits of measures.
We can now state the main theorem of this section.

**Theorem 4.5.** If $\Omega \subset \mathbb{R}^n$ is open, bounded and strictly convex, $\mu$ is a Borel measure in $\Omega$ with $\mu(\Omega) < +\infty$ and $g \in C(\partial \Omega)$, then there exists a unique convex generalized solution $\Phi$ to the problem (4.6).

We note that if $\mu$ is absolutely continuous with respect to the Lebesgue measure with density function $f$, where $f \in C(\Omega)$ is positive, then the Caffarelli’s regularity results give $\phi \in W^{2,p}(\Omega)$ for all $p \in (0, \infty)$. What is more, if $f \in C^\alpha(\Omega)$ for some $\alpha > 0$ then $u \in C^{2,\alpha}(\Omega)$. See Theorem 1 and Theorem 2 in Caffarelli [11].

**Proof (of Theorem 4.5; we paraphrase and simplify the proof from Gutiérrez [25], pp. 21-24).** The uniqueness follows by the comparison principle (Theorem 3.15).

By Lemma 4.3 there exists a sequence of measures $\{\mu_j\}$ converging weakly to $\mu$ such that each $\mu_j$ is a finite combination of delta masses with positive coefficients and $\mu_j(\Omega) \leq A$ for all $j$. If we solve the Dirichlet problem for each $\mu_j$ with boundary data $g$, then the theorem follows by Lemma 4.4. Therefore we assume from now on that

$$\mu = \sum_{i=1}^{N} a_i \delta_{x_i}, \quad x_i \in \Omega, a_i > 0.$$ 

We will show that the solution $\Phi$ is given by (4.7). We will show the following claims.

(a) $\mathcal{F}(\mu, g) \neq 0$ (so that $\Phi$ is well defined),

(b) If $\phi, \psi \in \mathcal{F}(\mu, g)$ then $\max(\phi, \psi) \in \mathcal{F}(\mu, g)$,

(c) $\Phi \in \mathcal{F}(\mu, g)$,

(d) $M\Phi \leq \mu$.

Clearly, these four claims prove the theorem.

(a) By Example 3.3 (b), $M(|x - x_i|) = \omega_n \delta_{x_i}$, with $\omega_n := |B(0, 1)|$. Let

$$f(x) := \frac{1}{\omega_n^{1/n}} \sum_{i=1}^{N} a_i^{1/n}|x - x_i|$$

and $\phi$ be a solution to the Dirichlet problem $M\phi = 0$ in $\Omega$ with $\phi = g - f$ on $\partial \Omega$ (see Theorem 4.2). We claim that $\psi := \phi + f \in \mathcal{F}(\mu, g)$. Indeed, it is clear
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Inhomogeneous Dirichlet problem (4.3) EXISTENCE THEOREMS

Let \( \Psi := \max(\phi, \psi) \). We first show that for each fixed \( y \in \Omega \), that \( \Phi(\psi) \) and so \( \psi \) is bounded. Since each \( \Psi \) and \( \phi \) in \( \Omega \), we also have \( \psi \geq \phi \) in \( \Omega \). In particular, \( \psi \) is well defined. 

(b) Let \( \Psi := \max(\phi, \psi) \), \( \Omega_1 := \{ x \in \Omega : \phi(x) \geq \psi(x) \} \), \( \Omega_2 := \{ x \in \Omega : \phi(x) < \psi(x) \} \). If \( E \subset \Omega_1 \), then \( \partial \Psi(E) \supset \partial \phi(E) \). Indeed, if \( p \in \partial \phi(x_0) \) for some \( x_0 \in E \), then \( \phi(x) \geq \phi(x_0) + p \cdot (x - x_0) \) for all \( x \in \Omega \) and, as \( \Psi(x_0) = \phi(x_0) \) and \( \Psi \geq \phi \) in \( \Omega \), we also have \( \Psi(x) \geq \Psi(x_0) + p \cdot (x - x_0) \) for all \( x \in \Omega \), i.e. \( p \in \partial \Psi(x_0) \subset \partial \Psi(E) \). Similarly, if \( E \subset \Omega_2 \), then \( \partial \Psi(E) \supset \partial \psi(E) \). For any \( E \subset \Omega \) we decompose \( E = (E \cap \Omega_1) \cup (E \cap \Omega_2) \) and write

\[
M\Psi(E) = M\Psi(E \cap \Omega_1) + M\Psi(E \cap \Omega_2) 
\geq M\phi(E \cap \Omega_1) + M\psi(E \cap \Omega_2) 
\geq \mu(E \cap \Omega_1) + \mu(E \cap \Omega_2) = \mu(E).
\]

(c) We first show that for each fixed \( y \in \Omega \) there exists a uniformly bounded sequence \( \psi_m \in \mathcal{F}(\mu, g) \) converging uniformly on compact subsets of \( \Omega \) to a function \( \psi \in \mathcal{F}(\mu, g) \) such that \( \psi(y) = \Phi(y) \). Indeed, by (a), let \( \psi_0 \in \mathcal{F}(\mu, g) \). For any \( \phi \in \mathcal{F}(\mu, g) \) we have \( \phi \leq \phi_0 \) with \( \phi_0 \) being a solution of the homogeneous Dirichlet problem (cf. explanation before Lemma 4.3). By definition of \( \Phi \) there exists a sequence \( \overline{\psi}_m \subset \mathcal{F}(\mu, g) \) such that \( \overline{\psi}_m(y) \to \Phi(y) \) as \( m \to \infty \). Let \( \psi_m := \max(\psi_0, \overline{\psi}_m) \). By (b), \( \psi_m \in \mathcal{F}(\mu, g) \) and therefore \( \overline{\psi}_m(y) \leq \psi_m(y) \leq \Phi(y) \) and so \( \psi_m(y) \to \Phi(y) \) as \( m \to \infty \). Because \( \psi_0 \leq \psi_m \leq \phi_0 \) in \( \Omega \), \( \psi_m \) is uniformly bounded. Since each \( \psi_m \) is convex in \( \Omega \), it follows from Corollary 2.8 that given \( K \subset \Omega \) compact, \( \psi_m \) are uniformly Lipschitz in \( K \) with constant \( C = C(K, \sup_{\Omega} |\psi_0|, \sup_{\Omega} |\phi_0|) \). Therefore \( \psi_m \) are equicontinuous on \( K \) and bounded in \( \Omega \). By Arzela-Ascoli theorem there exists a subsequence \( \psi_{m_j} \) converging uniformly on compact subsets of \( \Omega \) to a convex function \( \psi \), which satisfies \( \psi(y) = \Phi(y) \). By Corollary 3.8 we have that \( M\psi \geq \mu \) and hence \( w \in \mathcal{F}(\mu, g) \). In particular \( \psi \leq \Phi \) in \( \Omega \).

