University of Durham

Final year project

Quantum Field Theory:

The Higgs Mechanism

Author:
Simon Matthew Bignold

Supervisor:
Dr. Mukund Rangamani

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Abstract

In this report we will describe the Higgs mechanism, the technique used for giving mass to particles in the standard model of particle physics. We will start by introducing the theory required to understand the mechanism, continue to outline the mechanism itself and conclude with the implications of the mechanism and a brief summary of the attempts to measure its effects.

Contents

Chapter 1

Introduction

1.1 Notation

Below we give a brief discussion of the notation we will be using, the idea behind this is to define a set of notation that we will try and use consistently. We will leave aside universal notation such as basic addition etc, but we include vectors and the complex unit number as these can be denoted in several different ways.
We will use the energy scale where c (the speed of light) and $\hbar$ (planck's constant) are taken to be one. This is merely for convenience and we may restore these factors to compare the results of theory with those of experiment.

Vectors will be denoted $\vec{a}$
Operators will be denoted by a hat e.g. $\hat{A}$
Scalar fields will be generally denoted by ϕ
i,j will be used to denote the space-like indices of the four-vector
μ, ν will be used to denote the indices of the four-vector including the time indices
x_i will be used to denote position co-ordinates
$x_0 = t$ will be used to denote time x_0 will be generally used in the context of the four vector , and t when it is considered as a separate variable
normal script will be used to denote the four-vector restoring the index μ only when it is important
x, x' will be used to denote the space-time four-vector
k, k' will be used to denote the momentum four-vector , occasionally we will also use p,p' this should be clear from the context
$k.x = k_0x_0 - k_1x_1 - k_2x_2 - k_3x_3$, i.e we are using the mostly negative metric
$g^{\mu\nu}$ denotes the special-relativity metric
$\partial_\mu = \frac{\partial}{\partial x^\mu}$
Einstein summation is used i.e. $a_nb_n = \sum_i a_ib_i$

$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial {x_0}^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}$
a^T is the transpose of a
a^* is the complex conjugate of a
$a^\dagger = a^{*T}$ is the dagger of a

bra-ket notation is used where:
$\langle a \mid$ is the bra
$\mid a \rangle$ is the ket
$\langle a \mid = \mid a \rangle^\dagger$
$\langle a \mid b \rangle$ is the complex inner product of a and b
a!! = a(a-2)(a-4)... 1!!=1
Generally the measure of the integral e.g. dx will apply to all functions to the right of it up to the ending of a bracket
$\delta^3(\vec{k} - \vec{k}')$ is the dirac delta function of $\vec{k}, \vec{k}'$ for three-dimensions
i.e. $\int d^3k' f(\vec{k}) \delta^3(\vec{k} - \vec{k}') = f(\vec{k})$
This extends to four-dimensions i.e. $\int d^4k' f(k) \delta^4(k - k') = f(k)$
$i = \sqrt{-1}$
δS is the variation of S
confusingly δ_{ij} is the kronecker delta function $= \{ \begin{array}{c} 0 \ i \neq j \\ 1 \ i = j \end{array}$
$\dot{p} = \frac{dp}{dt}$
${}^*_*\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^{\dagger}{}^*_* = \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$ is normal ordering i.e. put all $\hat{a}^\dagger$ to the left of $\hat{a}$
$\theta(x)$ is the heavside function $= \{ \begin{array}{c} 1 \ x > 0 \\ 0 \ x < 0 \end{array}$
ϵ is the strictly positive small variable i.e. it can be arbitrarily small
T is the time ordering operator $T(\hat{\phi}(\mathbf{x}')\hat{\phi}^\dagger(\mathbf{x}')) = \theta(t' - t)\hat{\phi}(\mathbf{x}')\hat{\phi}^\dagger(\mathbf{x}) + \theta(t - t')\hat{\phi}^\dagger(\mathbf{x})\hat{\phi}(\mathbf{x}')$ the operator with the largest t comes first
λ is the small parameter used in the perturbative expansion of the interacting lagrangian

In the consideration of interacting fields
$\hat{\phi}_0(t, \vec{x})$ denotes the free field field i.e. when $\lambda = 0$
This also applies to the vacuum state $\mid 0\rangle_0$ is the free vacuum $\mid 0\rangle$ is the interacting vacuum $\phi_{t_0}(\vec{x})$ denotes the field in the Schrödinger picture

$O(x^2)$ denotes terms of order x^2 or higher
$\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$
$\Box = [\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2] = \partial_\mu \partial^\mu$
The 4-dimensional Levi-Cevita tensor $\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\nu\mu\rho\sigma}$ $\epsilon^{0123} = -\epsilon_{0123} = -1$
The 3-dimensional Levi-Cevita tensor $\epsilon^{ijk} = -\epsilon^{jik}$ $\epsilon^{123} = 1$

1.2 Required Mathematical Techniques

Obviously this is a mathematical paper and thus there will be much more mathematics outlined in this paper. Here we highlight some of the more commonly used mathematical techniques in this paper that we will take to be common knowledge but may be outside the general knowledge of the undergraduate mathematician. We expect a reasonable grounding in vector calculus and linear algebra. We hope to introduce and cover new techniques in detail throughout the paper and any confine any excessive calculations to the appendices.

<u>The Operator Exponent</u>
Taking our cue from the taylor series of the normal exponent we define :

$$e^{\hat{A}x} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n x^n \qquad n \in \mathbb{N} \tag{1.2.1}$$

<u>The Heavside function</u>
The Heavside function can be defined as the limit of an exponential :

$$\theta(t) = \lim_{\epsilon \to 0} \int \frac{d\omega}{2\pi i} \frac{e^{i\omega t}}{\omega - i\epsilon} \tag{1.2.2}$$

This can be seen by considering the contour-integral of the function over the two half-planes. As $\epsilon > 0$ the pole of the function ($\omega = i\epsilon$) is in the upper half -plane. Thus the integral over the lower half-plane ($t < 0$) is 0 as it contains no poles and the integral over the upper -half plane ($t > 0$)is the function evaluated at the pole which gives a value of 1 .

<u>The Taylor Expansion of a Function around a Minimum</u>
Consider the taylor expansion of a function V(x) about its minimum:

$$V(x) = V(x) \mid_0 + V'(x) \mid_0 x + \frac{V''(x)}{2} \mid_0 x^2 + O(x^3) \tag{1.2.3}$$

Since $V'(x) \mid_0 = 0$ at the minimum at low orders of x i.e. near the minimum.

$$V(x) \approx V(0) + \frac{k}{2} x^2 \qquad k = V''(x) \mid_0 \text{ is a constant} \tag{1.2.4}$$

Importantly for physics this means near a (stable) minimum many potential function V(x) can be approximated up to a constant by the harmonic oscillator potential $\frac{k}{2}x^2$.

<u>Variational Derivatives</u>
By analogue with normal derivatives for two fields $\phi(\vec{x}, t), \phi(\vec{y}, t)$ we define the variational derivative:

$$\frac{\delta\phi(\vec{x}, t)}{\delta\phi(\vec{y}, t)} = \delta^3(\vec{x} - \vec{y}) \tag{1.2.5}$$

Commutators

The commutator of two operators $\hat{a}, \hat{b}$ is:

$$[\hat{a}, \hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a} \tag{1.2.6}$$

If $[\hat{a}, \hat{b}] = 0$ $\hat{a}, \hat{b}$ are said to commute .

Anti-Commutators

The commutator of two operators $\hat{a}, \hat{b}$ is:

$$\{\hat{a}, \hat{b}\} = \hat{a}\hat{b} + \hat{b}\hat{a} \tag{1.2.7}$$

If $\{\hat{a}, \hat{b}\} = 0$ $\hat{a}, \hat{b}$ are said to anti-commute.

The Poisson Bracket

The Poisson bracket of two quantities f and g that are functions of p and q is defined:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{1.2.8}$$

Left and Right Derivatives

for the differential operator D we define $a\overleftarrow{D} = Da$ and :

$$f\overleftrightarrow{\partial_\mu}g \equiv f(\partial_\mu g) - (\partial_\mu f)g \tag{1.2.9}$$

Cross-Product

for two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ their cross product is defined by :

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \tag{1.2.10}$$

The Dirac Delta Function

The n-dimensional Dirac delta function is defined by $\delta^n(x)$ where for any functionf and any n-dimensional vector x we have:

$$\int d^n x f(x)\delta^n(x) = f(0) \tag{1.2.11}$$

We take the Fourier transform of this to give:

$$(2\pi)^n\delta^n(x - x') = \int d^n k e^{ik(x-x')} d^n k \tag{1.2.12}$$

1.3 Required Mathematics: Group Theory

We here introduce a few aspects of group theory that will be useful later . A group is a set of objects (e.g. numbers, vectors, matrices etc) with an operator *. That obeys the axioms of a group for group G operator *

i) $\forall A, B \in G$ If A*B =C $C \in G$ completeness.
ii) (A*B)*C = A*(B*C) $\forall A, B, C \in G$ associativity.
iii) $\exists I \in G$ such that I*A= A*I=A $\forall A \in G$ existence of the identity.
iv) $\forall A \in G$ $\exists B$ such that $A * B = B * A = I$ existence of the inverse.

We need only consider two groups U(1) and SU(2).
U(1) known as the circle group is the group of complex numbers modulus 1 with complex multiplication, i.e. it has members $z = e^{i\theta}$ $\mid z \mid= 1$.
Complex multiplication is associative, the identity is 1 given by $\theta = 0$, the inverse is given by $z* = e^{-i\theta}$. $e^{i\theta}$ has modulus one for all θ so $z_1 z_2 = e^{i(\theta_1 + \theta_2)}$ is in U(1) $\forall z_1, z_2$

The SU(2) group is the group of 2×2 unitary ($U^{-1} = U^{\dagger}$) matrices with determinant +1 (U(N) unitary matrices of dimension N have determinant ± 1 , SU(N) special unitary matrices of dimension N have determinant +1) under matrix multiplication.
matrix multiplication is clearly associative, the identity matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.1}$$

is clearly a member of SU(2). Since for any two matrices A, B det (AB) =det (A) det (B) if A and B have determinant 1 AB has determinant 1 and as matrix multiplication between two 2×2 matrices gives a 2×2 matrix , this group is closed. If the inverse A^{-1} exists then using the properties of determinants det(I) $=det(AA^{-1}) = det(A)det(A^{-1})$ since det(I) =det(A)=1 for A in SU(2) $det(A^{-1}) = 1$. For any 2×2 matrix :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3.2}$$

its inverse is :

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1.3.3}$$

We would now like to show the equivalence between O(3) and SU(2) where for our purposes O(3) is the group of all possible rotations in 3-space.

We follow now section 13.1 in [1]. Assuming our SU(2) matrix is of the form:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3.4}$$

then:

$$u^{\dagger} = u^{-1} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{1.3.5}$$

using:

$$\begin{aligned} uu^{-1} &= uu^{\dagger} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + b^*d & cc^* + dd^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1.3.6}$$

and det u = ad-bc=1 .
we now have four -equations:
1) $\mid a \mid^2 + \mid b \mid^2 = 1$.
2) $ac^* = -bd^*$.
3)$ad - bc = 1$.
4) $\mid c \mid^2 + \mid d \mid^2 = 1$.
These reduce to the one equation $\mid a \mid^2 + \mid b \mid^2 = 1$ with the solutions $c = -b^*, d = a^*$.
We now take our general matrix to be(13-3a p342 [1]):

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{1.3.7}$$

Consider the matrix:

$$h = \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \qquad x, y, z \in \mathbb{Z} \tag{1.3.8}$$

The equation $h' = uhu^{\dagger}$:

$$\begin{pmatrix} -z' & x' + iy' \\ x' - iy' & z' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \begin{pmatrix} a^* & -b \\ b & a \end{pmatrix} \tag{1.3.9}$$

clearly provides a transform :

$$r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rightarrow r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3.10}$$

which can be written in terms of a 3×3 matrix R:
i.e. $r' = Rr$

R is a rotation matrix as:
1) lengths are preserved:

$$det(h) = -z^2 - (x + iy)(x - iy) = -(z^2 + x^2 + y^2) \tag{1.3.11}$$

which is the negative of the length in free-space :

$$det(h') = det(uhu^{\dagger}) = det(u)det(h)det(u^{\dagger}) \tag{1.3.12}$$

using property of determinants det(AB) =det (A) det (B) (see Appendix A1)
so:

$$det(h') = det(h) \text{ as } det(u) = det(u^\dagger) = 1 \qquad (1.3.13)$$

by the properties of unitary matrices .
2) All its terms are real so R is real (see appendix A2)
3) Its determinant is 1 as $R \rightarrow I_3$ continuously as $u \rightarrow I_2$ ($a \rightarrow 1, b \rightarrow 0$) (see l9-13 p343 [1]) .

We will now show that some specific SU (2) matrices correspond using this method to rotations about the three co-ordinate axes (x,y,z) in 3-dimensional space. We here use introduce the Pauli-spin matrices(see p217 eq 5.4.18 [2]):

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.3.14)$$

we will use these in preference to the form in [1] for later clarity.

Assume u is diagonal, i.e. b=0 , $| \ a \ |^2 = 1 \rightarrow a = e^{\frac{-i\alpha}{2}}$ i.e. u becomes (l26 p343 [1]):

$$u_1 = \begin{pmatrix} exp[-\frac{i\alpha}{2}] & 0 \\ 0 & exp[\frac{i\alpha}{2}] \end{pmatrix} = exp[-\frac{i\alpha}{2}\sigma^3] \qquad (1.3.15)$$

Recall:

$$exp[-\frac{i\alpha}{2}\sigma^3] = 1 + -\frac{i\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (-\frac{i\alpha}{2})^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ... \qquad (1.3.16)$$

using the exponential for a matrix, which reduced to :

$$exp[-\frac{i\alpha}{2}\sigma^3] = \begin{pmatrix} 1 - \frac{i\alpha}{2} + (-\frac{i\alpha}{2})^2 + ... & 0 \\ 0 & 1 + \frac{i\alpha}{2} + (\frac{i\alpha}{2})^2 + ... \end{pmatrix} \qquad (1.3.17)$$

So:

$$exp[-\frac{i\alpha}{2}\sigma^3] = \begin{pmatrix} exp[-\frac{i\alpha}{2}] & 0 \\ 0 & exp[\frac{i\alpha}{2}] \end{pmatrix} \qquad (1.3.18)$$

using the taylor expansion of the exponential.
This means:

$$\begin{aligned}
\begin{pmatrix} -z' & x' + iy' \\ x' - iy' & z' \end{pmatrix} &= \begin{pmatrix} exp[-\frac{i\alpha}{2}] & 0 \\ 0 & exp[\frac{i\alpha}{2}] \end{pmatrix}\begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \\
&\times \begin{pmatrix} exp[\frac{i\alpha}{2}] & 0 \\ 0 & exp[-\frac{i\alpha}{2}] \end{pmatrix} \\
&= \begin{pmatrix} exp[-\frac{i\alpha}{2}] & 0 \\ 0 & exp[\frac{i\alpha}{2}] \end{pmatrix}\begin{pmatrix} -ze^{\frac{i\alpha}{2}} & e^{\frac{-i\alpha}{2}}(x + iy) \\ e^{\frac{i\alpha}{2}}(x - iy) & ze^{\frac{-i\alpha}{2}} \end{pmatrix} \\
&= \begin{pmatrix} -z & e^{-i\alpha}(x + iy) \\ e^{i\alpha}(x - iy) & z \end{pmatrix} \qquad (1.3.19)
\end{aligned}$$

This gives the four equations:
1) $-z' = -z$
2) $x + iy = cos\alpha x + sin\alpha y + i(-sin\alpha x + cos\alpha y)$
3)$x - iy = cos\alpha x + sin\alpha y + -i(-sin\alpha x + cos\alpha y)$
4) $z' = z$ using $e^{ix} = cosx + isinx$. which implies:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3.20}$$

So:

$$R(u_1) = \begin{pmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3.21}$$

which is a rotation about the z-axis.

