# Analysis I : Week II

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We meet twice per week for 2 hours,

Monday 10–12, A0.23
Friday 1–3, A0.23

How the module works:

* Workbook every week.
* For credit in weeks 2, 3, ..., 9.
* Hand in previous workbook at 10am Monday... no excuses!
* Work marked by your supervisor (not me) and they give feedback.

During classes:

* I give a short presentation.
* Then you work quietly on the questions.

① What is real analysis?

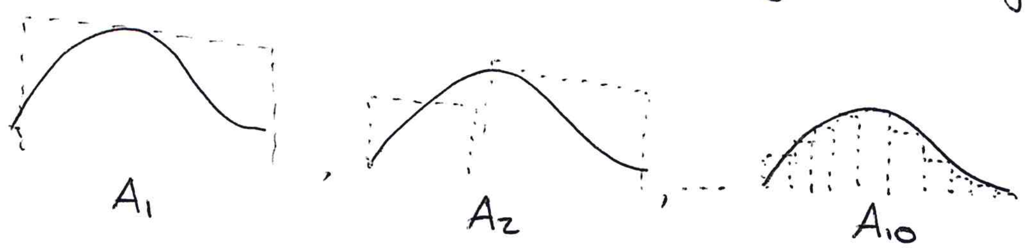
In short it is the study of sequences, series, and functions. Most importantly it is the study of limiting procedures.

* Analysis was introduced to make rigorous Newton's theory of calculus.
* Both differentiation, and integration require limit procedures.

Real analysis lays the foundation for other areas: functional analysis, harmonic analysis, stochastic analysis, complex analysis, topology, measure theory, ... it is essential for almost every module you will do!

② Beginning Analysis: Sequences.

Motivation: suppose we want to approximate the area under a curve, by taking n-rectangles



Our approximation only makes sense if the An (area of n-th approximation) approch the true area of the curve.

How do we make rigorous the idea of 'approaching'?

This is what we mean by studying limits.

Example. The sequence 1, 2, 4, 8, ... is given by $a_n = 2^n$. Clearly this keeps getting larger, i.e $\lim_{n\to\infty} a_n = \infty$, but how do we make this rigorous.

Example. The sequence 1, 1/2, 1/3, ... clearly gets closer to 0, i.e $\lim_{n\to\infty} a_n = 0$. How do we define this rigorously?

## ③ Class Notes

* Monotone sequences.

Remember monotonous means boring, repetitive, unvarying, etc. So in some sense a monotone sequence should be like that... i.e a sequence that is always getting "bigger" (increasing), "smaller" (decreasing), or "staying the same" (constant).

* Split sequences.

Normally our sequences can be written in the form $a_n = f(n)$, where f is a simple function. eg.

$$a_n = \sqrt{n} \quad , \quad f(x) = \sqrt{x}$$

$$a_n = \sin(n) \quad , \quad f(x) = \sin(x)$$

But we are also allowed to define 'split' sequences, eg.

$$a_n = \begin{cases} 0 & , \text{ if } n = 2k-1 \\ n & , \text{ if } n = 2k . \end{cases}$$

This sequence takes different values depending on whether n is odd or even.

We can also define sequences that only differ from a simple function at a single point eg.

$$a_n = \begin{cases} n & , n \in \mathbb{N}, n \neq 5 \\ 0 & , n = 5 \end{cases}$$

This is the sequence which begins

$$1, 2, 3, 4, 0, 6, 7, 8, \ldots$$

* $\mathbb{N} = \{1, 2, 3, \ldots\}$

In analysis we adopt the convention that 0 is not a natural number (i.e $0 \notin \mathbb{N}$). This simply makes it easier to define sequences like $a_n = 1/n$.

* Upper and lower bounds are not unique. Look at your definition of upper bound. Suppose $U$ is an upper bound, and that $V \geqslant U$. Then $V$ also satisfies the definition. Whilst it is nice when you can find the smallest upper bound (or the largest lower bound), this may not always be possible. So when asked for an upper bound, you don't have to give the smallest one, but it is good if you can.