We now show that \( M\Phi \geq \mu \) in \( \Omega \). It is enough to prove that \( M\Phi(\{x_i\}) \geq a_i \),
for \( i = 1, \ldots, N \). We prove for \( i = 1 \) (other cases are similar). By the above analysis, there exists a uniformly bounded sequence \( \psi_m \in \mathcal{F}(\mu, g) \) such that 
\[
\psi_m \xrightarrow{\infty} \psi \in \mathcal{F}(\mu, g)
\]
on uniformly on compacts of \( \Omega \) with \( \psi(x_1) = \Phi(x_1) \). We have \( M\psi(\{x_1\}) \geq a_1 \). If \( p \in \partial \psi(x_1) \), then \( \psi(x) \geq \psi(x_1) + p \cdot (x - x_1) \) in \( \Omega \) and hence \( \Phi(x) \geq \psi(x) \geq \psi(x_1) + p \cdot (x - x_1) = \Phi(x_1) + p \cdot (x - x_1) \), that is \( p \in \partial \Phi(x_1) \). Therefore \( \partial \Phi(\{x_1\}) \supset \partial \psi(\{x_1\}) \) and consequently \( M\Phi(\{x_1\}) \geq M\psi(\{x_1\}) \geq a_1 \).

(d) We first prove that the measure \( M\Phi \) is concentrated on the set \( \{x_1, \ldots, x_N\} \). Let \( x_0 \in \Omega \) with \( x_0 \neq x_i \) for all \( i = 1, \ldots, N \), and choose \( r > 0 \) so that \( |x_i - x_0| > r \) for \( i = 1, \ldots, N \) and \( B(x_0, r) \subset \Omega \). Let \( \psi \) be the convex solution of the problem \( M\psi = 0 \) in \( B(x_0, r) \) with \( \psi = \Phi \) on \( \partial B(x_0, r) \) (see Theorem 4.2) and let \( \phi \) be the "lifting of \( \Phi \)" , which is defined by

\[
\phi(x) := \begin{cases} 
\Phi(x) & \text{for } |x - x_0| \geq r, x \in \Omega, \\
\psi(x) & \text{for } |x - x_0| \leq r.
\end{cases}
\]

We claim that \( \phi \in \mathcal{F}(\mu, g) \). It is clear that \( \phi \in C(\overline{\Omega}) \) and \( \phi|_{\partial \Omega} = g \). Moreover, \( \phi \) is convex. Indeed, convexity in \( B(x_0, r) \) and in \( \Omega \setminus B(x_0, r) \) follows by definition. By (c) we have that \( M\Phi \geq \mu \geq 0 = M\psi \) in \( B(x_0, r) \) and hence by the comparison principle (Theorem 3.15), \( \Phi \leq \psi = \phi \) in \( B(x_0, r) \). To show convexity of \( \phi \) in \( \Omega \), let \( x \in B(x_0, r) \) and \( y \in \Omega \setminus \overline{B(x_0, r)} \) and let \( \nu \in (0, 1) \) be such that \( z := \nu x + (1 - \nu)y \in \partial B(x_0, r) \). Let \( \lambda \in [0, 1] \). If \( \lambda \geq \nu \) then \( \lambda x + (1 - \lambda)y \in \overline{B(x_0, r)} \) and take \( \overline{\lambda} := \frac{\lambda - \nu}{1 - \nu} \in [0, 1] \). Then \( \lambda x + (1 - \lambda)y = \lambda (1 - \overline{\lambda})(1 - \nu) = 1 - \lambda \) and, using convexity of \( \phi \) in \( B(x_0, r) \) and convexity of \( \Phi \) in \( \Omega \), we obtain

\[
\phi(\lambda x + (1 - \lambda)y) = \phi(\overline{\lambda}x + (1 - \overline{\lambda})(\nu x + (1 - \nu)y))
\]

\[
\leq \overline{\lambda}\phi(x) + (1 - \overline{\lambda})\phi(\nu x + (1 - \nu)y)
\]

\[
= \overline{\lambda}\phi(x) + (1 - \overline{\lambda})\Phi(\nu x + (1 - \nu)y)
\]

\[
\leq \overline{\lambda}\phi(x) + (1 - \overline{\lambda})(\nu \Phi(x) + (1 - \nu)\Phi(y))
\]

\[
\leq \overline{\lambda}\phi(x) + (1 - \overline{\lambda})(\nu \phi(x) + (1 - \nu)\phi(y))
\]

\[
= \lambda \phi(x) + (1 - \lambda)\phi(y).
\]

A similar calculation could be carried out for \( \lambda < \nu \) (in this case one should
take $\lambda := \frac{1}{\nu} \in [0, 1)$ and one does not need to use convexity of $\phi \in B(x_0, r)$.

Hence $\phi$ is convex in $\Omega$. We will now verify that $M\phi \geq \mu$ in $\Omega$. Noting that for $F \subset \Omega \setminus \overline{B(x_0, r)}$ we have $\partial \phi(F) \supset \partial \phi(F)$ (by the same argument as in showing $\partial \phi(\{x_1\}) \supset \partial \psi(\{x_1\})$ in the end of the proof of (d)), we write, for any Borel set $E \subset \Omega$,

\[
M\phi(E) = M\phi(E \cap B(x_0, r)) + M\phi(E \setminus B(x_0, r)) \\
\geq M\psi(E \cap B(x_0, r)) + M\Phi(E \setminus B(x_0, r)) = 0 + M\Phi(E \setminus B(x_0, r)) \\
\geq \mu(E \setminus B(x_0, r)) = \mu(E \cap \{x_1, \ldots, x_N\}) = \mu(E).
\]

Therefore indeed $\phi \in F(\mu, g)$.