We now assume u is real , we have (l30 p343 [1]):

$$u_2 = \begin{pmatrix} cos\frac{\beta}{2} & -sin\frac{\beta}{2} \\ sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} = exp[\frac{i\beta}{2}\sigma^2] \tag{1.3.22}$$

using the exponential for a matrix :

$$\begin{aligned} exp[\frac{i\beta}{2}\sigma^2] &= 1 + \frac{i\beta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + (\frac{i\beta}{2})^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ... \\ &= \begin{pmatrix} 1 - (\frac{\beta}{2})^2 + ... & -\frac{\beta}{2} + ... \\ \frac{\beta}{2} + ... & 1 - (\frac{\beta}{2})^2 + ... \end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\beta}{2} & -sin\frac{\beta}{2} \\ sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} \end{aligned} \tag{1.3.23}$$

using the taylor expansion of cos and sin in the last line. which implies:

$$\begin{aligned} \begin{pmatrix} -z' & x' + iy' \\ x' - iy' & z' \end{pmatrix} &= \begin{pmatrix} cos\frac{\beta}{2} & -sin\frac{\beta}{2} \\ sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \\ &\times \begin{pmatrix} cos\frac{\beta}{2} & sin\frac{\beta}{2} \\ -sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\beta}{2} & -sin\frac{\beta}{2} \\ sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} \\ &\times \begin{pmatrix} -zcos\frac{\beta}{2} - (x + iy)sin\frac{\beta}{2} & (x + iy)cos\frac{\beta}{2} - zsin\frac{\beta}{2} \\ (x - iy)cos\frac{\beta}{2} - zsin\frac{\beta}{2} & zcos\frac{\beta}{2} + (x - iy)sin\frac{\beta}{2} \end{pmatrix} \end{aligned} \tag{1.3.24}$$

This gives us four equations:
1)$-z' = -z(cos^2\frac{\beta}{2} - sin^2\frac{\beta}{2}) - x2sin\frac{\beta}{2}cos\frac{\beta}{2}$

2) $x' + iy' = -z2sin\frac{\beta}{2}cos\frac{\beta}{2} + (cos^2\frac{\beta}{2} - sin^2\frac{\beta}{2})x + iy(cos^2\frac{\beta}{2} + sin^2\frac{\beta}{2})$
3) $x' - iy' = -z2sin\frac{\beta}{2}cos\frac{\beta}{2} + (cos^2\frac{\beta}{2} - sin^2\frac{\beta}{2})x - iy(cos^2\frac{\beta}{2} + sin^2\frac{\beta}{2})$
4) $z' = z(cos^2\frac{\beta}{2} - sin^2\frac{\beta}{2}) + x2sin\frac{\beta}{2}cos\frac{\beta}{2}$
which using $cos^2\frac{\beta}{2} - sin^2\frac{\beta}{2} = cos\beta$ $cos^2\frac{\beta}{2} + sin^2\frac{\beta}{2} = 1$ $2sin\frac{\beta}{2}cos\frac{\beta}{2} = sin\beta$, gives in reduces to:
1)$-z' = -zcos\beta - xsin\beta$
2) $x' + iy' = -zsin\beta + cos\beta x + iy$
3) $x' - iy' = -zsin\beta + cos\beta x - iy$
4) $z' = zcos\beta + xsin\beta$
This implies :

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3.25}$$

So:

$$R(u_2) = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{pmatrix} \tag{1.3.26}$$

which is a rotation about the y-axis.

Assume b is imaginary, we get (l3 p344 [1]):

$$u_3 = \begin{pmatrix} cos\frac{\Delta}{2} & isin\frac{\Delta}{2} \\ isin\frac{\Delta}{2} & cos\frac{\Delta}{2} \end{pmatrix} = exp[\frac{i\Delta}{2}\sigma^1] \tag{1.3.27}$$

using the exponential for a matrix :

$$\begin{aligned} exp[\frac{i\Delta}{2}\sigma^1] &= 1 + \frac{i\Delta}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (\frac{i\Delta}{2})^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ... \\ &= \begin{pmatrix} 1 - (\frac{\Delta}{2})^2 + ... & i(\frac{\Delta}{2} + ...) \\ i(\frac{\Delta}{2} + ...) & 1 - (\frac{\Delta}{2})^2 + ... \end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\Delta}{2} & isin\frac{\Delta}{2} \\ isin\frac{\Delta}{2} & cos\frac{\Delta}{2} \end{pmatrix} \end{aligned} \tag{1.3.28}$$

using the taylor expansion of cos and sin in the last line. Our transformation is:

$$\begin{aligned} \begin{pmatrix} -z' & x' + iy' \\ x' - iy' & z' \end{pmatrix} &= \begin{pmatrix} cos\frac{\Delta}{2} & isin\frac{\Delta}{2} \\ isin\frac{\Delta}{2} & cos\frac{\Delta}{2} \end{pmatrix} \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \\ &\times \begin{pmatrix} cos\frac{\Delta}{2} & -isin\frac{\Delta}{2} \\ -isin\frac{\Delta}{2} & cos\frac{\Delta}{2} \end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\Delta}{2} & isin\frac{\Delta}{2} \\ isin\frac{\Delta}{2} & cos\frac{\Delta}{2} \end{pmatrix} \\ &\times \begin{pmatrix} -zcos\frac{\Delta}{2} - (x + iy)isin\frac{\Delta}{2} & (x + iy)cos\frac{\Delta}{2} + zisin\frac{\Delta}{2} \\ (x - iy)cos\frac{\Delta}{2} - zisin\frac{\Delta}{2} & zcos\frac{\Delta}{2} - (x - iy)isin\frac{\Delta}{2} \end{pmatrix} \end{aligned} \tag{1.3.29}$$

thus we get four equations:
1)$-z' = -z(cos^2\frac{\Delta}{2} - sin^2\frac{\Delta}{2}) + y2sin\frac{\Delta}{2}cos\frac{\Delta}{2}$
2) $x' + iy'z = 2isin\frac{\Delta}{2}cos\frac{\Delta}{2} + (cos^2\frac{\Delta}{2} + sin^2\frac{\Delta}{2})x + iy(cos^2\frac{\Delta}{2} - sin^2\frac{\Delta}{2})$
3) $x' - iy' = -z2isin\frac{\Delta}{2}cos\frac{\Delta}{2} + (cos^2\frac{\Delta}{2} + sin^2\frac{\Delta}{2})x - iy(cos^2\frac{\Delta}{2} - sin^2\frac{\Delta}{2})$
4) $z' = z(cos^2\frac{\Delta}{2} - sin^2\frac{\Delta}{2}) - y2sin\frac{\Delta}{2}cos\frac{\Delta}{2}$
using $cos^2\frac{\Delta}{2} - sin^2\frac{\Delta}{2} = cos\Delta\ cos^2\frac{\Delta}{2} + sin^2\frac{\Delta}{2} = 1\ 2sin\frac{\Delta}{2}cos\frac{\Delta}{2} = sin\Delta$, this reduces to:
1)$-z' = -zcos\Delta + ysin\Delta$
2) $x' + iy' = izsin\Delta + x + iycos\Delta$
3) $x' - iy' = -zisin2\Delta + x - iycos\Delta$
4) $z' = zcos\Delta - ysin2\Delta$

This implies:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\Delta & sin\Delta \\ 0 & -sin\Delta & cos\Delta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3.30}$$

So:

$$R(u_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\Delta & sin\Delta \\ 0 & -sin\Delta & cos\Delta \end{pmatrix} \tag{1.3.31}$$

Which is a rotation about the x-axis.

We assume that any rotation can be described by combinations of rotations about the three co-ordinate axes (see l12-13 p344 [1]) and thus the group O(3) is holomorphic to SU(2). Thus any rotation in three-space can be described by an appropriate combination of $exp[-\frac{i\alpha}{2}\sigma^3], exp[\frac{i\beta}{2}\sigma^2]$ and $exp[\frac{i\Delta}{2}\sigma^1]$. This will be useful later.

1.4 Introduction

In this paper we hope to give brief outline of quantum field theory and then move on to outlining the specifics of the Higgs Mechanism. In this introduction I hope to give a brief summary of the motivation behind quantum field theory and the central concepts underlying the theory. We will then proceed on to the main body of the paper where we will develop the conceptual framework from quantum field theory needed for understanding the Higgs mechanism before describing the mechanism itself and then finally highlighting the relevance of the Higgs mechanism to modern physics and performing some calculations to show how it can be used.

We begin with a brief summary of quantum mechanics and special relativity focusing on the aspects required for the development of quantum field theory and also highlighting the techniques from classical mechanics relevant to quantum field theory. We also highlight Maxwell's equations as a classical theory that can be easily adapted to a quantum theory. We then introduce dimensional analysis as a way of comparing quantities especially in the $c = \hbar = 1$ notation to ensure that equations are dimensionally correct. Next we introduce the five most common free fields starting with the most simple harmonic scalar field and then introducing the complex scalar field , we then move onto their spinor analogues in the Majorana fields and Dirac spinors, finally we introduce the concept of a vector field linking this to Maxwell's equations. We then introduce canonical quantisation and the path integral formalism as two standard ways of representing quantum field theory. We then introduce the concept of interacting fields and the techniques of dealing with these through perturbation theory and the LSZ formula and the graphical representations of these techniques in Feynman diagrams and their associated rules. We then introduce two important quantum field theory concepts the Ward identities and conserved currents the latter of which is very useful for our discussion of the Higgs mechanism. We then move onto corrections to Feynman diagrams to ensure accuracy specifically covering loop corrections and introducing 1PI vertices. Renormalisation is then discussed as a method of limiting the number of possible field theories and ensuring non-divergence of results.We introduce the quantum action as a useful concept for considering symmetries. We then discuss spontaneous symmetry breaking as a key component of the Higgs mechanism highlighting Goldstone's theorem and its prediction of the Goldstone Boson. We then expand on the concept of gauge theory and highlight the difference between abelian and non-abelian theories. We then give a brief summary of QED as the simplest useful quantum theory and an example of the use of different types of fields. Finally we discuss the Higgs mechanism and highlight Higgs detection paths explicitly explaining the connection to the LHC and the Higgs boson paths expected there. We then proceed to do some calculations to derive interesting quantities for a number of different predicted Higgs bosons. We conclude with a conclusion summarising the work above and follow with the Bibliography and appendices.

This paper attempts to be relatively self-contained and comprehensive, we hope to highlight issues which are not covered. Specifically we ignore the complexities associated with the strong force since all the new particles introduced by the strong force are massless and thus the Higgs mechanism does not apply to them.

The original aim of quantum field theory is to combine the two separate theories of quantum mechanics and special relativity. In essence this means creating a theory that allows for the Lorentz invariance of special relativity and quantisation of quantum mechanics, allowing us to simultaneously incorporate such key notions as the uncertainty relation of position and momentum and light-cones.

Initially we will consider the so-called free-field relation whose lagrangian density is given by $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$ which is analogous to the harmonic oscillator (this notation will be discussed later in further detail for now it is only the relationship to the harmonic oscillator that is important) . This can be derived by thinking of space as made up of a lattice of harmonic oscillators and taking the limit of an infinite number of harmonic oscillators. As we progress to more complicated field relations we will have to think of there being more complicated potentials at each point which is obviously less intuitive, but could be considered as interacting corrections between the different harmonic oscillators.

The initial consideration of harmonic oscillators is useful for two reasons. Firstly as will be shown below the equations for the harmonic oscillator can be solved exactly. Secondly by analogue with classical physics many potentials near their minimum can be approximated at low order by a harmonic potential. This second point is something we will come back to when we consider perturbative expansions leading to Feynman diagrams.

1.5 Brief Summary of Required Quantum Mechanics

For this section we retain $\hbar$'s to make it easier for the reader to recognise concepts that they may already have covered.
One of the fundamental tenants of quantum mechanics is that fundamental physical quantities such as energy and momentum do not form a continuous spectrum but are instead quantised in discrete levels . This is a concept realised by the operator notation of quantum mechanics where each physical quantity we wish to measure is promoted to an operator whose eigenvalues are the possible measurable values (observables) of the given physical quantity . These operators are considered to be hermitian ($\hat{A} = \hat{A}^\dagger$) so that in matrix representation their eigenvectors form a basis and their eigenvalues are real. The eigenvectors forming a basis means that they span the required space and thus any vector in this space can be written as a linear combination of these eigenvectors, these two general properties of hermitian matrices are proven in the appendix A1. The eigenvectors are taken to be real so that observables are real which is what we hope as we always observe physical quantities as being real.

In generalised notation the eigenvectors are represented by the dirac bras i.e:

$$\hat{A} \mid n\rangle = a \mid a\rangle \tag{1.5.1}$$

where $\hat{A}$ is an operator, a is an eigenvalue $\mid n\rangle$ is the eigenvector

There are a specific class of operators that will be very useful to us known as the creation and annihilation operators.They have an analogue in the standard quantum mechanical treatment of the harmonic oscillator which I will outline below.

The expectation value of a given operator $\hat{A}$ is given by :

$$\langle\hat{A}\rangle = \langle\psi \mid \hat{A} \mid \psi\rangle \tag{1.5.2}$$

where $\mid \psi\rangle$ is a arbitrary state vector

Another important quality of quantum mechanics are uncertainty relations, these essentially say that if we have two quantities and one is measured to a certain degree of accuracy , the other cannot be measured to arbitrary accuracy. The most famous of these is the uncertainty relation between position and momentum which can be written as:

$$\Delta P \Delta Q \geq \frac{\hbar}{2} \tag{1.5.3}$$

where ΔP is the uncertainty in P and ΔQ is the uncertainty in Q

It can be shown that all uncertainty relations between two quantities is equivalent to the two operators representing them not commuting i.e for two operators $\hat{A}, \hat{B}$

$$\Delta A \Delta B \neq 0 \Rightarrow [\hat{A}, \hat{B}] \neq 0 \tag{1.5.4}$$

This follows from a proof that shows that if $(\Delta\hat{A})^2$ is defined as:
$\langle\psi \mid (\hat{A} - \langle\hat{A}\rangle)^2 \mid \psi\rangle$ for arbitrary operators $\hat{A}$ and $\hat{B}$ then :

$$\Delta\hat{A}\Delta B \geq \frac{1}{2} \mid \langle\psi \mid [\hat{A}, \hat{B}] \mid \psi\rangle \mid \tag{1.5.5}$$

(see for example [3])
this shows that if if $[\hat{Q}, \hat{P}] = i\hbar\hat{I}$ then $\Delta\hat{Q}\Delta\hat{P} \geq \frac{\hbar}{2}$ and vice versa
.
For the harmonic oscillator in quantum mechanics :
$\hat{a} \mid n\rangle = \sqrt{n} \mid n-1\rangle$ annihilation operator as $\hat{a} \mid 0\rangle = 0$
$\hat{a}^\dagger \mid n\rangle = \sqrt{n+1} \mid n+1\rangle$ creation operator as $\hat{a}^\dagger \mid 0\rangle = \mid 1\rangle$
see the appendix A2 for details

In quantum mechanics as stated above each physical quantity is promoted to an operator and each poisson bracket is promoted to a commutator and multiplied by $i\hbar$ so the poisson bracket $\{q, p\} = 1$ implies

$$[\hat{Q}, \hat{P}] = i\hbar \tag{1.5.6}$$

The Pauli spin matrices will also prove useful .They have the property that :

$$\sigma^i \sigma^j = i\epsilon^{ijk}\sigma^k \;\; i \neq j \tag{1.5.7}$$

which implies :

$$\{\sigma^i, \sigma^j\} = 0 \tag{1.5.8}$$

together with :

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.5.9}$$

the Pauli matrices form a basis for the 2 -dimensional matrices with the properties :

$$\sigma^\mu \sigma^\mu = \sigma^0 \tag{1.5.10}$$

(see appendix A3)

1.6 Brief Summary of Required Special Relativity

The principle idea of special relativity we wish to preserve is the concept of Lorentz invariance. This is the idea that there exists a transformation called a Lorentz transformation which when it is performed on a on certain quantities (Lorentz invariant) gives the quantity after transformation as before.