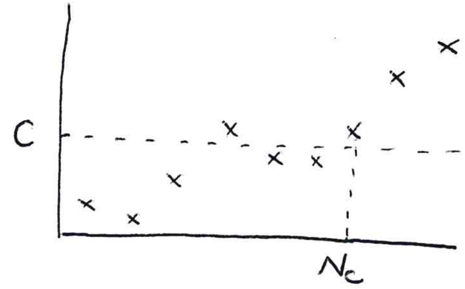
* Convergence (by Picture!)

$a_n \to \infty$

$\forall C > 0, \exists N = N_C$

st. $\forall n > N$

$a_n > C$.



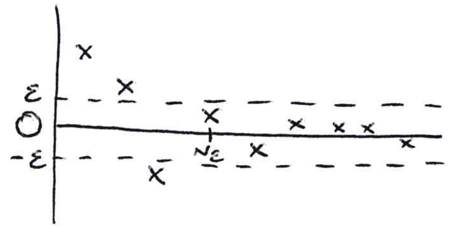

(beyond $N_C$, all points in the picture are above the dotted line.

$a_n \longrightarrow 0$

$\forall \varepsilon > 0, \exists N = N_\varepsilon$

st $\forall n > N$

$|a_n| < \varepsilon$.




(beyond $N_\varepsilon$, all points lie inside the dotted tube).

$\boxed{\text{Remember}}$

When proving convergence, your proof must be true for all choices of $C$ (or $\varepsilon$), not just for one choice!

* Showing Convergence (directly)

We will learn many ways to show $a_n \to 0$, or $a_n \to \infty$. At the moment we are working 'directly' from definition.

When doing questions like this it is best to do the calculation 'backwards' in rough, and then redo the answer neatly after.

Recall if we want to show $a_n \to 0$, then our answer should look like.

Fix $\varepsilon > 0$. Let $N_\varepsilon = \underline{\quad\quad}$, then
$\forall n > N$

$$|a_n| = \underline{\ \dots\ } \le |a_N| < \varepsilon \quad \square$$

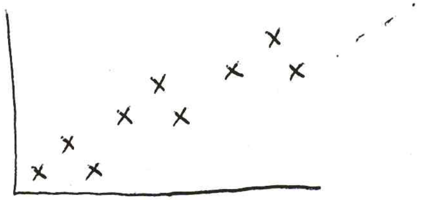
We of course must fill in the blanks.
To do this,

i) First find the $N = N_\varepsilon$ for which $a_n < \varepsilon$ when $n > N_\varepsilon$. Normally this can be done by solving a simple inequality.

ii) Now 'rework' the inequality to show $|a_n| < |a_N|$

Part (i) is done in your rough work, part (ii) is then put in your answer... they are effecchively the same step.

Ex. $a_n = \sin\left(\frac{n\pi}{2}\right) + n - \frac{1}{2}\left(1-(-1)^n\right)$



We guess that this goes to $\infty$. Now find $N_C$.

i) [Rough work] Want $N_C$ st

$$\sin\left(\frac{n\pi}{2}\right) + n - \frac{1}{2}\left(1-(-1)^n\right) > C \quad \text{if } n \ge N$$

Note $\sin\left(\frac{n\pi}{2}\right) \ge -1$, $-\frac{1}{2}\left(1-(-1)^n\right) \ge -1$

so
$$a_n \ge n - 2$$

If we choose $N_C = C + 2$, then

$$a_n \ge n - 2 \ge N_C - 2 = C, \quad \text{if } n \ge N_C$$

Now we write our solution:

Fix $C > 0$, let $N_C = C + 2$. Then, $\forall n > N_C$

$$\begin{aligned} a_n &= \sin\left(\tfrac{n\pi}{2}\right) + n - \tfrac{1}{2}\left(1-(-1)^n\right) \\ &\ge n - 2 \\ &\ge N_C - 2 \\ &> C \end{aligned} \quad \square$$

And we are done!