Hence $\phi \leq \Phi$ and, since $\phi = \psi \geq \Phi$ in $B(x_0, r)$, we get $\psi = \Phi$ in $B(x_0, r)$.

This means that $M\Phi = M\psi = 0$ in $B(x_0, r)$. Choosing any $B(x_0, r)$ such that $\overline{B(x_0, r)} \cap \{x_1, \ldots, x_N\} = \emptyset$ we get that $M\Phi(E) = 0$ for any Borel set $E \subset \Omega$ such that $E \cap \{x_1, \ldots, x_N\} = \emptyset$. Therefore $M\Phi$ is concentrated on the set $\{x_1, \ldots, x_N\}$, that is

$$M\Phi = \sum_{i=1}^{N} \lambda_i \delta_{x_i},$$

with $\lambda_i \geq 1$, $i = 1, \ldots, N$.

We claim that $\lambda_i = 1$ for each $i = 1, \ldots, N$. Suppose by contradiction that $\lambda_i > 1$ for some $i$. Without loss of generality, we may assume that $i = 1$ and $x_1 = 0$ (otherwise apply renumbering of $\{x_i\}$ and translate $\Phi$). We have $|\partial \Phi(\{0\})| = \lambda a > 0$. The set $\partial \Phi(\{0\})$ is convex. Indeed, if $\Phi(0) + p_1 \cdot x \leq \Phi(x)$ and $\Phi(0) + p_2 \cdot x \leq \Phi(x)$ for all $x \in \Omega$, then $\Phi(0) + (\lambda p_1 + (1 - \lambda)p_2) \cdot x = \lambda \Phi(0) + p_1 \cdot x + (1 - \lambda)(\Phi(0) + p_2 \cdot x) \leq \Phi(x)$. Hence, as the set $\partial \Phi(\{0\})$ is convex and has positive measure, it contains an open ball, i.e. there exists a ball $B(p_0, 2\epsilon) \subset \partial \Phi(\{0\})$. This means that $\Phi(x) \geq \Phi(0) + p \cdot x$ for all $p \in B(p_0, 2\epsilon)$ and for all $x \in \Omega$. Let $\Psi(x) = \Phi(x) - p_0 \cdot x$ (see Fig. 6). Then $\Psi(x) \geq \Psi(0) + (p - p_0) \cdot x$ for all $x \in \Omega$ and $p \in B(p_0, 2\epsilon)$. Given $x \in \Omega$ take $p = p_0 + \epsilon \frac{x}{|x|} \in B(p_0, 2\epsilon)$ and so

$$\Psi(x) \geq \Psi(0) + \epsilon |x| \quad \forall x \in \Omega.$$

Let $\alpha$ be a constant such that $\Psi(0) - \alpha$ is negative and $\Psi(0) - \alpha > -\epsilon \min_{i=2,\ldots,N} |x_i|$, and define $\overline{\Psi}(x) := \Psi(x) - \alpha$ (see Fig. 6). Note that the choice of $\alpha$ implies that
4.3 Inhomogeneous Dirichlet problem

$x_i \notin \{\Psi < 0\}$ for all $i = 2, \ldots, N$. We have that $\Psi(0)$ is negative and small, and $\Psi(x) \geq \Psi(0) + \epsilon |x|$ for all $x \in \Omega$. If $r = -\frac{\Psi(0)}{\epsilon}$, then $\Psi(x) \geq \Psi(0) + \epsilon |x| \geq 0$ for all $|x| \geq r$. Let

$$\varphi(x) := \begin{cases} 
\Psi(x) & \text{if } \Psi(x) \geq 0, \\
\lambda^{-1/n} \Psi(x) & \text{if } \Psi(x) < 0,
\end{cases}$$

see Fig. 6. We will show that $\varphi \in \mathcal{F}(\mu, \bar{g})$, where $\bar{g}$ are the boundary values of $\Psi(x) \equiv \Phi(x) - p_0 \cdot x - \alpha$. Notice that since $\lambda > 1$, we have $\lambda^{-1/n} \Psi(x) > \Psi(x)$ on the set $\{\Psi(x) < 0\}$. Therefore, similarly to the proof of convexity of $\phi$,

![Figure 6: The functions $\Phi(x)$, $\Psi(x)$, $\Psi(x)$, $\varphi(x)$ (note this is a 1D sketch of an $n$-dimensional situation).](image)

the function $\varphi$ is convex in $\Omega$. Also, on the set $\{\Psi(x) < 0\}$, we have $M\varphi = M(\lambda^{-1/n} \Psi) = \frac{1}{\lambda} M\Psi = \frac{1}{\lambda} M\Phi = a_1 \delta_0 = \mu$, where we have used Example 3.3 (d) and Corollary 3.11. On the other hand $\varphi = \Psi$ on the set $\{\Psi(x) \geq 0\}$, so $M\varphi = M\Psi = M\Phi \geq \mu$ on the same set. Consequently, $M\varphi \geq \mu$ in $\Omega$. Hence indeed $\varphi \in \mathcal{F}(\mu, \bar{g})$.

We note that, by Corollary 3.11, the function $v'(x) := v(x) - p_0 \cdot x - \alpha$ belongs to $\mathcal{F}(\mu, \bar{g})$ if and only if $v \in \mathcal{F}(\mu, g)$. Hence, by definition of $\Phi$,

$$\Psi(x) = \Phi(x) - p_0 \cdot x - \alpha = \sup_{v \in \mathcal{F}(\mu, g)} \{v(x) - p_0 \cdot x - \alpha\} = \sup_{v' \in \mathcal{F}(\mu, \bar{g})} v'(x).$$

Since $\varphi \in \mathcal{F}(\mu, \bar{g})$, we get that $\varphi(x) \leq \Psi(x)$ for all $x \in \Omega$. In particular, $\varphi(0) \leq \Psi(0)$ or, equivalently, $\lambda^{-1/n} \Psi(0) \leq \Psi(0)$. As $\Psi(0) < 0$ we obtain

$$\lambda^{-1/n} \geq 1$$

a contradiction since $\lambda > 1$. \hfill \Box

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4.4 At most two $C^2$ solutions in 2D

We will now show that in dimension $n = 2$ we have at most 2 solutions to the inhomogeneous Dirichlet problem (4.6) with measure $\mu$ given by a density $f$ with respect to the Lebesgue measure, where $f \in C(\Omega)$ is strictly positive. In other words we have the following theorem.