A very useful quantity in special relativity is the metric $g^{\mu\nu}$ which can be taken to be represented by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \quad \text{negative} \tag{1.6.1}$$

This can be used to switch between up and down notation.
For special relativity we define the contravariant four vector:

$$a^\mu = (a^0, \vec{a}) \tag{1.6.2}$$

where $\vec{a}$ is the normal three vector and the covariant four vector:

$$a_\mu = (a^0, -\vec{a}) \;\; a^\mu = g^{\mu\nu} a_\nu \tag{1.6.3}$$

and the square of the vector is defined as :

$$a^2 = a_\mu a^\mu = a^\mu a_\mu = (a^0)^2 - (a^1)^2 - (a_2)^2 - (a_3)^2 \qquad (1.6.4)$$

Some texts use the negative of the metric and so the four-vectors have time-components which change sign, but the vector remains the same sign.
For the position four-vector x^μ we define its Lorentz transformed quantity $x^{\mu\prime}$ by:

$$x^{\mu\prime} = \Lambda^{\mu\nu} x^\nu \qquad (1.6.5)$$

where :

$$\Lambda^{\mu\nu} = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \qquad (1.6.6)$$

is a Lorentz transformation. Similarly:

$$\begin{aligned} x_\mu{}' &= \Lambda_{\mu\nu} x_\nu && (1.6.7) \\ x^{\mu\prime} &= \Lambda^\mu{}_\nu x_\nu && (1.6.8) \\ x_\mu{}' &= \Lambda_\mu{}^\nu x^\nu && (1.6.9) \end{aligned}$$

where:

$$\begin{aligned} \Lambda_{\mu\nu} &= \frac{\partial x_\mu{}'}{\partial x_\nu} && (1.6.10) \\ \Lambda^\mu{}_\nu &= \frac{\partial x^{\mu\prime}}{\partial x_\nu} && (1.6.11) \\ \Lambda_\mu{}^\nu &= \frac{\partial x_\mu{}'}{\partial x^\nu} && (1.6.12) \end{aligned}$$

Where $\Lambda^\mu{}_\rho \Lambda_\mu{}^\nu = g^\nu_\rho = \delta^\nu_\rho$.
This means that Lorentz transforms can be considered as unitary transforms, we will get back to this later. This delta relation which follows from commutativity of partial derivatives means that all products of four vectors chosen like this are Lorentz invariant, i.e. any $a_\mu b^\mu$ for a^μ, b^μ four-vectors is Lorentz invariant.

A tensor of rank 2 is a quantity that transforms like:

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} \qquad (1.6.13)$$

where the Λ are Lorentz transformations as above. It can be seen that for two tensors like this $F_{\mu\nu} G^{\mu\nu}$ is Lorentz invariant.

We now define some useful four-vector quantities which are Lorentz invariant. The first is the position four-vector:

$$x^\mu = (ct, x, y, z) \qquad (1.6.14)$$

where t is the time co-ordinate, x,y,z are the spatial co-ordinates. c is the speed of light.
The momentum four -vector is defined as :

$$p^\mu = (\frac{E}{c}, \vec{p}) \tag{1.6.15}$$

where $E = mc^2\gamma, \vec{p} = \gamma m_0 \vec{v}$ where m_0 is the mass of the object measured in its rest frame $\vec{v}$ is its velocity and :

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.6.16}$$

We also define the derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.6.17}$$

and vice versa

$$\partial^\mu = \frac{\partial}{\partial x_\mu} \tag{1.6.18}$$

We also note that the normal vector should now be written $\vec{a} = (a^1, a^2, a^3)$ and that $a^0 = a_0$, $a_i = -a^i$ $i \neq 0$.

1.7 Brief Summary of Maxwell's Equation

We first state the normal differential form of Maxwell's equations (see p120 eqs 5.1 -5.4 [4]):

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \frac{\rho_0}{\epsilon_0} && (1.7.1) \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && (1.7.2) \\ \vec{\nabla}.\vec{B} &= 0 && (1.7.3) \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} && (1.7.4) \end{aligned}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field density , $\vec{J}$ is the current density and μ_0 and ϵ_0 are constants with SI values
$\epsilon_0 = 8.85 \times 10^{-12} Fm^{-1}, \mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ with relation $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$
These equation can be simplified using the vector $\vec{A}$ and potential ϕ where :

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.7.5}$$

and

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{1.7.6}$$

since $\vec{\nabla}.\vec{\nabla}\times\vec{C}=0$ and $\vec{\nabla}\times\vec{\nabla}a=0$ for all vectors $\vec{C}$ and potentials a (proof in appendix A4,A5)
$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$,$\vec{\nabla}.\vec{B}=0$ are automatically satisfied and thus maxwell's four equations reduce to two:

$$\vec{\nabla}^2\phi+\frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho_0}{\epsilon_0} \tag{1.7.7}$$

$$(\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2})-\vec{\nabla}(\nabla.\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}) = -\mu_0\vec{J} \tag{1.7.8}$$

using that $\vec{\nabla}\times\vec{\nabla}\vec{\times}A=\vec{\nabla}^2\vec{A}-\vec{\nabla}\vec{\nabla}.\vec{A}$ (see appendix A6,A7)

Obviously the choice of $\vec{A}$ and ϕ is not unique any $\vec{A}'$ that differs from $\vec{A}$ by an amount $\vec{D}$ such that $\vec{\nabla}\times\vec{D}=0$ will give the same result for $\vec{B}$ and any ϕ' that differs from ϕ by an amount d such that $\vec{\nabla}d=-\frac{\partial\vec{D}}{\partial t}$ gives the same result for $\vec{E}$ so $\vec{A}',\phi'$ can be used instead of $\vec{A},\phi$ providing d and $\vec{D}$ are chosen as described above. This is called choosing a gauge.

If we use the Lorenz gauge :

$$\vec{\nabla}\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0 \tag{1.7.9}$$

and define the four vector:

$$A^\mu=(\frac{\phi}{c^{\text{`}}},\vec{A}) \tag{1.7.10}$$

and the four current:

$$J^\mu=(\rho c,\vec{J}) \tag{1.7.11}$$

maxwell's equations become:

$$\Box A^\mu=\mu_0 J^\mu \tag{1.7.12}$$

(see appendix A8)

Suppose:

$$F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha \tag{1.7.13}$$

with $A_\alpha=(\frac{\phi}{c},-\vec{A})$

Then in matrix form (see appendix A9) :

$$F_{\alpha\beta}=\begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{1.7.14}$$

by definition :

$$F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} \tag{1.7.15}$$

So in matrix from (see appendix A10):

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{1.7.16}$$

We will generally be taking c as one but we leave in factors of c so we can compare to usual results. The choice in four-vectors and tensors may mean difference of a factors of c to other literature.
We can define a new quantity :

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{1.7.17}$$

which in matrix form becomes(see appendix A11 :

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix} \tag{1.7.18}$$

Thus the Maxwell's equations become (appendix A12):

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu_0 J^{\beta} \tag{1.7.19}$$

$$\frac{\partial \tilde{F}^{\alpha\beta}}{\partial x^{\alpha}} = 0 \tag{1.7.20}$$

We can show that the lagrangian density:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_{\alpha} J^{\alpha} \tag{1.7.21}$$

leads to the equation of motion (see appendix A13):

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu_0 J^{\beta} \tag{1.7.22}$$

In a vacuum with no sources i.e. $J^{\mu} = 0$ the lagrangian becomes :

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} \tag{1.7.23}$$

1.8 Dimensional Analysis

(This section follows Srednicki Chapter 12)
In the set of units where $\hbar$=c=1 time, inverse mass and length can be thought of as having the same units as for a given length L there exits a time T such that L=cT (since c is simply a velocity) and since $\hbar$ has units Joule seconds in SI (from E=$\hbar\omega$) and a joule is a kgm^2s^{-2} ($F = \frac{d\vec{p}}{dt}, \vec{p} = m\vec{v}, E = Fd$)
$\hbar$ has units kgm^2s^{-1} and thus $\hbar c^{-1}$ has units kgm thus for every inverse mass M^{-1} there exists a length such that $M^{-1} = c\hbar^{-1}L$.

This means that any quantity that could previously been given with units in terms of powers of length, time and mass can now been given in terms of a power of just one of these conventionally taken as mass. The quantity covered include all dynamic variables. The power in terms of mass of a quantity C will be denoted $[C]$. eg:

$$[m] = +1 \tag{1.8.1}$$

by definition (see p104 eq 12.1 [5]).

$$[x^\mu] = -1 \tag{1.8.2}$$

(see p104 eq 12.2 [5]) as the space parts of x^μ have dimension length and the time parts have dimension time.

$$[\partial^\mu] = +1 \tag{1.8.3}$$

(see p104 eq 12.3 [5]) the inverse of the previous as this differentiates a quantity with respect to x^μ and thus reduces the power in terms of x^μ by 1 and hence increase the power of mass by 1.

$$[d^dx] = -d \tag{1.8.4}$$

(see p104 eq 12. 4[5]) this is the inverse process of the previous one and thus has the inverse sign.

Knowing these and similar results we can derive the dimension in terms of mass for any new quantity such as the field. We can also invert the process so that if we know the dimension in terms of time, mass and length of a quantity and have computed it in this regime we substitute in c's and $\hbar$'s to restore the original dimension and thus we have the correct number of c's and $\hbar$'s.

1.9 Summary

In this section we have introduced the notation that we will be use and given a brief summary of the basic physics and mathematics that this report draws on. We have also introduced the topic and given a brief summary of the report and the motivations for it. Finally we have introduced dimensional analysis as a useful concept and a pseudo-justfication for taking $\hbar = c = 1$ as well as a process that we can invert to get quantities in terms of SI units.

Chapter 2

The Types of Field

2.1 The Free Harmonic Field

The simplest possible free theory involves a scalar field ϕ given a real value at every point in space-time i.e. :

$$\phi = \phi(t, \vec{x}) \tag{2.1.1}$$

We identify the Lagrangian L as the three-integral of the lagrangian-density $\mathcal{L}$ i.e. :

$$L = \int d^3x \mathcal{L} \tag{2.1.2}$$

and similarly identify the Hamiltonian H as the three-integral of the hamiltonian density$\mathcal{H}$ i.e. :

$$H = \int d^3x \mathcal{H} \tag{2.1.3}$$

The simplest useful lagrangian density is (p16 , eq 2.6 citeP+S):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2.1.4}$$

The similarity between this and the harmonic potential of classical physics is obvious. The use of this lagrangian is limited as it has no self-interaction terms for the field above harmonic order, so it does not simulate the interaction between particles, i.e. it models particles that hit each other but are not affected by their mutual interaction. However the great advantage of this model is it can be exactly solved and provides a simple example of the procedure that can then be extended to be more complicated lagrangian densities.

The following derivation is fairly standard and can be found for example in Section 2.2 of [7] (with the caveat that we use the mostly negative metric $g^{\mu\nu}$ rather then the mostly positive $\eta^{\nu\mu}$).

The action S is defined as the four integral of the Lagrangian density, i.e.:

$$S = \int d^4x \mathcal{L} \tag{2.1.5}$$

where $\int d^4x = \int dt \int d^3x$ or equivalently the time integral of the Lagrangian $S = \int dtL$.
The principle of least action suggests that particles follow paths in which the action is minimised i.e. the variational derivative of S is zero ($\delta S = 0$) . Thus we have the equation:

$$0 = \delta S = \int d^4x [\frac{\delta \mathcal{L}(\mathbf{x})}{\delta \phi(\mathbf{y})} \delta \phi(\mathbf{y})] \tag{2.1.6}$$

Using the definition of variational derivatives above (Section 1.2 (1.2.5)):

$$\delta S = \int d^4x [\frac{\partial \delta^4(x-y)}{\delta x^\nu} \frac{\partial \phi(x)}{\partial x^\mu} g^{\nu\mu} - m^2 \phi(x) \delta^4(x-y)] \delta \phi(y) \tag{2.1.7}$$

where the halves have disappeared as the differentiation is applied twice to the same term to give double the outcome and $g^{\mu\nu}$ comes from switching ∂_μ to ∂^ν integration by parts on the first term then gives:

$$\delta S = -\int d^4x [\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x)] \delta^4(x-y) \delta \phi(y) \tag{2.1.8}$$

if we assume the derivative term:

$$\delta^4(x-y) \frac{\partial \phi(x)}{\partial x^\mu} g^{\nu\mu} \tag{2.1.9}$$

vanishes when evaluated at the boundaries. Thus :

$$\delta S = -[\partial_\mu \partial^\mu \phi(y) + m^2 \phi(y)] \delta \phi(\mathbf{y}) = 0 \tag{2.1.10}$$

since $\delta\phi(y)$ is arbitrary. We have quite simply the Klein-Gordan equation(p17 , eq 2.7 [9]):

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{2.1.11}$$

Analogously to classical physics we define conjugate momentum :

$$\pi(\mathbf{x}) = \frac{\delta S}{\delta \dot{\phi}(x)} \tag{2.1.12}$$

and the hamiltonian,"the legendre transformation of the lagrangian"',(p12 l23 [7]) as (p16 , eq 2.5 [9]):

$$H = \int d^4x (\pi(x) \dot{\phi}(x) - \mathcal{L}(x)) \tag{2.1.13}$$

We defer further discussion of ϕ and creation and annihilation operators to the next section as they are a special case of the complex system which is more illuminating and useful later.

2.2 The Complex Scalar Field

The complex scalar field is the complex analogue of the real scalar field i.e. it assigns a complex number to every point in space-tme. As might be expected the complex field allows for two different types of particle, it also allows us to introduce the concept of current. The analogous lagrangian is(p90 eq 3.51 [6]):

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2} m^2 \mid \phi \mid^2 \tag{2.2.1}$$

for which the equation of motion is again the klein-Gordan equation (see appendix B1) (2.1.11):

$$\partial_\mu \partial^\mu \phi + m^2 \phi \tag{2.2.2}$$

the conjugate momentum:

$$\pi(\mathbf{x}) = \frac{\delta S}{\delta \dot{\phi}} = \dot{\phi}^*(\mathbf{x}) \tag{2.2.3}$$

Since the plane waves:

$$e^{i\vec{k}.\vec{x} \pm i\omega_k t} \tag{2.2.4}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$ (seen by substitution into the equation) are solutions of the Klein-Gordan equation. Using the analogue of fourier transforms we can postulate that the field can be written as (p20 eq3.26 [7]):

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} (a(\vec{k}) e^{i\vec{k}.\vec{x} - i\omega_k t} + \tilde{b}^*(\vec{k}) e^{i\vec{k}.\vec{x} + i\omega_k t}) \tag{2.2.5}$$

where $\frac{1}{(2\pi)^3\sqrt{2\omega_k}}$ is a normalisation constant and $a(\vec{k})$ and $\tilde{b}^*(\vec{k})$ are arbitrary coefficients.
Separating out the $\tilde{b}^*(\vec{k})$ integral taking $\tilde{b}^*(\vec{k}) = -b^*(-\vec{k})$ and swapping $\vec{k}$ for $-\vec{k}$ gives (p20 eq3.27 a [7]):

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} (a(\vec{k}) e^{i\vec{k}.\vec{x} - i\omega_k t} + b^*(\vec{k}) e^{-i\vec{k}.\vec{x} + i\omega_k t}) \tag{2.2.6}$$

We then derive:

$$\phi^*(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} (a^*(\vec{k}) e^{-i\vec{k}.\vec{x} + i\omega_k t} + b(\vec{k}) e^{i\vec{k}.\vec{x} - i\omega_k t}) \tag{2.2.7}$$

thus for a real scalar field where $\phi = \phi^*$ we get by comparison of the coefficients of the exponentials $a(\vec{k}) = b(\vec{k})$. Then the conjugate momentum and its transpose are:

$$\pi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} (i)(a^*(\vec{k}) e^{-i\vec{k}.\vec{x} + i\omega_k t} - b(\vec{k}) e^{i\vec{k}.\vec{x} - i\omega_k t}) \tag{2.2.8}$$

$$\pi^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} (-i)(a(\vec{k}) e^{i\vec{k}.\vec{x} - i\omega_k t} - b^*(\vec{k}) e^{-i\vec{k}.\vec{x} + i\omega_k t}) \tag{2.2.9}$$

We can thus derive $a(\vec{k})$ and $b(\vec{k})$ (see appendix B2) and get :

$$a(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\phi(\mathbf{x}) + \frac{i}{\sqrt{2\omega_k}}\pi^*(\mathbf{x}))e^{-i\vec{k}.\vec{x}+i\omega_k t} \tag{2.2.10}$$

$$b(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\phi^*(\mathbf{x}) + \frac{i}{\sqrt{2\omega_k}}\pi(\mathbf{x}))e^{-i\vec{k}.\vec{x}+i\omega_k t} \tag{2.2.11}$$

As the Poisson bracket of position and conjugate momentum is 1. We postulate that:

$$[\hat{\phi}(t,\vec{x}), \pi(t',\vec{x}')]_{t=t'} = i\hbar\delta^3(\vec{x}-\vec{y}) \tag{2.2.12}$$

Where we have promoted our fields to field operators and used Dirac's prescription that we should then multiply their commutator by $i\hbar$, this is an example of canonical quantisation which we will come back to. Previously all our results would have been true of classical scalar fields. Using this we can then derive (see appendix B3) :

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}') \tag{2.2.13}$$

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{b}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{2.2.14}$$

$$[\hat{a}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{2.2.15}$$

We can show that the hamiltonian becomes (p21 eq3.32 [7]) (see appendix B4).:

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\omega_k[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) + const] \tag{2.2.16}$$

This constant integrates out to an infinite number which can be ignored by choosing our measuring system so that this constant is zero by making measurements relative to this constant.
One can see that this can be separated into two Hamiltonians which depend on $\hat{a}$ and $\hat{b}$ of the same form as the one acquired if ϕ was a scalar field hence $\hat{a}$ and $\hat{b}$ can be considered two represent two different types of particles.
It will be shown later that these two particles are anti-particles i.e. they have the same energy but opposite charge.

2.3 Majorana Fields and Dirac Spinors

(This chapter follows from Chapter 3 Peskin and Schroeder)
First we need to define the four gamma matrices (given in the Weyl representation) these are four dimensional matrices given in terms of the Pauli spin matrices by (p41 , eq 3.25 [9]) :

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \tag{2.3.1}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} i = 1, 2, 3 \quad (2.3.2)$$

these have the property that:

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij} \quad (2.3.3)$$

These matrices satisfy the Dirac algebra (p40 , eq 3.22 [9]) for 4 dimensions:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \hat{I}_{nxn} \quad (2.3.4)$$

It can be shown that :

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad (2.3.5)$$

satisfies the Lorentz algebra (see appendix B6) and thus boosts can be represented by S^{0i} and rotations by S^{ij} .
One can show that:

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_\nu\gamma^\nu \quad (2.3.6)$$

p41 ,eq 3.29 [9] (see appendix B7)
where (p42 eq3.30 [9]):

$$\Lambda_{\frac{1}{2}} = exp(\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}) \quad (2.3.7)$$

We can now show that the Dirac equation (p42 eq 3.31 [9]):

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad (2.3.8)$$

is Lorentz invariant where $\psi(x)$ is a Dirac spinor, a four-component field that transforms under boost according to S^{ij} and under rotations according to S^{0i}p41 , l25-26 [9] , i.e it has Lorentz transformation:

$$\psi(x) \rightarrow \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \quad (2.3.9)$$

We can show the Klein-Gordon equation (2.1.11) $(\partial^2 + m^2)\psi = 0$ follows from the Dirac equation (see appendix B8) .
We now want to define a lagrangian that is Lorentz invariant . It can be shown that :

$$\overline{\psi} = \psi^\dagger\gamma^0 \quad (2.3.10)$$

Lorentz transforms to (see appendix B9) :

$$\overline{\psi}\Lambda_{\frac{1}{2}}^{-1} \quad (2.3.11)$$

this implies that (p43 eq 3.34 [9]):

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \quad (2.3.12)$$

is Lorentz invariant (see appendix B10) and we can show that the Euler -Lagrange equation for $\overline{\psi}$ implies the Dirac equation (see appendix B11).

We consider some of the forms of the solution for the free theory.
As ψ satisfies the klein-gordon equation we know that ψ can be written as a linear combination of plane waves $u(\mathbf{p})e^{-i\mathbf{p}.\mathbf{x}}$ and $v(\mathbf{p})e^{i\mathbf{p}.\mathbf{x}}$ where $p^2 = m^2 \quad p^0 > 0$ and thus one can write the field as (p54 eq 3.92 [9]):

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} (a^s_{\vec{p}} u^s(\mathbf{p}) e^{-i\mathbf{p}.\mathbf{x}} + b^{s\dagger}_{\vec{p}} v^s(\mathbf{p}) e^{i\mathbf{p}.\mathbf{x}}) \tag{2.3.13}$$

s are the spin states.

The rotation and boosts generators $S^{\mu\nu}$ can be shown to be in block diagonal form (Weyl representation) (see appendix B12) and thus we can write the Dirac spinor as(p43 eq 3.36 [9]):

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.3.14}$$

where each block of the matrices acts on ψ_L or ψ_R where ψ_L and ψ_R are the left and right-handed Weyl spinors,a two component spinor that transforms with the pauli matrices under rotations.

Defining (p44 eq 3.41 [9]) :

$$\sigma^\mu \equiv (\hat{I}, \vec{\sigma}) \; \overline{\sigma}^\mu \equiv (\hat{I}, -\vec{\sigma}) \tag{2.3.15}$$

one can show (see for example section 3.3 [9]):

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{\mathbf{p}.\sigma}\xi \\ \sqrt{\mathbf{p}.\sigma}\xi \end{pmatrix} \tag{2.3.16}$$

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{\mathbf{p}.\sigma}\eta \\ -\sqrt{\mathbf{p}.\sigma}\eta \end{pmatrix} \tag{2.3.17}$$

where ξ is a weyl spinor. We normalise u and v with (see appendix B9):

$$\begin{aligned} \overline{u}^r(\mathbf{p})u^s(\mathbf{p}) &= 2m\delta^{rs} \text{ or } u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) = 2E_{\vec{p}}\delta^{rs} && (2.3.18) \\ \overline{v}^r(\mathbf{p})v^s(\mathbf{p}) &= -2m\delta^{rs} \text{ or } v^{r\dagger}(\mathbf{p})v^s(\mathbf{p}) = 2E_{\vec{p}}\delta^{rs} && (2.3.19) \end{aligned}$$

and

$$\begin{aligned} \overline{u}^r(\mathbf{p})v^s(\mathbf{p}) &= \overline{v}^r(\mathbf{p})u^s(\mathbf{p}) = 0 && (2.3.20) \\ u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) &= v^{r\dagger}(-\mathbf{p})v^s(\mathbf{p}) = 0 && (2.3.21) \end{aligned}$$

The spin sums give (see appendix B13 for derivation) :

$$\sum_s u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) = \gamma.\mathbf{p} + m \tag{2.3.22}$$

and

$$\sum_s v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) = \gamma.\mathbf{p} - m \tag{2.3.23}$$

Defining:

$$\gamma^\mu p_\mu = \not{p} \tag{2.3.24}$$

and (p50 eq 3.68 [9]):

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.3.25}$$

we can show that in the weyl representation (Appendix B15):

$$\gamma^5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.26}$$

We find our conjugate momentum to ψ is:

$$i\psi^\dagger \tag{2.3.27}$$

The Hamiltonian of the Dirac field can then be shown to be (Appendix B15) :

$$\int d^3x\overline{\psi}(-i\vec{\gamma}.\vec{\nabla} + m)\psi \tag{2.3.28}$$

We now introduce the Majorana field defined as (p101 l15 [8]):

$$\psi_c = C\gamma^0\psi^* \tag{2.3.29}$$

where C is the charge conjugation operator(p70 eq 3.143 [9]):

$$C{a^s}_pC = {b^s}_p \qquad C{b^s}_pC = {a^s}_p \tag{2.3.30}$$

it can be shown that (see appendix B16):

$$\psi_c = \gamma^2\psi^* \tag{2.3.31}$$

It can also be shown that the Dirac equation implies the Majorana equation (Appendix B17):

$$i\not{\partial}\psi = m\psi_c \tag{2.3.32}$$

This type of field can be used in the description of neutrinos but will not be of great use to us.

2.4 Vector Fields

We now look at the concept of vector fields. Above we introduced an example of this for electro-magnetism A^μ and its associated Lagrangian:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_\alpha J^\alpha \tag{2.4.1}$$

Using the source-less case (J=0).
We rewrite $\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$ as :

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\mu_0}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= -\frac{1}{4\mu_0}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu + \partial^\nu A^\mu \partial_\nu A_\mu) \\
&= \frac{-1}{2\mu_0}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\nu A_\mu)
\end{aligned} \tag{2.4.2}$$

by changing the order of variables on two of the terms.

We seperate out the time and space co-ordinates to get :

$$\begin{aligned}
\mathcal{L} &= \frac{-1}{2\mu_0}(\partial^0 A^i \partial_0 A_i - \partial^0 A^i \partial_i A_0 + \partial^i A_0 \partial_i A_0 - \partial^i A^0 \partial_0 A_i + \partial^j A^i \partial_j A_i - \partial^j A^i \partial_i A_j \\
&= \frac{-1}{4\mu_0}(\dot{A}_i \dot{A}_i - \nabla_i A_j \nabla_i A_j + \nabla_i A_j \nabla_j A_i - 2\dot{A}_i \nabla_i A_0 + \nabla_i A_0 \nabla_i A_0)
\end{aligned} \tag{2.4.3}$$

Considering :

$$\nabla_i A_j \nabla_j A_i - 2\dot{A}_i \nabla_i A_0 \tag{2.4.4}$$

We can integrate by parts to get

$$A_j \nabla_i \nabla_j A_i - 2\nabla_i \dot{A}_i A_0 \tag{2.4.5}$$

assuming the field components vanish at the boundaries. By choosing the gauge :

$$\nabla_i A_i = 0 \tag{2.4.6}$$

This expression vanishes and we get :

$$\mathcal{L} = \frac{-1}{4\mu_0}(\dot{A}_i \dot{A}_i - \nabla_i A_j \nabla_i A_j + \nabla_i A_0 \nabla_i A_0) \tag{2.4.7}$$

which has no propagation term for the A_0 components and is essentially the scalar lagrangian upto a constant with mass equal to zero for the other $A_i's$ (If any mass terms did exist they would by analogue with the scalar field of the form $\frac{1}{2}m^2 A_i^2$ since scalar fields have a mass term $-\frac{1}{2}m^2\phi^2$ (2.2.1) and there is arelative minus sign between the propagation terms of the two lagrangians)

Using an analogue with our scalar case (section 2.1 and 2.2) we can rewrite $A_\mu(x)$ as (p336 eq 55.1 [5]) :

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \sum_{r=0}^{3} a^r_{\vec{p}} \epsilon^r_\mu(p) e^{-ip.x} + a^{r\dagger}_{\vec{p}} \epsilon^{r*}_\mu(p) e^{ip.x} \tag{2.4.8}$$

where $\epsilon^r_\mu(p)$ are the polarisation 4-vectors and r labels the basis of these vectors and thus there are as many as there are space-dimensions. We define $\epsilon^\lambda_0(p) = 0$ So we can safely ignore the A^0 terms.

We can in analogue to the scalar case define creation and annihilation operators for each A_i and for each A_i we find the canonically conjugate momentum :

$$\Pi_i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = \dot{A}_i \tag{2.4.9}$$

As demonstrated above $A_\mu(x)$ is gauge invariant a property that will be important for vector fields as we will see below.

Bosonic particles (particles with integer spin) can be represented by vector fields. The carriers of each fundamental force are represented by gauge bosons, here we only consider electro-week interactions and ignore the gluons associated with the strong force.

The two types of vector fields we are interested in are those associated with electromagnetism and those associated with weak interactions. $A_\mu(x)$ is associated with the U(1) symmetry group and the three fields associated with the SU(2) symmetry group $B^a_\mu(x)$. The group SU(3) is the extra group we need for strong force interactions but we will not consider it here.

The number of vector fields required for each symmetry group corresponds to the number of infinitesimal generators required for that group, we will enlarge on this later. As discussed above the U(1) symmetry group has one generator and the SU(2) symmetry group which has 3 generators (see also above) and thus the SU(2) group has three vector fields. The SU(3) group has 8 generators corresponding to the eight vector fields used to describe the gluons but we will not discuss that further here.

For the lagrangian terms of the form $m^2 A^2_\mu(x)$ correspond to mass terms (kinetic energy like terms) the extra minus sign that we would see by comparison to the normal scalar field comes from the relative minus sign between spatial and time co-ordinates, since normal mass terms come from the spatial co-ordinate.

2.5 Summary

In this section we have introduced the field types we will need to consider the Higgs mechanism. We have tried to give for each type the free-field lagrangian and free-field solutions. We will consider these fields again when we consider the interacting case.

Chapter 3

Different Formulations of Quantum Field Theory

3.1 Canonical Quantisation

Although we have already used some of the concepts of canonical quantisation, however we now give a more detailed summary to ensure clarity. The basic idea behind canonical quantisation is to use a set of relations relating classical and quantum representations of the same quantity to derive quantum field theory results from known classical results.

We define the canonically conjugate momentum to a field $\phi(x)$ by:

$$\pi(x) = \frac{\delta S}{\delta \dot{\phi}} \tag{3.1.1}$$

by analogue with the classical case

For this process we use the Poisson bracket as defined above with p and q momentum and position respectively. Dirac postulated that given these classical relations we can replace the poison bracket of two quantities with the commutator (for bosons) or anti-commutator (for fermions) of their operators at equal times and multiply by $i\hbar$.

Defining the position and conjugate momentum fields in this way allows us to derive the relation between creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators for fields. These commutation relations between creation and annihilation operators make it much easier to commute vacuum expectaton values a property which will become important later. An example of this with the scalar fields was given earlier.

Care is needed to apply canonical quantisation in the case of the Dirac spinor and Majorana fields. Using the appropriate anti-commutator relation (as we are representing fermions) :

$$\{\psi_a(\vec{x},t), \pi^c(\vec{y},t)\} = i\delta_a^c \delta^3(\vec{x}-\vec{y}) \tag{3.1.2}$$

ignoring factors of $\hbar$.
The conjugate momentum of the dirac field is:

$$i\psi^\dagger \tag{3.1.3}$$

This leads to for the Dirac field (p237 eq 37.14 [5]):

$$\{\Psi_\alpha(\vec{x},t), \overline{\Psi}_\beta(\vec{y},t)\} = (\gamma^0)_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{3.1.4}$$

and for the majorana field (p237 eq 37.21 [5]):

$$\{\Psi_\alpha(\vec{x},t), \overline{\Psi}_\beta(\vec{y},t)\} = (\gamma^0)_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{3.1.5}$$

by using the definition of the conjugate momentum. We also gain for the majorana fields (p237 eq 37.22 [5]) (see appendix C1):

$$\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} = (C\gamma^0)_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{3.1.6}$$

3.2 Path Integral Formalisation

Another way of describing quantum mechanics is to use the path integral formalism. Although this is theoretically equivalent to the formalism found by using canonical quantisation (see p275 l5-7 [9]) there are many results that are easier to derive in one formalism then another, thus for the sake of convenience we here introduce this second formalism.