**Theorem 4.6.** Let $\Omega \subset \mathbb{R}^2$ be an open set. Let $f : \Omega \to \mathbb{R}$ be continuous and everywhere positive and let $g : \partial\Omega \to \mathbb{R}$ be continuous. Then the problem

$$
\begin{aligned}
\det D^2\phi &= f & \text{in } \Omega, \\
\phi &= g & \text{on } \partial\Omega
\end{aligned}
$$

(4.10)

has at most two $C^2(\Omega) \cap C(\overline{\Omega})$ solutions.

We note that if $f \in C^\alpha(\Omega)$ for some $\alpha > 0$ then the problem (4.10) has exactly two solutions (in a generalised sense), which are of regularity $C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ (see the remark after the statement of Theorem 4.5).

The proof of Theorem 4.6 is adapted from Courant & Hilbert [15], pp. 324-325, where a more general equation is considered. The proof we will present is a simplification of the analysis of this more general equation. Before the proof, we first need to recall a uniqueness theorem for elliptic differential equations. Let $\Omega$ be a bounded and connected domain in $\mathbb{R}^n$. Let the differential operator

$$
L\psi \equiv a_{ij}(x)D_{ij}\psi + b_i(x)D_i\psi + c(x)\psi
$$

(4.11)

be elliptic in $\Omega$, i.e. let the matrix $\{a_{ij}(x)\}_{i,j=1,...,n}$ be positive definite for each $x \in \Omega$ (i.e. $a_{ij}(x)\xi_i\xi_j > 0 \forall x \in \Omega, \forall \xi \in \mathbb{R}^n$) and let $a_{ij}$, $b_i$, $c$, for $i, j \in \{1, \ldots, n\}$, be continuous. Let us also assume that

$$
\frac{|b_i(x)|}{\lambda(x)} \leq C \quad \forall x \in \Omega, \forall i = 1, \ldots, n,
$$

(4.12)

where $\lambda(x) > 0$ is the smallest eigenvalue of the matrix $\{a_{ij}(x)\}_{i,j=1,...,n}$ ($\lambda > 0$ since all eigenvalues of a positive definite matrix are positive) and $C > 0$. We have the following maximum principle.

**Lemma 4.7** (Weak maximum principle). Suppose $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $L\psi \geq 0$
in $\Omega$ with $c \leq 0$. Then $\psi$ attains on $\partial\Omega$ its nonnegative maximum in $\overline{\Omega}$, i.e.

$$\sup_{\Omega} \psi \leq \sup_{\partial\Omega} \psi^+.$$

**Proof.** See Gilbarg & Trudinger [24], p. 33. \qed

**Corollary 4.8.** There exists at most one solution $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ to the problem

$$\begin{cases}
L\phi = f & \text{in } \Omega, \\
\phi = g & \text{on } \partial\Omega,
\end{cases}$$

(4.13)

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$ and $L$ is as in Lemma 4.7.

**Proof.** If $\phi, \psi$ are two solutions to (4.13) then $\varphi := \phi - \psi$ satisfies (4.13) with $f \equiv 0$, $g \equiv 0$ and lemma 4.7 implies $\sup_{\Omega} \varphi \leq \sup_{\partial\Omega} \varphi^+ = 0$. Analogously, taking $-\varphi$ gives $\sup_{\Omega} (-\varphi) \leq 0$. Hence $0 \leq \varphi \leq 0$ and consequently $\phi = \psi$. \qed

We can now prove Theorem 4.6.

**Proof (of Theorem 4.6; adapted from Courant & Hilbert [15], pp. 324-325).** If $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to (4.10), then

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 > 0$$

(4.14)

in $\Omega$. In particular neither $\phi_{xx}$ nor $\phi_{yy}$ can vanish at any point $x \in \Omega$. Therefore it remains to show that there exists at most one solution to (4.10) for which

$$\phi_{xx} > 0 \quad \text{(and then also } \phi_{yy} > 0)$$

(4.15)

and at most one solution to (4.10) for which

$$\phi_{xx} < 0 \quad \text{(and then also } \phi_{yy} < 0)$$

(4.16)

holds in $\Omega$ (cf. (1.9)). In what follows we will consider the case (4.15) (the case (4.16) follows in a similar way). Suppose there exist two solutions $\phi, \psi \in C^2(\Omega) \cap C(\overline{\Omega})$ of the problem (4.10) satisfying (4.15). Then, letting $\varphi := \phi - \psi$ we get

$$0 = \det D^2(\psi + \varphi) - \det D^2 \psi = (\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2) + \psi_{xx}\varphi_{yy} + \psi_{yy}\varphi_{xx} - 2\psi_{xy}\varphi_{xy},$$

$$0 = \det D^2(\phi - \varphi) = -(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2) + \phi_{xx}\varphi_{yy} + \phi_{yy}\varphi_{xx} - 2\phi_{xy}\varphi_{xy}.$$
Adding the two equalities we obtain \( L(\psi + \phi)\varphi = 0 \), where we define
\[
L_w\varphi := w_{yy}\varphi_{xx} - 2w_{xy}\varphi_{xy} + w_{xx}\varphi_{yy}
\] (4.17)
for \( w \in C^2(\Omega) \cap C(\overline{\Omega}) \). Hence \( \varphi \) is a solution to the problem
\[
\begin{align*}
\begin{cases}
L(\psi + \phi)\varphi = 0 & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
\] (4.18)

We note that \( L(\psi + \phi) \) is a differential operator of the form (4.11). Indeed, we have \( L(\psi + \phi) = L_\psi + L_\phi \) and each of \( L_\psi, L_\phi \) is of the form (4.11) because of (4.14) and (4.15) (note that (4.14) and (4.15) together are equivalent to the positive definiteness of the matrix \( \{\phi_{ij}\}_{i,j=x,y} \)). We also note that \( L(\psi + \phi) \) satisfies condition (4.12) (trivially). Therefore, noting that \( \varphi \equiv 0 \) is a solution of (4.18), we may use Corollary 4.8 to conclude that it is the only solution of this problem. Consequently \( \phi \equiv \psi \).

We note that assumption \( f > 0 \) in Theorem 4.6 is crucial in a sense that it implies convexity or concavity of the solution \( \phi \), i.e. it enables us to use the trick (4.15-4.16).

### 4.5 Similarities to the Laplace equation

This is a good moment to reflect on the similarities between the Monge-Ampère equation and the Laplace equation.

Firstly, both the Laplace operator \( \Delta \phi \) of a function \( \phi : \mathbb{R}^n \supset \Omega \to \mathbb{R} \) and the determinant of the Hessian matrix \( \det D^2 \phi \) are invariant under orthogonal transformations of variables. That is, if \( y := [y_1, \ldots, y_n]^T \) and \( x := [x_1, \ldots, x_n]^T \) are such that \( y = Qx \) for some orthogonal \( Q \in \mathbb{R}^{n \times n} \) (i.e. \( Q^{-1} = Q^T \)), then letting \( \overline{\phi}(y) := \phi(x) \) we have \( \Delta \overline{\phi}(y) = \Delta \phi(x) \) and \( \det D^2 \overline{\phi}(y) = \det D^2 \phi(x) \). This property is a consequence of the fact that the trace and the determinant are two of the \( n \) scalar invariants of orthogonal transformations of a \( n \times n \) symmetric matrix (hence in 2D these are the only such invariants), which can be proved using the characteristic polynomial (see Horn & Johnson [27], pp. 49-51).

Secondly, one can compare the comparison principle, Theorem 3.15, to the maximum principle for subharmonic functions (and minimum principle for superharmonic functions), that is: if \( v \in C^2(\Omega) \) (for \( \Omega \) bounded) satisfies \( \Delta v \geq 0 \) (i.e. \( v \) is subharmonic) then \( \max_{\Omega} v = \max_{\partial \Omega} v \) (see e.g. Gilbarg & Trudinger [24], p. 15). One could try to compare convex solutions of Monge-Ampère equation to superharmonic
functions (and concave solutions to subharmonic functions), but one needs to be careful, because "superharmonicity" and "subharmonicity" relate to the inequality in the partial differential equation (i.e., in the Laplace equation), while convexity and concavity of solutions of the Monge-Ampère equation correspond to the two different "modes" of solution of the same equation. It is interesting to observe how these two modes appear as the only two solutions in the Dirichlet problem in Theorem 4.6 in the section above.

Furthermore, the fact that the convex solution \( \phi \) to the homogeneous Dirichlet problem for the Monge-Ampère equation is given by (4.3) is reminiscent of the Perron’s method for solving the Dirichlet problem for Laplace equation. Namely, for \( \Omega \subset \mathbb{R}^n \) open, bounded and with \( C^2 \) boundary and for \( g \in C(\partial \Omega) \), let

\[
S := \{ v \in C(\overline{\Omega}) : v \text{ is subharmonic and } v \leq g \text{ on } \partial \Omega \}
\]

(we note that one can also define subharmonic (superharmonic) functions in the space \( C(\overline{\Omega}) \)). Then the solution \( u \) to the problem \( \Delta u = 0 \) in \( \Omega \), \( u = g \) on \( \partial \Omega \) is given by

\[
u(x) = \sup_{v \in S} v(x),
\]

see Han & Lin [26], pp. 125-130, for the details.

Finally, let us look at the part (d) of the proof of Theorem 4.5 (the existence of solution to inhomogeneous Dirichlet problem for the Monge-Ampère equation). The "lifting procedure" we used there (i.e., we lifted a convex function \( \Phi \) with \( M\Phi \geq \mu \) on a ball \( B(x_0, r) \) using the convex solution \( \psi \) of the problem \( M\psi = 0 \) in \( B(x_0, r) \), \( \psi = \Phi \) on \( \partial B(x_0, r) \)) is analogous to the lifting procedure used in the Perron’s method (where one lifts a subharmonic function \( v \) on a ball \( B(x_0, r) \) using the unique solution \( w \) of the problem \( \Delta w = 0 \) in \( B(x_0, r) \), \( w = v \) on \( \partial B(x_0, r) \)), see Han & Lin [26], pp. 125-130, for the details.

We note that these similarities together with the application of the Monge-Ampère equation in the 2D Navier-Stokes equation, which we described in the motivation section (Section 1.1), were the author’s inspiration to undertake this topic in the dissertation.
5 Bounds on the solution to the Monge-Ampère equation

Here we state the main results of the project. We extend the approach presented in [39] into the case of the inhomogeneous Dirichlet problem. In this section we will be concerned with the case of even $n$ only. We comment on the case of odd $n$ at the end of this section.

5.1 Generalised Comparison Principle

Let $[\nu]^+ := \max(\nu, 0)$ denote the positive part of a Borel measure $\nu$.

**Theorem 5.1** (Generalised comparison principle). Let $\Omega$ be a bounded, strictly convex set in $\mathbb{R}^n$. Let $\psi \in C(\overline{\Omega})$ be a convex function in $\Omega$ and let $\phi \in W^{2,n}(\Omega)$ be such that

$$M\psi \geq [M\phi]^+ \quad \text{on } \Omega.$$  \hfill (5.1)

Then

$$\min_{x \in \Omega}(\phi - \psi)(x) = \min_{x \in \partial \Omega}(\phi - \psi)(x).$$  \hfill (5.2)

Similarly, if $\psi$ is concave in $\Omega$ and $\phi \in W^{2,n}(\Omega)$ is such that $M\psi \geq [M\phi]^+$ on $\Omega$, then

$$\max_{x \in \Omega}(\phi - \psi)(x) = \max_{x \in \partial \Omega}(\phi - \psi)(x).$$  \hfill (5.3)

If $\phi, \psi \in C^2(\overline{\Omega})$ then (5.1) is equivalent to

$$\det D^2\psi \geq (\det D^2\phi)^+$$  \hfill (5.4)

**Proof** (of Theorem 5.1; we modify the proof from Rauch & Taylor [39], pp. 362-364). Let $\psi$ be convex (the case of $\psi$ concave proceeds similarly).