The idea behind the path integral is that a particle traveling between two points can be considered to travel along any path between the two points with a given probability for each path .Hence the amplitude to propagate between these two points is the sum over all these paths.

The propagator from vacuum to vacuum can be considered as equivalent to a given path integral. By analogue with the discrete case where we would sum over paths, we integrate over all fields (only scalar fields considered) i.e.:

$$\int D\phi e^{\frac{i}{\hbar}S[\phi]-\frac{\epsilon}{\hbar}\int d^4x\phi(x)^2} \tag{3.2.1}$$

the ϵ term means that if ϕ does not go to zero at the boundaries the integral is still suppressed and thus we can calculate it.

To justify the exponential dependence on S we first discuss the classical case for a discrete number of paths the above amplitude is given by :

$$\sum_{\text{all paths} X(\tau)} e^{i.\theta[X(\tau)]} \tag{3.2.2}$$

where θ is the phase for the path $X(\tau)$ this expression means that we end up assigning an amplitude square of $\mid e^{i\phi(\tau)} \mid^2$ to each path which can be considered as a probability as it is always between one and zero and is chosen for each path . Generally the larger contributions to this sum come from the stationary phase, in the classical limit only one path will contribute, which will thus be stationary hence (p30 eq 4.5 [7]):

$$\frac{\delta}{\delta X(\tau)}(\theta[X(\tau)]) = 0 \tag{3.2.3}$$

The classical path is also described by the principle of least action hence:

$$\frac{\delta}{\delta X(\tau)}(S[X(\tau)]) = 0 \tag{3.2.4}$$

for a path dependent action.
We know that classical physics is the $\hbar \to 0$ limit of quantum physics. Thus by identifying:

$$\theta[X(\tau)] = \frac{1}{\hbar}S[X(\tau)] \tag{3.2.5}$$

we get an agreement that becomes better and better as we reach the classical limit. Hence the exponent in the sum becomes:

$$e^{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f} Ld\tau} \tag{3.2.6}$$

where L is the lagrangian and $X(\tau_i)$ is the start of the path and $X(\tau_f)$ is the end of the path and τ runs along the path.

Ignoring the ϵ term we can add in a source term that allows us to write terms of the form:

$$\int D\phi\phi(x_1)\phi(x_2)e^{\frac{i}{\hbar}S[\phi]} \tag{3.2.7}$$

in terms of functional derivatives of the path integral (the importance of these terms will become clear later) .
The expression we want is then:

$$Z[J] = \int D\phi e^{\frac{i}{\hbar}S[\phi] + \int d^4\phi(x)J(x)} \tag{3.2.8}$$

thus $Z[J = 0] = \langle 0 \mid 0 \rangle$ vacuum to vacuum propagator.
Hence for example:

$$\int D\phi\phi(x_1)\phi(x_2)e^{\frac{i}{\hbar}S[\phi]} = \frac{1}{Z[J]}(\frac{\delta}{\delta J(x_1)})(\frac{\delta}{\delta J(x_2)})Z[J] \mid_{J=0} \tag{3.2.9}$$

For the free scalar field we define:

$$Z_0[J] = \int D\phi\phi(x_1)\phi(x_2)e^{\frac{i}{\hbar}\int d^4x\frac{1}{2}\phi(\partial^2-m^2+i\epsilon)\phi+\frac{\hbar}{i}J\phi} \tag{3.2.10}$$

restoring the ϵ and the Klein-Gordan equation [?].
We identify $G_F(x-y)$ as the green function of the Klein-Gordan equation i.e. :

$$(\partial^2 - m^2 + i\epsilon)G_F(x-y) = \frac{\hbar}{i}\delta^4(x-y) \tag{3.2.11}$$

and using the parameter shift :

$$\phi(x) = \tilde{\phi}(x) + \int d^4yG_F(x-y)J(y) \tag{3.2.12}$$

we obtain:

$$Z_0[J] = \int D\tilde{\phi}e^{\frac{i}{\hbar}\int d^4x[\frac{1}{2}\tilde{\phi}(\partial^2-m^2+i\epsilon)\tilde{\phi}]+\frac{1}{2}\int d^4xd^4yJ(x)G_F(x-y)J(y)} \tag{3.2.13}$$

which simplifies to:

$$Z_0[J] = Z_0[J=0] \times e^{\frac{1}{2}\int d^4xd^4yJ(x)G_F(x-y)J(y)} \tag{3.2.14}$$

using the definition of $Z_0[J=0]$. The use of this form will become clear later , we are also able to remove the ϕ dependence by division by $Z_0[J=0]$

3.3 Summary

In this section we have clarified the techniques of canonical quantisation and path integrals. These are two useful formalism for describing quantum field theory and we will make use of both forms later, primarily following the technique of canonical quantisation

Chapter 4

Interacting Theory

4.1 Perturbation Theory

(This chapter follows from Chapter 4 of Peskin and Schroeder)
We have introduced the basic fields and some of the Lagrangians. We will now add in self-interacting terms, we want the action to be dimensionless (a definite requirement of the path integral formalism since the action appears in the exponent) using our discussion on the $d^d x$ we know its dimension is -4 so our lagrangian density has dimension 4. We know that we have the correct dimension lagrangian density for the scalar fields, the spinor fields and the vector fields. The $m^2\phi^2$ has dimension 4 so ϕ has dimension 1 as we are working in dimensions of mass. The dirac lagrangian has a term $\overline{\psi}m\psi$ so $\overline{\psi}\psi$ has dimension $\frac{3}{2}$. The vector lagrangain has a term like $\mid \partial_\mu A^\mu \mid^2$ since ∂_μ (see above) has dimension 1 this means A^μ has dimension 1 . We will discuss renormalisation later for now we will just state that our constant must have positive dimension. This means there are only a finite number of interaction terms we need to consider for the three types of field. Clearly we can't have an self- interaction of the dirac spinors higher then $\psi\overline{\psi}$ as this already dimension 3 and adding another ψ makes it dimension 4.5 which is not allowed. Since the differential ψ term has order dimension $\frac{5}{2}$ it can't interact with terms of higher dimension then a spinor. The scalar and vector derivative terms have order 2 so we can have at most two of them interacting together, the scalars and vectors themselves have dimension one so we can have at most four of them interacting together. We assume only positive integer powers of the fields are allowed (by analogue with Taylor expansions) and we can recombine any lower order terms into our already existing quadratic terms for dirac spinors and scalar fields up to a constant , i.e. we can remove linear terms for these fields by changing the variables to absorb them into the quadratic term (compare the technique of completing the square). The condition of Lorentz invariance allows us to place even more restrictions on the allowable interactions.

Despite this massive simplification there are still very few lagrangians that we can solve exactly, we therefore consider lagrangians with small coupling constants which we can then expand around the free solution that we know exactly (small perturbations hence perturbation theory).We can thus assume that:

$$H = H_0 + H_{int} \tag{4.1.1}$$

where H_0 is the non-interacting hamiltonian and H_{int} is the interacting hamiltonian

4.2 Correlators

To demonstrate the idea behind perturbation theory we will consider the two-point correlator for the scalar field, the analysis will follow similarly for other fields. The use of the two-point correlator in measurable quantities will discussed later. The two-point correlator for the scalar field is given by:

$$\langle \Omega \mid T\phi(x)\phi(y) \mid \Omega \rangle \tag{4.2.1}$$

where $\langle \Omega \mid$ represents the ground state of the theory in the case of interactions (the perturbed vacuum).
Using the definition of the Heisenberg field we obtain (p30 eq 4.8 [7]):

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{iH(t-t_0)} \tag{4.2.2}$$

we will use the free field:

$$\phi_0(t, \vec{x}) = e^{iH_0(t-t_0)}\phi(t_0, \vec{x})e^{i(H_0(t-t_0)} \tag{4.2.3}$$

and then find that:

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_0(t, \vec{x})e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.4}$$

we define :

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.5}$$

We define:

$$H_I(t) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)} \tag{4.2.6}$$

Thus :

$$i\frac{\partial}{\partial t}U(t, t_0) = e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)} = H_I(t, t_0)U(t, t_0) \tag{4.2.7}$$

by inserting the identity :

$$I = e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)} \tag{4.2.8}$$

We thus postulate a relationship that can be shown retrospectively:

$$U(t,t_0) = T(exp[-i\int_{t_0}^{t} dt' H_I(t')]) \tag{4.2.9}$$

and thus generally:

$$U(t,t') = T(exp[-i\int_{t'}^{t} dt'' H_I(t'')]) \tag{4.2.10}$$

We now consider the state $\langle \Omega \mid$ by assuming $\langle 0 \mid \Omega \rangle \neq 0$:

$$e^{-iHT} \mid 0\rangle = e^{-iE_0T} \mid \Omega\rangle\langle\Omega \mid 0\rangle + \sum_{n\neq 0} e^{-iE_nT} \mid n\rangle\langle n \mid 0\rangle \tag{4.2.11}$$

where $E_0 = \langle\Omega \mid H \mid \Omega\rangle$ ($H_0 \mid 0\rangle = 0$)
by expanding in the basis of the eigenstates of E ($\mid n\rangle$) .
As $E_n > E_0$ if we let T tend to infinity with a slight imaginary component i.e $T \to \infty(1 - i\epsilon)$ then the e^{-iE_0T} dominates as the other terms die faster. Thus we get a limiting expresion for $\mid \Omega\rangle$ i.e.:

$$\mid \Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} (e^{-iE_0T}\langle\Omega \mid 0\rangle)^{-1} e^{-iHT} \mid 0\rangle \tag{4.2.12}$$

Since T is close to infinity we can alter the exponentials by a small amount t_0 , also since:

$$H_0 \mid 0\rangle = 0 \tag{4.2.13}$$

which implies using the taylor expansion of the exponential :

$$e^{aH_0} \mid 0\rangle = \mid 0\rangle \tag{4.2.14}$$

inserting this into our expression and using the definition of U above gives:

$$\mid \Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} (e^{-iE_0(t_0-(-T)}\langle\Omega \mid 0\rangle)^{-1} U(t_0,-T) \mid 0\rangle \tag{4.2.15}$$

similarly (p87 eq 4.29 [9]):

$$\langle\Omega \mid = \lim_{T\to\infty(1-i\epsilon)} \langle 0 \mid U(T,t_0)(e^{-iE_0(t_0-(-T)}\langle 0 \mid \Omega\rangle)^{-1} \tag{4.2.16}$$

We can now find an overall expression for the two-point correlator, by assuming $x^0 > y^0 > t_0$ we get we find using our definitions of $\langle\Omega \mid$ and $\phi_0(x)$ we find:

$$\langle\Omega \mid \phi(x)\phi(y) \mid \Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} (e^{-iE_0(T-t_0)}\langle 0 \mid \Omega\rangle)^{-1}\times$$

$$\langle 0 \mid U(T,t_0)U(x^0,t_0)^\dagger\phi_0(x)U(x^0,t_0)U(y^0,t_0)^\dagger\phi_0(y)U(y^0,t_0)U(t_0,-T) \mid 0\rangle$$

$$\times(e^{-iE_0(t_0-(-T))}\langle\Omega \mid 0\rangle)^{-1} \tag{4.2.17}$$

which by rearrangement and the relations (see Appendix D1):

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) \tag{4.2.18}$$

and

$$U(t_1, t_3)U(t_2, t_3)^\dagger = U(t_1, t_3) \tag{4.2.19}$$

simplifies to:

$$\lim_{T\to\infty(1-i\epsilon)}(e^{-i2E_0T} \mid \langle 0 \mid \Omega\rangle \mid^2)^{-1}\langle 0 \mid U(T, x^0)\phi_0(x)U(x^0, y^0)\phi_0(y)U(y^0, -T) \mid 0\rangle$$

dividing through by:

$$\langle\Omega \mid \Omega\rangle =$$

$$\lim_{T\to\infty(1-i\epsilon)}(e^{-i2E_0T} \mid \langle 0 \mid \Omega\rangle \mid^2)^{-1}\langle 0 \mid U(T, t_0)U(t_0, -T) \mid 0\rangle \tag{4.2.20}$$

which is equal to one.
Thus:

$$\langle\Omega \mid \phi(x)\phi(y) \mid \Omega\rangle =$$

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 \mid U(T, x^0)\phi_0(x)U(x^0, y^0)\phi_0(y)U(y^0, -T) \mid 0\rangle}{\langle 0 \mid U(T, -T) \mid 0\rangle} \tag{4.2.21}$$

This is a time ordered expression and so by using our exponential formula for U

$$\langle 0 \mid T(\phi(x)\phi(y)) \mid \Omega\rangle =$$

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 \mid T(\phi_0(x)\phi_0(y)exp[-i\int_{-T}^{T} dtH_I(t)]) \mid 0\rangle}{\langle 0 \mid T(exp[-i\int_{-T}^{T} dtH_I(t)]) \mid 0\rangle} \tag{4.2.22}$$

The derivation has not relied on the number of fields or any property of the fields expect the relation between their Schrödinger and Heisenberg expressions, we can insert any number of fields on the left and gain the same number of free fields ϕ_0's in the numerator on the right . The specific advantage of this function is if H_I is small i.e it depends on a coupling constant that is small we can taylor expand the exponential in terms of this coupling constant to give an expression for the two-point correlator in the perturbed theory in terms of the two-point correlator for the free field going to arbitrary accuracy by taking more terms of the expansion.