1. First suppose that $\phi \in C^2(\overline{\Omega})$.

   Suppose that there exists $x_0 \in \Omega$ such that

   $$(\phi - \psi)(x_0) = \min_{x \in \Omega}(\phi - \psi)(x) < \min_{x \in \partial \Omega}(\phi - \psi)(x)$$

   and consider the function $\psi_\epsilon(x) := \psi(x) + \epsilon|x - x_0|^2$ for $\epsilon > 0$. It is clear that if $\epsilon$ is sufficiently small then the function $\phi - \psi_\epsilon$ still does not attain its minimum on $\partial \Omega$ (see Fig. 7). Fix such $\epsilon$. 

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5.1 Generalised Comparison Principle

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\[ \phi - \psi \]

\[ x_0, x_\epsilon \]

Figure 7: The points \( x_0 \) and \( x_\epsilon \) (note this is a 1D sketch of a 2D situation).

Hence there exists \( x_\epsilon \in \Omega \) such that

\[ (\phi - \psi_\epsilon)(x_\epsilon) = \min_{x \in \Omega} (\phi - \psi_\epsilon)(x) < \min_{x \in \partial \Omega} (\phi - \psi_\epsilon)(x). \]  (5.5)

We note that \( x_\epsilon \) can be chosen such that the Hessian matrix \( D^2 \phi(x_\epsilon) \) is not positive definite. Indeed, letting \( \Omega_1 := \{ x \in \Omega : D^2 \phi > 0 \} \) we can see that \( \phi \) is convex on \( \Omega_1 \). Therefore, using Corollary 3.12, \( M \psi_\epsilon \geq M \psi \geq M \phi \) on \( \Omega_1 \).

From Theorem 3.15, it follows that the minimum of \( \phi - \psi_\epsilon \) over \( \Omega_1 \) is attained on \( \partial \Omega_1 \). Hence, as \( (\phi - \psi_\epsilon)(x_\epsilon) \) is the minimum of \( \phi - \psi_\epsilon \) over \( \Omega \supset \Omega_1 \), we can assume that \( D^2 \phi(x_\epsilon) \) is not positive definite.

As any symmetric matrix is positive definite iff all its eigenvalues are positive (see e.g. Theorem 7.2.1 on p. 438 of Horn & Johnson [27]), we conclude that \( D^2 \phi(x) \) has at least one nonpositive eigenvalue for each \( x \in \Omega \setminus \Omega_1 \). Thus let \( \lambda_\epsilon \leq 0 \) be an eigenvalue of \( D^2 \phi(x_\epsilon) \) and let \( \alpha_\epsilon \in \mathbb{R}^n, |\alpha_\epsilon| = 1 \), be the respective eigenvector. Then, by the Taylor expansion in the \( \alpha_\epsilon \) direction, we can write, for small values of \( t \in \mathbb{R} \),

\[ \phi(x_\epsilon + t\alpha_\epsilon) - \phi(x_\epsilon) = l(t) + \lambda t^2 + o(t^2), \]  (5.6)

where \( l_1(t) \equiv l_1 t, l_1 \in \mathbb{R}^n, \) is a linear functional on \( \mathbb{R} \) and \( o(\cdot) : \mathbb{R} \to \mathbb{R} \) is any function such that \( o(y)/y \xrightarrow{y\to0} 0 \). As \( \psi_\epsilon \) is convex, it has a supporting hyperplane at \( x_\epsilon \) (see Lemma 2.7 (a) ). Hence we have

\[ \psi_\epsilon(x_\epsilon + t\alpha) - \psi_\epsilon(x_\epsilon) = \psi(x_\epsilon + t\alpha) - \psi(x_\epsilon) + \epsilon(|x_\epsilon + t\alpha - x_0|^2 - |x_\epsilon - x_0|^2) \geq l_2(t) + \epsilon(|x_\epsilon + t\alpha - x_0|^2 - |x_\epsilon - x_0|^2) = l_3(t) + \epsilon t^2, \]

where \( l_2(t) \equiv l_2 t, l_3(t) \equiv l_3 t, l_2, l_3 \in \mathbb{R}^n, \) are linear functions. Combining this
inequality with (5.6), we get

\[
(\phi - \psi_e)(x_1) \leq (\phi - \psi)(x_1 + t\alpha) \\
\leq (\phi - \psi_e)(x_1) + l(t) + (\lambda - \epsilon)t^2 + o(t^2)
\]

for small values of \(t\). This means that the quadratic polynomial \(l(t) + (\lambda - \epsilon)t^2\) attains its minimum at \(t = 0\). Hence \(l(\cdot) \equiv 0\) and \(\lambda - \epsilon \geq 0\), which contradicts \(\lambda \leq 0 < \epsilon\). Therefore the minimum principle (5.2) holds for \(\phi \in C^2(\Omega)\).

2. Now let \(\phi \in W^{2,n}(\Omega)\).

Note that for \(\phi \in W^{2,n}(\Omega)\), we have \(\det D^2\phi \in L^1(\Omega)\) by Hölder’s inequality. We are going to use an approximation argument, i.e. according to Lemma 3.7 we need a sequence \(\phi_j \in C(\overline{\Omega})\) converging to \(\phi\) uniformly on compact sets. Since \(\Omega\) is only taken to be convex, \(\partial \Omega\) is not in general \(C^2\) (so we cannot use the standard extension theorem (see e.g. Theorem 4.26 in Adams [1])) and we have to justify that \(u\) can be extended to a \(W^{2,n}(\mathbb{R}^n)\) function of compact support. Indeed, by Theorem 5 on p. 181 and Example 2 on p. 189 of Stein [41] we get a continuous extension of \(\phi\) to \(\mathbb{R}^n\) with \(\|\phi\|_{W^{2,n}(\mathbb{R}^n)} \leq C\|\phi\|_{W^{2,n}(\Omega)}\) and we can then consider \(\phi\xi\), where \(\xi \in C_0^\infty(B(0,2R))\) is such that \(\xi \equiv 1\) on \(B(0,R)\) and \(R > 0\) is chosen such that \(\overline{\Omega} \subset B(0,R)\). It is clear that \(\phi\xi\) is the required extension.

Hence, by the approximation property of Sobolev spaces (see e.g. Theorem 3.18 in Adams [1]), there exist a sequence of functions \(\phi_j \in C^\infty(\mathbb{R}^n)\) such that each \(\phi_j\) vanishes for \(|x| > 2R\) and \(\phi_j \rightharpoonup^j \phi\) in \(W^{2,n}(\Omega)\). Hence also, by Sobolev embedding \(W^{2,n}(B(0,2R)) \hookrightarrow C(\overline{B(0,2R)})\) (see e.g. Evans [20], pp. 284-285), we have \(\phi_j \rightharpoonup^j \phi\) in \(C(\overline{\Omega})\).

Let \(\mu_j := [M\phi_j]^+\) and \(\mu := [M\phi]^+\), so that \(\mu_j \rightharpoonup \mu\) by Lemma 3.7. Let \(\psi_j\) be the unique convex function in \(C(\overline{\Omega})\) such that

\[
\begin{cases}
M\psi_j = \mu_j & \text{in } \Omega, \\
\psi_j = \phi_j & \text{on } \partial \Omega.
\end{cases}
\]

The existence and uniqueness of \(\psi_j\) is given by Theorem 4.5. As \(\phi_j \in C^2(\overline{\Omega})\) we
have (from 1.) that
\[ \psi_j \leq \phi_j \quad \text{in } \Omega. \tag{5.7} \]

Furthermore, since \( \phi_j \to \phi \) in \( C(\partial \Omega) \) and \( M\psi_j \to \mu \), the functions \( \psi_j \) converge in \( C(\bar{\Omega}) \) to the unique convex function \( \bar{\psi} \) in \( C(\bar{\Omega}) \) such that \( \bar{\psi} = \phi \) on \( \partial \Omega \) and \( M\bar{\psi} = \mu \) in \( \Omega \) (again by Lemma 3.7). Taking the limit \( j \to \infty \) in (5.7) we get \( \bar{\psi} \leq \phi \) in \( \bar{\Omega} \). Moreover, since \( M\psi \geq [M\phi]^+ = \mu = M\bar{\psi} \) and both \( \psi, \bar{\psi} \) are convex, we can use the usual comparison principle, Theorem 3.15, to obtain
\[ \min_{\bar{\Omega}} (\bar{\psi} - \psi) \geq \min_{\partial \Omega} (\bar{\psi} - \psi). \tag{5.8} \]

Hence, for each \( x \in \Omega \) we have
\[
\phi(x) - \psi(x) = \underbrace{(\phi(x) - \bar{\psi}(x)) + (\bar{\psi}(x) - \psi(x))}_{\geq 0} \geq \min_{y \in \partial \Omega} (\bar{\psi}(y) - \psi(y)) = \min_{y \in \partial \Omega} (\phi(y) - \psi(y)).
\]

For a concave \( \psi \) use the same reasoning with \( \phi, \psi \) and \( \psi_c \) replaced by \( -\phi, -\psi \) and \( -\psi_c \) respectively. \( \square \)

### 5.2 Bounds on the solution

The following are the author’s own results.

**Theorem 5.2.** Let \( \Omega \) be a bounded, strictly convex set in \( \mathbb{R}^n \). Suppose that \( \phi \in W^{2,n}(\Omega) \) is a solution to
\[
\begin{cases}
  \det D^2 \phi = f & \text{in } \Omega, \\
  \phi = g & \text{on } \partial \Omega,
\end{cases}
\tag{5.9}
\]

where \( f \in L^1(\Omega) \). Then
\[
\Phi_{\text{conv}} \leq \phi \leq \Phi_{\text{conc}} \quad \text{in } \bar{\Omega},
\tag{5.10}
\]

where \( \Phi_{\text{conv}} \in C(\bar{\Omega}) \) is the unique convex solution (\( \Phi_{\text{conc}} \in C(\bar{\Omega}) \) is the unique concave solution) of the problem
\[
\begin{cases}
  \det D^2 \Phi = f^+ & \text{in } \Omega, \\
  \Phi = g & \text{on } \partial \Omega.
\end{cases}
\tag{5.11}
\]
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Proof. Clearly $M\Phi_{\text{conv}} \geq M\phi$ and $M\Phi_{\text{conc}} \geq M\phi$. Hence, by Theorem 5.1, we get

$$\min_{x \in \Omega}(\phi - \Phi_{\text{conv}})(x) = \min_{x \in \partial\Omega}(\phi - \Phi_{\text{conv}})(x) = 0,$$
$$\max_{x \in \Omega}(\phi - \Phi_{\text{conc}})(x) = \max_{x \in \partial\Omega}(\phi - \Phi_{\text{conc}})(x) = 0$$

and the claim follows. \qed

This theorem is a rather surprising result, which says that no matter how negative $\det D^2\phi$ is inside the domain $\Omega$, the solution $\phi$ (if exists) will still be bounded above and below by the two solutions: convex and concave, of the Dirichlet problem (5.11). One of the applications of this result, used in the theory of thin shells (see Rauch & Taylor [39], pp. 352, 364), corresponds to the case $f \equiv 0$. In short, it enables one to prove that, in some parameter regimes, the potential energy of possible shell states has a negative minimum (see Remark 2.2 in Rabinowitz [38]). The argument is based on an observation that the only 2D surface with everywhere zero Gaussian curvature and homogeneous boundary data is a zero surface (see Remark 2.2 in Rabinowitz [38], Theorem 4.1 in Berger [8] and Lemma 3.2 in Berger [7]), i.e. a surface of a zero function. Another consequence of Theorem 5.2, and the main result of the project, is the following non-existence result for the Monge-Ampère equation with constant boundary data and non-positive right-hand side.

**Corollary 5.3.** Let $C \in \mathbb{R}$ and $f \in L^1(\Omega)$ be a nonpositive function such that $f \not\equiv 0$. Then the problem

$$\begin{cases} 
\det D^2\phi = f & \text{in } \Omega, \\
\phi = C & \text{on } \partial\Omega
\end{cases} \tag{5.12}$$

has no $W^{2,n}(\Omega)$ solution.