Providing we have the correct n-point correlator on the left for any other type of field we expect we can use this expression to give a similar expression on the right

4.3 Contractions

The advantage of the free field two-point correlator is it can be expressed in a relatively simple way. We split our free field into an annihilation and creation part i.e.:

$$\phi_0(x) = \phi_0^{cr}(x) + \phi_0^{an}(x) \tag{4.3.1}$$

where:

$$\langle 0 \mid \phi_0^{cr} = 0, \phi_0^{an} \mid 0 \rangle = 0 \tag{4.3.2}$$

This means:

$$\begin{aligned} \phi_0(x)\phi_0(y) &= \phi^{cr}(x)\phi^{cr}(y) + \phi^{cr}(x)\phi^{an}(y) + \phi^{an}(x)\phi^{cr}(y) \\ &+ \phi^{an}(x)\phi^{an}(y) \end{aligned} \tag{4.3.3}$$

We want normal ordering where creation operators are on the left of annihilation operators :

$$\begin{aligned} \phi_0(x)\phi_0(y) &= \phi^{cr}(x)\phi^{cr}(y) + \phi^{cr}(y)\phi^{an}(x) + \phi^{cr}(x)\phi^{an}(y) \\ &+ \phi^{an}(x)\phi^{an}(y) + [\phi_0^{an}(x), \phi_0^{cr}(y)] \end{aligned} \tag{4.3.4}$$

This is :

$$N(\phi_0(x)\phi_0(y)) + [\phi_0^{an}(x), \phi_0^{cr}(y)] \tag{4.3.5}$$

where N is normal ordering, since the first four terms before the commutator are the same if y and x are exchange, the time-ordered form of these terms is the same for these terms. We define the contraction of ϕ as the time ordered form of the commutator of the creation and annihilation fields als equivalent to the Feynman propagator p33 eq 4.24 [7] :

$$D_F(x-y) = \overbracket{\phi_0(x)\phi_0(y)} = \left\{ \begin{array}{l} [\phi_0^{an}(x), \phi_0^{cr}(y)] \ x_0 > y_0 \\ [\phi_0^{an}(y), \phi_0^{cr}(x)] \ x_0 < y_0 \end{array} \right. \tag{4.3.6}$$

We let:

$$N(\overbracket{\phi_0(x)\phi_0(y)}) = \overbracket{\phi_0(x)\phi_0(y)} \tag{4.3.7}$$

recalling the commutator relation between annihilation and creation operators in the scalar field. We find that :

$$T(\phi_0(x)\phi_0(y)) = N(\phi_0(x)\phi_0(y) + \overbracket{\phi_0(x)\phi_0(y)}) \tag{4.3.8}$$

since N puts annihilation operators on the right and creation operators on the left for n-fields :

$$\langle 0 \mid N(\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)) \mid 0 \rangle = 0 \tag{4.3.9}$$

and thus :

$$\langle 0 \mid T(\phi_0(x)\phi_0(y)) \mid 0\rangle = \langle 0 \mid \wick{\c{\phi}_0(x)\c{\phi}_0(y)}) \mid 0\rangle \tag{4.3.10}$$

we want to extend this concept to correlators of n-fields. Wick's theorem (proved in the Appendix D2) generalises this so that :

$$T(\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)) = N(\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)+\text{all possible contractions}) \tag{4.3.11}$$

to demonstrate this we will do the time-ordered product of four-fields i.e. :

$$\begin{aligned}
T \quad & (\phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\phi_0(x_4)) = N(\phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\phi_0(x_4) \\
+ \quad & \wick{\c{\phi}_0(x_1)\c{\phi}_0(x_2)\phi_0(x_3)\phi_0(x_4)} + \wick{\c{\phi}_0(x_1)\phi_0(x_2)\c{\phi}_0(x_3)\phi_0(x_4)} \\
+ \quad & \wick{\c{\phi}_0(x_1)\phi_0(x_2)\phi_0(x_3)\c{\phi}_0(x_4)} + \wick{\phi_0(x_1)\c{\phi}_0(x_2)\c{\phi}_0(x_3)\phi_0(x_4)} \\
+ \quad & \wick{\phi_0(x_1)\c{\phi}_0(x_2)\phi_0(x_3)\c{\phi}_0(x_4)} + \wick{\c1{\phi}_0(x_1)\c1{\phi}_0(x_2)\c2{\phi}_0(x_3)\c2{\phi}_0(x_4)} \\
+ \quad & \wick{\c1{\phi}_0(x_1)\c1{\phi}_0(x_2)\c2{\phi}_0(x_3)\c2{\phi}_0(x_4)} + \wick{\c1{\phi}_0(x_1)\c2{\phi}_0(x_2)\c1{\phi}_0(x_3)\c2{\phi}_0(x_4)}) \quad (4.3.12)
\end{aligned}$$

Taking the contractions through the normal ordering symbol , we get one normal ordered product of four fields, six terms involving a contraction and a normal ordered product of two fields and three terms involving just contractions. For any normal ordered product of two fields the annihilation operators are on the right and creation fields on the right this means if we take the vacuum expectation value, i.e. take $\langle 0 \mid$ on the right and $\mid 0\rangle$ on the left, we gain 0 by the definition of our creation and annihilation fields. This means our four-point correlator only has the terms dependent only on the contractions i.e. :

$$\begin{aligned}
\langle 0 \mid \quad & T(\phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\phi_0(x_4)) \mid 0\rangle = \langle 0 \mid \wick{\c1{\phi}_0(x_1)\c1{\phi}_0(x_2)\c2{\phi}_0(x_3)\c2{\phi}_0(x_4)} \\
+ \quad & \wick{\c1{\phi}_0(x_1)\c1{\phi}_0(x_2)\c2{\phi}_0(x_3)\c2{\phi}_0(x_4)} + \wick{\c1{\phi}_0(x_1)\c2{\phi}_0(x_2)\c1{\phi}_0(x_3)\c2{\phi}_0(x_4)} \mid 0\rangle \quad (4.3.13)
\end{aligned}$$

In terms of Feynman propagators this becomes:

$$\begin{aligned}
D_F(x_1 - x_2)D_F(x_3 - x_4) \quad + & \quad D_F(x_1 - x_3)D_F(x_2 - x_4) \\
+ & \quad D_F(x_1 - x_4)D_F(x_2 - x_3) \quad (4.3.14)
\end{aligned}$$

This concept generalises to n fields therefore we can see that the only terms that come out in our correlator are the terms where all the fields are contracted all other terms going to zero. This means we can ignore correlators with odd numbers of fields as these will not all contract and that even fields will have (n-1)!! terms made up of the product of $\frac{n}{2}$ Feynman propagators where each product has the field co-ordinates occurring only once in the propagator

4.4 LSZ formula

We now introduce the LSZ formula. This is a way of relating probabilities for particles created in the far past going to particles in the far future given in terms of the correlators discussed above. For generality we will try to discuss all three types of field together, since the method is similar for all three fields we concentrate on the details for the scalar field and only highlight the results for the other two field types, consigning the specifics to the appendices. For the scalar field we defined (see Appendix D3 for the equivalence to our previous definition (B.3)):

$$a(\vec{k}) = i\int d^3x \frac{1}{\sqrt{2\omega_k}} e^{ikx} \overleftrightarrow{\partial_0} \phi_0(x) \tag{4.4.1}$$

for the free theory, where :

$$\mid k\rangle = \hat{a}^\dagger(\vec{k}) \mid 0\rangle \tag{4.4.2}$$

and

$$\hat{a}(\vec{k}) \mid 0\rangle = 0 \tag{4.4.3}$$

By analogue we define a time-independent operator which creates a particle which is localised in momentum-space around $\vec{k}_1$:

$$a_1^\dagger \equiv \int d^3x g_1(\vec{k}) a^\dagger(\vec{k}) \tag{4.4.4}$$

where $g_1(\vec{k}) \propto e^{\frac{-(\vec{k}-\vec{k}_1)^2}{4\sigma^2}}$ is a gaussian function which localises the wave-packet (σ is the width in momentum space) . If we assume this works similarly for interacting theory in the far-past and far-future, then defining:

$$a_1(t)^\dagger \equiv \int d^3x g_1(\vec{k})(-i)\int d^3x \frac{1}{\sqrt{2\omega_k}} e^{-ikx} \overleftrightarrow{\partial_0} \phi_0(x) \tag{4.4.5}$$

For n particles created with initial momentum $\vec{k}_1, \vec{k}_2, .., \vec{k}_n$ our initial state is :

$$\mid i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t) .... a_n^\dagger(t) \mid 0\rangle \tag{4.4.6}$$

We finish with n' particles with momentum $\vec{k}_1', \vec{k}_2', ..., \vec{k}_n'$ the final state is :

$$\mid f\rangle = \lim_{t\to\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t) .... a_{n'}^\dagger(t) \mid 0\rangle \tag{4.4.7}$$

Hence our scattering amplitude is:

$$\langle f \mid i\rangle = \langle 0 \mid a_{1'}(\infty)a_{2'}(\infty)....a_{n'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)...a_n^\dagger(-\infty) \mid 0\rangle \quad (4.4.8)$$

as the operators are already time-ordered we can insert the time-ordering symbol and write :

$$\langle f \mid i\rangle = \langle 0 \mid T(a_{1'}(\infty)a_{2'}(\infty)....a_{n'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)...a_n^\dagger(-\infty)) \mid 0\rangle \quad (4.4.9)$$

We can show that(see Appendix D4):

$$a_1^\dagger(-\infty) = a_1^\dagger(\infty) + i\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial^2 + m^2)\phi(x) \quad (4.4.10)$$

and:

$$a_1(-\infty) = a_1(\infty) - i\int d^3k g_1(\vec{k})\int d^4x \frac{1}{\sqrt{2\omega_k}} e^{ikx}(\partial^2 + m^2)\phi(x) \quad (4.4.11)$$

by substituting these into our expression we realise that the time ordering symbol now moves the annihilation operators to the left where they annihilate the vacuum and thus are not counted. We also localise the particle by letting $\sigma \to 0$ thus $g_1(\vec{k}) \to \delta(\vec{k} - \vec{k}_1)$. We are then left with

$$\begin{aligned}\langle f \mid i\rangle &= i^{n+n'}\int d^4x_1 e^{ik_1x_1}(\partial_1^2 + m^2).....\int d^4x_{1'} e^{-ik_1'x_1'}(\partial_{1'}^2 + m^2)\\ &\times \langle 0 \mid T(\phi(x_1).....\phi(x_1')....) \mid 0\rangle \frac{1}{\sqrt{2\omega_1}}.....\frac{1}{\sqrt{2\omega_{1'}}}.... \quad (4.4.12)\end{aligned}$$

Defining (see Appendix for equivalence to previous definition (B.3)) :

$$b(\vec{k}) = i\int d^3x \frac{1}{\sqrt{2\omega_k}} e^{ikx}\overleftrightarrow{\partial_0}\phi_0^\dagger(x) \quad (4.4.13)$$

for the free theory.

We now consider the dirac field using the definition in terms of $a_{\vec{p}}^s$ and $b_{\vec{p}}^s$ we find (see Appendix D5).:

$$a_{\vec{p}}^s = \int \frac{d^3x}{\sqrt{2\omega_k}} e^{ipx}\overline{u}^s(\vec{p})\psi(x) \quad (4.4.14)$$

which by transposing gives:

$$a_{\vec{p}}^{s\dagger} = \int \frac{d^3x}{\sqrt{2\omega_k}} e^{-ipx}\overline{\psi}(x)\gamma^0 u^s(\vec{p}) \quad (4.4.15)$$

We also have:

$$b_{\vec{p}}^{\dagger s} = \int \frac{d^3x}{\sqrt{2\omega_k}} e^{-ipx}\overline{v}^s(\vec{p})\gamma^0\overline{\psi}(x) \quad (4.4.16)$$

and its transpose:

$$b_{\vec{p}}^{s} = \int \frac{d^3x}{\sqrt{2\omega_k}} e^{ipx}\overline{\psi}(x)\gamma^0 v_{\vec{p}}^{s} \tag{4.4.17}$$

These two creation operators then act on the vacuum to produce particles and their anti-particles i.e.:

$$| \, p, s, + \rangle = a_{\vec{p}}^{s\dagger} \mid 0 \rangle \tag{4.4.18}$$

$$| \, p, s, + \rangle = b_{\vec{p}}^{s\dagger} \mid 0 \rangle \tag{4.4.19}$$

By analogue with above we define a:

$$a_1^{\dagger} = \int d^3p g_1(\vec{p}) a_{\vec{p}}^{s\dagger} \tag{4.4.20}$$

which creates a particle with spin s, charge + localised in momentum space near $\vec{p}_1$ where $g_1(\vec{p}) \propto e^{\frac{-(\vec{p}-\vec{p}_1)}{4\sigma^2}}$ which is a gaussian-like function that localises the wave-packet. Our initial state for n particles is then :

$$| \, i \rangle = \lim_{t \to -\infty} a_1^{\dagger}(t) a_2^{\dagger}(t) .... a_n^{\dagger}(t) \mid 0 \rangle \tag{4.4.21}$$

and our final state for n' particles is:

$$| \, f \rangle = \lim_{t \to \infty} a_{1'}^{\dagger}(t) a_{2'}^{\dagger}(t) .... a_{n'}^{\dagger}(t) \mid 0 \rangle \tag{4.4.22}$$

We then find by a similar method that (see Appendix D6) :

$$a_1^{\dagger}(-\infty) = a_1^{\dagger}(\infty) + i \int d^3p g_1(\vec{p}) \int t \frac{d^4x}{\sqrt{2\omega_k}} \overline{\psi}(x)(i \overleftarrow{\not{\partial}} + m) u_{\vec{p}}^{s} e^{-ipx} \tag{4.4.23}$$

conjugate transposing gives :

$$a_1(\infty) = a_1(-\infty) + i \int d^3p g_1(\vec{p}) \int \frac{d^4x}{\sqrt{2\omega_k}} e^{ipx} \overline{u}_{\vec{p}}^{s}(-i \not{\partial} + m)\psi(x) \tag{4.4.24}$$

We can also see that (see Appendix D7) :

$$b_1^{\dagger}(\infty) = b_1^{\dagger}(-\infty) - i \int d^3p g_1(\vec{p}) \int \frac{d^4x}{\sqrt{2\omega_k}} e^{-ipx} \overline{v}_{\vec{p}}^{s}(-i \not{\partial} + m)\psi(x) \tag{4.4.25}$$

and conjugate transposing:

$$b_1(-\infty) = b_1(\infty) + i \int d^3p g_1(\vec{p}) \int \frac{d^4x}{\sqrt{2\omega_k}} \overline{\psi}(x)(i \overleftarrow{\not{\partial}} + m) v_{\vec{p}}^{s} e^{ipx} \tag{4.4.26}$$

Our initial state will be given as the vacuum state $| \, 0 \rangle$ acted on by $b_1^{\dagger}(-\infty)$ for every anti-particle and a $a_1^{\dagger}$ for every particle and our final state in terms of the vacuum state $\langle 0 \mid$ acting on $b_1(\infty)$ for every anti-particle and a a_1 for every particle. Thus our required quantity $\langle f \mid i\rangle$ is time ordered, as before replacing the a's and b's with the formulas we have found for them with the help of the time-ordering eliminates explicit a and b dependence due to terms annihilating or being annihilated by the vacuum.We then gain the prescription that within the vacuum expectation, i.e. acted on from the left by $\langle 0 \mid$ and from the right by $\mid 0\rangle$, where in terms are at $-\infty$, out terms at ∞:

$$
\begin{aligned}
a^{s\dagger}_{\vec{p}\ in} &\rightarrow i\int \frac{d^4x}{\sqrt{2\omega_k}}\overline{\psi}(x)(i\overleftarrow{\not\partial}+m)u^s_{\vec{p}}e^{-ipx} && (4.4.27)\\
a^{s}_{\vec{p} out} &\rightarrow i\int \frac{d^4x}{\sqrt{2\omega_k}}e^{ipx}\overline{u}^s_{\vec{p}}(i\ \not\partial+m)\psi(x) && (4.4.28)\\
b^{s\dagger}_{\vec{p}\ in} &\rightarrow -i\int \frac{d^4x}{\sqrt{2\omega_k}}e^{ipx}\overline{v}^s_{\vec{p}}(-i\ \not\partial+m)\psi(x) && (4.4.29)\\
b^{s}_{\vec{p} out} &\rightarrow i\int \frac{d^4x}{\sqrt{2\omega_k}}\overline{\psi}(x)(i\overleftarrow{\not\partial}+m)v^s_{\vec{p}}e^{ipx} && (4.4.30)
\end{aligned}
$$

This applies similarly for majorana fields. We will clarify this idea when we do calculations later

From our calculations above showing the link between the spatial components of a vector field and the scalar field and assuming two polarisation states (+ and -), we have :

$$\vec{A}(x) = \sum_{\lambda=\pm}\frac{d^3k}{(2\pi)^3\sqrt{2\omega}}[\vec{\epsilon}^{*}_{\lambda}a_\lambda(\vec{k})e^{ikx}+\vec{\epsilon}_\lambda a_\lambda(\vec{k})e^{-ikx}] \qquad (4.4.31)$$

where the $\vec{\epsilon}_\lambda$'s are the polarisation vectors and $k^0 = \omega = \mid \vec{k} \mid$.
Thus we get very similar result as the scalar case:

$$
\begin{aligned}
a_\lambda(\vec{k}) &= i\vec{\epsilon}_\lambda(\vec{k}).\int \frac{d^3x}{\sqrt{2\omega}}e^{ikx}\overleftrightarrow{\partial_0}\vec{A}(x) && (4.4.32)\\
a^\dagger_\lambda(\vec{k}) &= -i\vec{\epsilon}^{*}_\lambda(\vec{k}).\int \frac{d^3x}{\sqrt{2\omega}}e^{-ikx}\overleftrightarrow{\partial_0}\vec{A}(x) && (4.4.33)
\end{aligned}
$$

If we define $\epsilon^0_\lambda(\vec{k}) \equiv 0$ we get:

$$
\begin{aligned}
a_\lambda(\vec{k}) &= i\epsilon^\mu_\lambda(\vec{k}).\int \frac{d^3x}{\sqrt{2\omega}}e^{ikx}\overleftrightarrow{\partial_0}A_\mu(x)\\
a^\dagger_\lambda(\vec{k}) &= -i\epsilon^{*\mu}_\lambda(\vec{k}).\int \frac{d^3x}{\sqrt{2\omega}}e^{-ikx}\overleftrightarrow{\partial_0}A_\mu(x) && (4.4.34)
\end{aligned}
$$

By analogue with the scalar case and defining $a^\dagger_\lambda(\vec{k})_{in}$ as the operator that creates a photon in the λ polarisation at $-\infty$ and $a_\lambda(\vec{k})_{out}$ as the operator that creates a photon in the λ polarisation at ∞ we get:

$$a^\dagger_\lambda(\vec{k})_{in} \rightarrow -i\epsilon^{\mu*}_\lambda(\vec{k})\int \frac{d^4x}{\sqrt{2\omega}} e^{ikx}\partial^2 A_\mu(x) \quad (4.4.35)$$

$$a_\lambda(\vec{k})_{out} \rightarrow i\epsilon^{\mu}_\lambda(\vec{k})\int \frac{d^4x}{\sqrt{2\omega}} e^{-ikx}\partial^2 A_\mu(x) \quad (4.4.36)$$

The process of contractions and the LSZ formula means that for we can write its general interacting field correlator in terms of its two-point correlator in the free-field theory. Hence we now derive a more mathematically succinct expression for the two-point corelator. We can show that (see Appendix D8):

$$\begin{aligned}\langle 0 \mid T(\phi_0(x')\phi^\dagger_0(x)) \mid 0\rangle &= G_F(x,x') \\ &= \lim_{\epsilon\to 0}(-i\hbar)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik_\mu(x'-x)^\mu}}{-k_\mu k^\mu + m^2 - i\epsilon} \quad (4.4.37)\end{aligned}$$

which is nicely in momentum space:

$$G_F = \frac{-i\hbar}{-k_\mu k^\mu + m^2 - i\epsilon} \quad (4.4.38)$$

for a scalar field.
For fields given by Dirac spinor we want a similar propagator:

$$\langle 0 \mid T\psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle \quad (4.4.39)$$

We define the time-ordering symbol acting on Fermion fields to pick up an extra minus sign for each interchange of fields required to take it from the original order to the time order p116 l2-3 [9] .We define this to be true for Normal ordering p116 l7 [9] as well therefore Wick's theorem remains the same, but the expression for the free-field correlator in terms of Dirac propagators picks up an extra minus sign for each pair of fields $\psi\overline{\psi}$.

$$T\psi_\alpha(x)\overline{\psi}_\beta(y) = \theta(x^0 - y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0 - x^0)\overline{\psi}_\beta(y)\psi_\alpha(x) \quad (4.4.40)$$

we can then find the Dirac propagator as (see Appendix D9) :

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-p.(x-y)} \quad (4.4.41)$$

or in momentum space as:

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \quad (4.4.42)$$

The propagator for gauge fields :

$$\langle 0 \mid TA^i(x)A^j \mid 0\rangle \quad (4.4.43)$$

is given by an analogous calculation as for the complex scalar field and thus we obtain (see Appendix D10):

$$-i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \sum_{\lambda=\pm} \epsilon_{\lambda}^{i*}(\vec{k})\epsilon_{\lambda}^{j}(\vec{k}) \tag{4.4.44}$$

and in momentum space:

$$-i\hbar \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \sum_{\lambda=\pm} \epsilon_{\lambda}^{i*}(\vec{k})\epsilon_{\lambda}^{j}(\vec{k}) \tag{4.4.45}$$

By defining the two-point correlator of the time-part of the four vector (A^0) as the source of the coulomb potential:

$$(\langle 0 \mid A^0(x)A^0(y) \mid 0\rangle = \frac{1}{4\pi \mid \vec{x} - \vec{y} \mid}) \tag{4.4.46}$$

we can extend to :

$$\Delta^{\mu\nu}(k) = \frac{1}{k^2} + \frac{1}{k^2 - i\epsilon} \sum_{\lambda=\pm} \epsilon_{\lambda(\vec{k})}^{\mu*}\epsilon_{\lambda}^{\nu}(\vec{k}) \tag{4.4.47}$$

(This non-rigorous explanation is expanded on in the appendix where we use path integral formalism to show these results)

4.5 Feynman Diagrams and Rules

Feynman diagrams are a diagrammatic way of representing the terms in a correlator or more generally in a state to state transition in the non-interacting vacuum. For a given set of interaction hamiltonians we can draw a set of Feynman diagrams consisting of lines and vertices which have an associated set of Feynman rules. The lines are associated with a particle propagating from one place to another and each particle type is given its own type of line (typically dotted lines for scalar particles, wavy lines for vector particles and filled lines for fermions and each line has a four-momentum associated with it). The types of vertices allowed are determined by our interaction hamiltonian e.g. if our interaction term is ϕ^4 our only vertex will be one where four scalar particles meet, if it is $g\overline{\psi}\psi\phi$ (yukawa theory) then each vertex will be the meeting of two fermions and a scalar particle. Feynman rules can be formulated in either momentum or position space. Here we will give the rules associated with the propagators (lines) for our three types of fields and some example vertices which we will choose for their use later. There are also rules excluding what diagrams we can draw. Generally our interaction Hamiltonian will be a sum of terms. Since our perturbation is small H_I is usually considered for each interaction type as

a small coupling constant multiplied by some product of the fields. Since H_I only occurs in the exponential we can write $e^{-i\int_{-\infty}^{\infty} H_I(t)}$ as a Taylor expansion in terms of the small parameter. Thus our Feynman diagrams are given an order in this small parameter corresponding to that term in the Taylor expansion. The order corresponds to the number of vertices of that type in the diagram as H_I is the term that allows the particle interactions represented by vertices. Considering the free case if we start with particles with momenta $\vec{p}_1, \vec{p}_2$... and end with a state with momenta $\vec{p}_1', \vec{p}_2'$ Then considering a given interaction the order of our coupling constant must be so large that the total number of the fields representing that particle in the interaction is greater then or equal to the total number of momenta we have specified. Since there is always time-ordering in our expression we contract off fields with momenta until we have exhausted all momenta and then we can contract the fields with themselves only contracting fields of the same type. The contractions of fields with momenta are represented by internal lines and the contractions of fields with themselves are represented by lines between vertices. Four-momentum is conserved in all our diagrams. We need to consider $\mathcal{H}$ of the form $\lambda\phi^4$, $g\overline{\psi}\psi\phi$,$a\phi A_\mu A^\mu$, $e\overline{\psi}\gamma^\mu A_\mu$, . The reason for considering only these will be explained later :
Finally we have the arrows representing incoming particles as discussed above these are obtained by contracting fields with momentum states. To get the value one applies the free field onto the momentum state which is essentially the annihilation operator with a given eigenvalue applied to the vacuum the commuting properties of the annihilation/creation operators and the cancelation of the normalising factor isolates the exponential (and the spin (spinors) or polarisation (vectors) variables)in position space and thus has the simple representation in momentum space given below (see appendix for calculation)

for ϕ each line gives a factor of 1
for ψ coming into a vertex (a particle) we get a factor $u^s(p)$
for $\overline{\psi}$ coming into a vertex (a particle) we get a factor $\overline{u}^s(p)$
for $\overline{\psi}$ coming out of a vertex (a anti-particle) we get a factor $\overline{v}^s(p)$
for ψ coming out of a vertex (a anti-particle) we get a factor $v^s(p)$
for A_μ into a vertex we get $\epsilon_\mu(p)$
for A_μ out of a vertex we get $\epsilon_\mu^*(p)$
If we start with the vacuum we can just pair up all the fields inside the vacuum expectation and no longer worry about particles coming into vertices

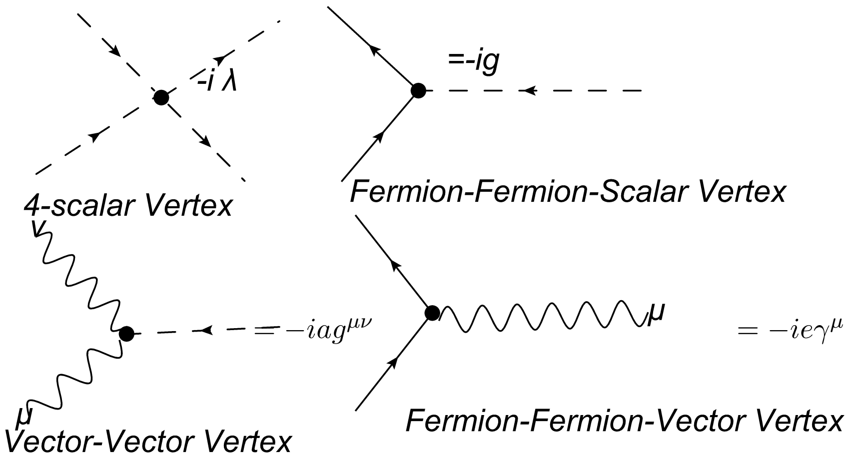


Figure 4.5.1: Above are given the four-vertices required. In momentum-space we associate with each of these a coefficient. In all but one of the cases this is simple since the number of vertices is the same as the order of the expansion and the expansion is in (-i) times the coupling constant we just associate (-i) times the coupling constant with the vertex in the case of two fermion's interacting with a vector field we require a γ^{μ} for each vertex to make the quantity scalar and so we associate this with each vertex as well as (-i) times the coupling constant . We also require the standard metric to convert the μ vector to the ν vector in the vector-vector scalar vertex

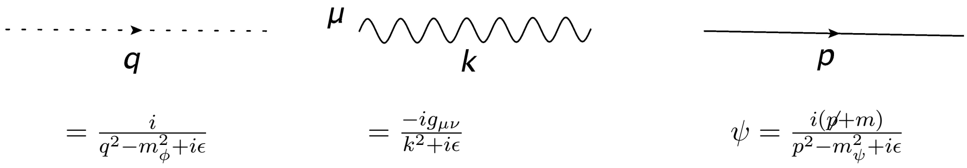


Figure 4.5.2: Above are shown the propagators for each type of particle . These are the lines that pass between two vertices they have associated values in momentum-space for $\phi \Rightarrow \frac{i}{q^2-m_{\phi}^2+i\epsilon}$, for $\psi = \frac{i(\not{p}+m)}{p^2-m_{\psi}^2+i\epsilon}$ and for $A_{\mu} \Rightarrow \frac{-ig_{\mu\nu}}{k^2+i\epsilon}$ (Shown in the Appendix D10)

In general our rules will be something like
1.Associate a line of the correct type with each particle that is incoming or outgoing
2. Assign an arrow to each line to denote whether it is in-going or out-going
3. Put in the number of vertices equivalent to the order we want
4. Join each line to a vertex

5. Depending on the interaction Hamiltonian create enough internal lines so that each vertex has the correct total amount of fields
6. Assign a four-momentum to each line the external lines are already given the rest can be calculated by four-momentum conservation or are undetermined
7. Assign the factors to external, internal lines and vertices as discussed above
8. Integrate over all undetermined momentum (i.e. one for each closed loop)
9. Divide by the symmetry factor, i.e if there are changes of internal propagators that leave the diagram unchanged
10. Summing over all the different diagrams thus produced gives the required contribution
11. We consider each possible configuration at this order
There is also a problem to do with disconnected diagrams i.e diagrams where vertices exist that are not joined by external lines. However it can be shown that in a free field the sum of all diagrams for a given number of particles is the sum of all the connected diagrams in that set multiplied by the exponential of the sum of the disconnected diagrams in that set (see). Thus we normalise away the disconnected pieces

4.6 Summary

In this section we have introduced the idea of an interacting field theory as a perturbation on a free-field theory. We have then used the idea of contractions and the LSZ formula to give formulations of complicated correlators and particle interactions in terms of propagators. Finally we have introduced the idea of Feynman diagrams as a diagrammatic representation of terms in the expansion

Chapter 5

Ward Identities and Conserved Currents

5.1 Ward Identities

The Ward-Takahashi identity applies only to QED processes i.e. those with (eq 4.3 p78 [9]) :

$$\mathcal{L} = -\frac{1}{4\mu_0}(F_{\mu\nu})^2 + \overline{\psi}(i\not{\partial} - m)\psi - e\overline{\psi}\psi\gamma^\mu A_\mu \tag{5.1.1}$$

with an external photon with momentum $\vec{k}$. If we add up the diagram using the Feynman rules in position space and call this quantity :

$$\mathcal{M}(k) = \epsilon_\mu(k)\mathcal{M}^\mu(k) \tag{5.1.2}$$

we get:

$$k_\mu \mathcal{M}^\mu(k) = 0 \tag{5.1.3}$$

. For later convenience we consider 1PI (defined later) contributions to the photon propagator and the two-photon to two-photon scattering. We show these two diagrams below we call the photon propagator $i\Pi^{\mu\nu}(q)$ and the two-photon scattering $A^{\mu\nu\sigma\rho}$. We make the large assumption that neither of these has a pole at $q^2 = 0$ (following sections 7.4 ,7.5 [9]) for momentum q. Thus for Π considering both photons μ and ν we get:

$$q_\mu \Pi^{\mu\nu}(q) = 0 \tag{5.1.4}$$

and

$$q_\nu \Pi^{\mu\nu}(q) = 0 \tag{5.1.5}$$

Thus Γ is proportional to $(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2})$
since $q_\mu(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) = (q^\nu - \frac{q_\mu q^\mu q^\nu}{q^2}) = q^\nu - q^\nu = 0$
and

$q_\nu(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) = (q^\mu - \frac{q_\nu q^\mu q^\nu}{q^2}) = q^\mu - q^\mu = 0.$
Our requirement for no poles means that :

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^{\mu\nu})\Pi(q^2) \tag{5.1.6}$$

and thus this has no terms of order less then q^2. Giving our momentum labels μ, ν, σ, ρ we get the conditions :

$$k_\mu A^{\mu\nu\rho\sigma}(k) = k_\nu A^{\mu\nu\rho\sigma}(k) = k_\sigma A^{\mu\nu\rho\sigma}(k) = k_\rho A^{\mu\nu\rho\sigma}(k) = 0 \tag{5.1.7}$$

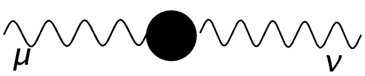
these each lead to a proportionality term of the form:

$$(g^{\mu\nu} k^\sigma - g^{\mu\sigma} k^\nu) \tag{5.1.8}$$

cycling (clockwise with respect to the diagram) through the photons so the coefficients change by one i.e we get:

$$(g^{\nu\sigma} k^\rho - g^{\nu\rho} k^\sigma), (g^{\sigma\rho} k^\mu - g^{\sigma\mu} k^\rho) \text{ and } (g^{\rho\mu} k^\nu - g^{\rho\nu} k^\mu) \tag{5.1.9}$$

Our no poles condition means that these four constants multiply a term constant or higher power k term so that our original A has only terms k^4 or above.