Proof. Suppose there exists a $W^{2,n}(\Omega)$ solution $\phi$ of the problem (5.12). The constant function $\Phi \equiv C$ satisfies $\det D^2\Phi = 0 = f^+$ with $\Phi|_{\partial\Omega} = C$. Therefore, by Theorem 5.2, $C \leq \phi \leq C$, i.e. $\phi \equiv C$. Hence $0 \equiv \det D^2\phi \equiv f \not\equiv 0$, a contradiction. \qed

We note that, since the regularity results for the Monge-Ampère equation (see the comment after Theorem 4.5) are concerned with a nonnegative right-hand side of the Monge-Ampère equation only, this corollary does not state nonexistence of less regular solutions. What is more, it is not clear what kind of less regular notion of solution would apply here (note that the concept of generalised solutions also corresponds to nonnegative right-hand side only).
Finally, for odd space dimension $n$, the results of this section no longer hold. This is, fundamentally, because for any matrix $A \in \mathbb{R}^{n \times n}$ with odd $n$ we have $\det (-A) = -\det A$. Consequently, the proof of Theorem 5.1 would be no longer valid as the negative definiteness of $D^2\phi$ would imply $M\phi < 0$ and hence we wouldn’t be able to use the comparison principle (Theorem 3.15), as we did in part 1. of the proof of Theorem 5.1. Instead, in the case of odd $n$, one could prove a modification of Theorem 5.1, in which, in the case of concave $\psi$, the positive part $[M\phi]^+$ of the measure $M\phi$ is replaced by the negative part $[M\phi]^-$ (one would also have to be careful with defining a negative Monge-Ampère measure; for instance $M\phi$ for a concave $\phi$ could be defined as a Monge-Ampère measure of the convex $-\phi$). This would lead to the analogue of Theorem 5.2 of the following form.

**Theorem 5.4.** Let $\Omega$ be a bounded, strictly convex set in $\mathbb{R}^n$ with $n$ odd. Suppose that $\phi \in W^{2,n}(\Omega)$ is a solution to

\[
\begin{cases}
\det D^2\phi = f & \text{in } \Omega, \\
\phi = g & \text{on } \partial \Omega,
\end{cases}
\]

where $f \in L^1(\Omega)$. Then

\[
\phi_{\text{conv}} \leq \phi \leq \phi_{\text{conc}} \quad \text{in } \overline{\Omega},
\]

where $\phi_{\text{conv}} \in C(\overline{\Omega})$, $\phi_{\text{conv}}$ is convex, $\phi_{\text{conc}} \in C(\overline{\Omega})$, $\phi_{\text{conc}}$ is concave, are solutions of the problems

\[
\begin{cases}
\det D^2\phi_{\text{conv}} = f^+ & \text{in } \Omega, \\
\phi_{\text{conv}} = g & \text{on } \partial \Omega
\end{cases}
\quad \text{and} \quad
\begin{cases}
\det D^2\phi_{\text{conc}} = -f^- & \text{in } \Omega, \\
\phi_{\text{conv}} = g & \text{on } \partial \Omega
\end{cases},
\]

respectively.

However, we note that no nonexistence result of the form of Corollary 5.3 would follow from the above theorem.

6 Conclusion

In summary, we have that if $\Delta p > 0$ in $\Omega$ or $\Delta p \leq 0$ in $\Omega$ (for $\Omega \subset \mathbb{R}^2$ strictly convex) then there is no divergence-free $u \in H^1_0(\Omega)$ such that (1.4) holds. Therefore, for any velocity field $u$ satisfying the 2D Navier-Stokes equations, the pressure $p$ cannot be superharmonic in $\Omega$ nor strictly subharmonic in $\Omega$ at any moment of time.
Nevertheless, although this fact gives us insight about the pressure $p$, it is still not clear whether we can determine the velocity field $u$ from the pressure $p$.

We note that the theory developed in the dissertation is, in the most part, disjoint with the analysis of Navier-Stokes equations. It would be interesting to learn which of the above results could be obtained from the Navier-Stokes theory (i.e. without using the substitution (1.5)). In fact, using substitution (1.5), one can calculate that $\det D^2 \phi = \det \nabla u$, which translates the Monge-Ampère equation for $\phi$ into the Jacobian equation for $u$. We note, however, that the Jacobian equation is, generally speaking, often considered with boundary conditions of the form $u|_{\partial \Omega} = \text{id}$ (as the Jacobian equation is usually studied within the context of diffeomorphisms of a given domain) and a strictly positive right-hand side (see e.g. Dacorogna & Kneuss [17] to understand why a sign-changing Jacobian determinant is a problem; see also Dacorogna & Moser [18] for an interesting dynamical systems approach to solving a Jacobian equation). An attempt to generalise the presented results to the case of Jacobian equation $\det \nabla u = \frac{1}{2} \Delta p$ for the velocity field $u$ could possibly give us a valuable insight into the properties of the velocity in Navier-Stokes equations.

There are also several directions of a further study of Monge-Ampère equation in the presented context.

1. Generalised solutions to (5.12). One could try to construct a generalised notion of solutions of the Monge-Ampère equation with a negative right-hand side (note that Definition 4.1 corresponds only to a positive right-hand side) and hence investigate the existence of solutions to (5.12) that are not $W^{2,2}$.

2. Nonconstant boundary data in (5.12). Theorem 5.2 suggests that the amplitude of "oscillations" of $f$ inside $\Omega$ is bounded by the "positivity" of $\det D^2 \phi$. Hence one could "hope" for the nonexistence result of the following form. If $\text{osc}_{\partial \Omega} g := \sup_{\partial \Omega} g - \inf_{\partial \Omega} g \leq \epsilon$ then there exists $\delta > 0$ such that the problem

$$\begin{cases}
\det D^2 \phi = f & \text{in } \Omega, \\
\phi = g & \text{on } \partial \Omega,
\end{cases}$$

has no solution when $f \leq -\delta$. Note that Corollary 5.3 corresponds to the case $\epsilon = \delta = 0$.

3. Complex Monge-Ampère equation. One could investigate the theory of complex Monge-Ampère theory (e.g. Aubin [4]) and possibly understand the geometrical picture behind the presented results, particularly behind Corollary 5.3.
References