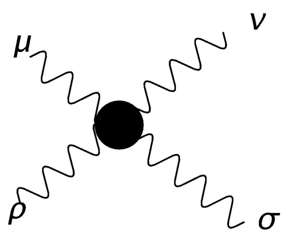


Figure 5.1.1: The photon propagator is shown on the left, the two-photon scattering is shown on the right

It now remains to proof the Ward identity
We consider the diagram with our photon removed and postulate there are two ways it could join the diagram, either joining onto a line that runs from an external point to an external point or joining onto an internal photon loop. If we consider the photon added to an external line we can arrange the line so that the photon travels from right to left and thus every propagator left of the point our photon joins will gain momentum $\vec{k}$. The contribution to the diagram multiplied by k_μ is given by $-iek_\mu\gamma^\mu$. Thus if our photon occurs at position j our diagram has structure:

$$...(\frac{i}{\not{p}_{j+1} - m})\gamma^{\lambda_{j+1}}[(\frac{i}{\not{p}_j + \not{k} - m})(-ie\not{k})(\frac{i}{\not{p}_j - m})\gamma^{\lambda_j}](\frac{i}{\not{p}_{j-1} - m})\gamma^{\lambda_{j-1}}...$$

which can be written in the form :

$$...(\frac{i}{\not{p}_{j+1}-m})\gamma^{\lambda_{j+1}}[\frac{i}{\not{p}_j+\not{k}-m}-\frac{i}{\not{p}_j-m}]\gamma^{\lambda_j}(\frac{i}{\not{p}_{j-1}-m})\gamma^{\lambda_{j-1}}... \quad (5.1.10)$$

by writing :

$$\not{k}=\not{p}+\not{k}-m-(\not{p}-m) \quad (5.1.11)$$

separating out the terms in the numerator and canceling .
This is true for all j , the minus sign between the two terms means that if we do all j unique insertions and then sum over the j we get cancellation of all terms except for the ones where we insert the photon at either of the extreme ends. These two terms have a minus sign between them and are both multiplied by e from the vertex, the one where the photon is inserted at the far right (the beginning) has all momentum terms increased by k relative to the other diagram. Using q=p'+k where p' is the final momentum of the fermion without the photon being inserted, we see that our original diagram where we have ingoing momentum p and outgoing momentum q should have a term proportional to:

$$\frac{i}{\not{q}-m}\frac{i}{\not{p}-m} \quad (5.1.12)$$

since for the photon inserting on the far right momentum is always greater than or equal to p+k and for our photon inserted on the far left final momentum is p' which is less then q , our contributions from the propagators which depend on the momentum cannot correspond to this on either side and thus the contribution must be zero

Clearly the individual j terms will be exactly the same for photons joining the fermion loop if we choose an initial vertex and an order in which to enumerate our j's either clockwise or anti-clockwise. By direct analogue we are only left with two terms which are products of propagators and in which the momentum differs by k in each propagator between the two terms. The two terms also have a relative minus sign between. Since for each loop we are integrating over undetermined momentum for a given k we can in our second term shift the integration variable meaning we have two terms that are equivalent with a relative minus sign between them and thus cancel

5.2 Conserved Currents

The main result we use from conserved currents is Noether's Theorem. This Theorem states that for each continuous symmetry of $\mathcal{L}$ we have a conserved current $j^\mu(x)$. This is important as we will be using spontaneous symmetry breaking as an integral part of the Higgs mechanism. We start with a approximate proof of the Noether's theorem using the scalar field and then outline the conserved currents for the complex scalar and Dirac fields

We consider the transformation of the field :

$$\phi \rightarrow \phi + \alpha \Delta \phi(x) \tag{5.2.1}$$

and decide that the field is symmetric under this transformation i.e. this is a symmetry, if this transformation leaves the equations of motion unchanged. This means that the action will be unchanged except for a possible surface term and that the Lagrangian density will be unchanged except for a four-divergence (which integrates up to this surface term) . So the Lagrangian transforms as:

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu(x) \tag{5.2.2}$$

We suggest that this can also be written:

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \mathcal{L}(x) \tag{5.2.3}$$

By varying the fields we obtain :

$$\alpha \Delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi}(\alpha \Delta \phi) + (\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)})\partial_\mu(\alpha \partial \phi) \tag{5.2.4}$$

as $\mathcal{L}$ is a function of ϕ and $\partial_\mu \phi$ only.
Using that :

$$\partial_\mu(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\Delta \phi) = \partial_\mu(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)})\Delta \phi + (\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)})\partial_\mu \Delta \phi \tag{5.2.5}$$

we can rewrite $\alpha \Delta \mathcal{L}$ as:

$$\alpha \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta \phi) + \alpha[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})]\Delta \phi \tag{5.2.6}$$

Our standard Euler-lagrange equations for $\mathcal{L}$ give:

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}) = 0 \tag{5.2.7}$$

Then equating $\alpha \partial_\mu \mathcal{J}^\mu$ and $\alpha \Delta \mathcal{L}$ we find $\partial_\mu j^\mu = 0$ where our conserved current $j^\mu(x)$ is given as :

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta \phi - \mathcal{J}^\mu \tag{5.2.8}$$

For every such continuous symmetry of the lagrangian density we have a similar conservation law. If we choose a symmetry involving more then one field, the derivative of the lagrangian density will now be a sum over the fields of the derivatives of the lagrangian density for each field multiplied by the change in that field.

If we consider the free complex scalar field lagrangian, we can see that this is invariant under the transformation:

$$\phi \to e^{i\alpha}\phi(\text{ also } \phi^\dagger \to e^{-i\alpha}\phi^\dagger \tag{5.2.9}$$

considering ϕ and $\phi^\dagger$ as independent fields this becomes the infinitesimal transform :

$$\alpha\Delta\phi = i\alpha\phi \text{ and } \alpha\Delta\phi^\dagger = -i\alpha\phi^\dagger \tag{5.2.10}$$

since the lagrangian is unchanged :

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi \tag{5.2.11}$$

which is:

$$\alpha i[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)] \tag{5.2.12}$$

We can also consider a infinitesimal transformation on the spacetime coordinates :

$$x^\mu \to x^\mu - a^\mu \tag{5.2.13}$$

Given the dependence of $\mathcal{L}$ upon x^μ we obtain (p19 l2 [9]):

$$\mathcal{L} + a^\mu\partial_\mu\mathcal{L} = \mathcal{L} + a^\nu\partial_\mu(\delta^\mu_\nu\mathcal{L}) \tag{5.2.14}$$

where the second expression gives the change in the form $\partial_\mu\mathcal{J}^\mu$ we then define four conserved currents one for each component of ν :

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{5.2.15}$$

as:

$$\Delta\phi = \partial_\nu\phi \tag{5.2.16}$$

for the ν the component of a^ν from above.
We note that the Hamiltonian (p19 eq2.18 [9]):

$$H = \int T^{00}d^3x \tag{5.2.17}$$

and the physical momentum (p19 eq 2.19 [9]) :

$$P^i = \int T^{0i}d^3x \tag{5.2.18}$$

We now consider the Dirac field, we obtain two possible conserved currents by considering the transformations :

$$\psi(x) \to e^{i\alpha}\psi(x) \text{ and } \psi(x) \to e^{i\alpha\gamma^5}\psi(x) \tag{5.2.19}$$

where as in (2.3.25) :

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5.2.20}$$

Since the first transformation leaves the Lagrangian unchanged we expect the current aged associated with it to be unchanged. Using the infinitesimal form of the transformation:

$$\psi \to \psi + i\alpha\psi \tag{5.2.21}$$

for the first transformation and :

$$\psi \to \psi + i\alpha\gamma^5\psi \tag{5.2.22}$$

for the second and that :

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\overline{\psi}\gamma^\mu \tag{5.2.23}$$

we expect our first conserved current to be of the form :

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi^\mu \tag{5.2.24}$$

ignoring the constant term and it is shown in Appendix E1 that this in fact conserved. This current will be important later. It can be shown (see Appendix E2) that :

$$\gamma^5\gamma^\mu = -\gamma^5\gamma^\mu \tag{5.2.25}$$

We know that (see Appendix E3) :

$$\gamma^{5\dagger} = \gamma^5 \tag{5.2.26}$$

and that (Appendix E4):

$$\gamma^5\gamma^5 = 1 \tag{5.2.27}$$

Thus :

$$\overline{\psi} \to \psi e^{-i\alpha\gamma^5}\gamma^0 = -\overline{\psi}e^{-i\gamma^5\alpha} \tag{5.2.28}$$

using the definition of overline (2.3.10) and the commutator relation for γ^5 . Hence the lagrangian is invariant in the derivative term where the γ^μ leads to an extra minus sign but not in the mass term, since the extra term is not in the form of a four-divergence. By comparison with the other current we expect :

$$j^{\mu5}(x) = \overline{\psi}(x)\gamma^\mu\gamma^5\psi^\mu \tag{5.2.29}$$

we show in Appendix E5 that the divergence is :

$$\partial_\mu j^{\mu5}(x) = 2im\overline{\psi}\gamma^5\psi \tag{5.2.30}$$

and thus as expected the current is only conserved when m=0

5.3 Summary

In this chapter we have introduced the idea of conserved currents and through the use of Nother's theorem linked this to symmetries of our lagrangian. This will be important in our study of symmetry later. We have also derived some examples of conserved currents for the free-scalar and Dirac field models.

We have also introduced the idea of ward identities as a useful concept this will be used when we discuss the renormalisation of QED.

Chapter 6

Loops and Renormalisation

6.1 Loop Corrections

When drawing Feynman diagrams it becomes obvious that we can have loops, i.e an internal propagators that starts and ends at the same vertex or two or more propagators that start and end at the same vertices. This means that their internal momentum cannot be completely determined by the conservation of four-momentum and we introduced the prescription that we should determine over all undetermined momementa.

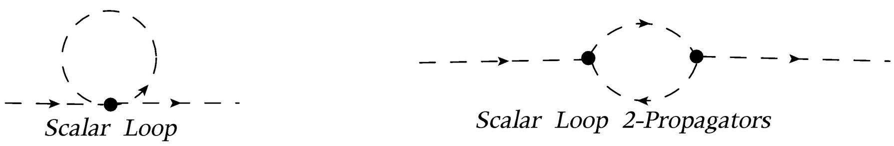


Figure 6.1.1: Here we illustrate the two simplest types of loop on the left we use ϕ^4 theory on the right we have ϕ^3 theory. We illustrate here with scalar fields for simplicity the phenomena of loops can occur for all field types

The presence of these loops will lead to a term coming from our Feynman diagram where the propagator or propagators are integrated over the momentum. The forms of our propagator mean that each individual propagator has in terms of momentum dimension p^{-2} (scalar and vector propagators) or dimension p^{-1} (the spinor propagator) the momentum four integral has dimension p^4 thus this integral is proportional to a positive power of p and thus diverges

We attempt to solve this problem by introducing a large cut-off momentum Λ which we integrate upto and specify that we don't expect the theory to work above $p > \Lambda$ some suitably large energy scale

We ideally want our results to be independent of this Λ .So far the parameters we have introduced in the Lagrangian have been artificially selected and need loop corrections to correspond to physical observables

6.2 1PI Diagrams

One-particle irreducible (1PI) diagrams are diagrams that cannot be split into two by removing a single line.

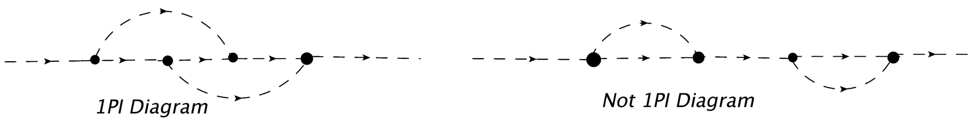


Figure 6.2.1: here we illustrate a 1PI diagram on the left and a one-particle reducible diagram on the right

6.3 Renormalisation

We define a theory as renormalisable if its physically observable quantities are independent of our cut-off scale lambda. To assess renormalisability we introduce a parameter D corresponding to the superficial power of momentum which will tell us whether a theory is superficially renormalisable or not. We get superficially :

$$(\int d^4p)^L(\frac{1}{p^a})^P \tag{6.3.1}$$

,a =1 for fermions, 2 for vectors or scalars P is the number of propagators, L is the number of loops D=4L-aP, clearly if $D > 0$ this diverges and if $D < 0$ it converges if D=0 we get a term like:

$$\frac{\int d^N p}{p^N} \tag{6.3.2}$$

which diverges logarithmically. Superficially divergent diagrams may not be divergent but we will use other methods to assess this. Superficially undivergent diagrams may diverge but they will have a divergent sub-diagram.

We here cheat slightly by only assessing the renormalisability of the diagrams given by the QED Lagrangian (5.1.1):

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\psi}(i \not{\partial} - m_0)\psi - e_0\overline{\psi}\psi\gamma^\mu A_\mu \tag{6.3.3}$$

the Yukawa lagrangian (see for example p79 eq4.9 [9]):

$$\overline{\psi}(i\ \not{\partial} - m_0)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - g\overline{\psi}\psi\phi \tag{6.3.4}$$

and the general n-dimensional interacting scalar lagrangian (similar to section 4.4)

$$\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{n!}\phi^n \tag{6.3.5}$$

where we have denoted the non-physical quantities with the sub-script zero. We also only consider one-loop divergences which is a more serious problem and ignore the issues associated with divergent sub-diagrams.

Starting with the scalar-field, the superficial divergence is given by the power of momentum in the numerator minus the power of momentum in the denominator. Since the powers of momentum in the denominator come from the propagators each of which have a momentum squared operator in the denominator, the powers of momentum in the numerator come from the loops each of which give a integral over all the dimensions. So the superficial divergence is given by:

$$D = 4L - 2P \tag{6.3.6}$$

where 4 is the number of dimensions.
The number of loops:

$$L = P - V + 1 \tag{6.3.7}$$

where V is the number of vertices , using the fact that the number of loops is the same as the number of undetermined momenta and each vertex determines the momenta for one propagator leaving it, except for one vertex required for overall conservation.
Since each vertex has n lines coming out, each propagator joins twice to a vertex and each external line joins once to a vertex:

$$nV = N + 2P \tag{6.3.8}$$

N = number of external lines.
Combining these we get:

$$D = 2P + 4 - 4V = (n-4)V - N + 4 \tag{6.3.9}$$

We see that for a given number of external lines,if $n > 4$ there will be a number of vertices such that the diagram will diverge. For $n \leq 4$ choosing enough vertices and external lines, will mean the diagrams will not diverge, we can then enumerate these diagrams and assess whether they do actually diverge. For the special case d=4 we have:

$$D = 4 - N \tag{6.3.10}$$

and thus we only need to look at the diagrams with between 0 and four external lines.

Consider all diagrams with N external legs to have the same amplitude and remember from above that our field has dimension 1. An interaction can be written as:

$$\eta\phi^N \tag{6.3.11}$$

with one vertex which thus picks out just η and thus has dimension 4-N or in terms of $\lambda\phi^N$ where there are V vertices that each pick up a factor λ and may be divergent i.e. the term is proportional to :

$$\lambda^V \Lambda^D \tag{6.3.12}$$

re-introducing the cut-off momentum Λ for the divergent quantity.
If λ now has dimension r this gives for the right-hand side a dimension of

$$rV + D \tag{6.3.13}$$

and thus by comparing both sides

$$D + rV = 4 - N \tag{6.3.14}$$

hence:

$$D = 4 - rV - N \tag{6.3.15}$$

Clearly for $r > 0$ we can pick a number of vertices such that D is always negative and for $r < 0$ we can pick a number of vertices such that $D > 0$ this is our result that the coupling constant must have dimension greater then one

We now define the three case, We take N to be the total number of incoming lines not worrying about their type, as for each case we only consider one vertex type V is a reasonable number (positive integer), this should hopefully be clear from the calculations:
Super-Renormalisable : There are only a finite number of Feynman diagrams which diverge superficially (i.e. there are only certain values of N and V we can pick so that $D > 0$)
Renormalisable: Only a finite number of amplitudes superficially diverge (i.e. as amplitudes are defined by external lines N and not by V , there are only certain N's for which diagrams diverge) but there are diverging diagrams at all orders (we get these by picking an N such that the diagram diverges and then varying V through the orders) i.e. D is independent of V . It is difficult to show this is equivalent to our initial definition so we will just show this for the special cases out-lined above
Non-Renormalisable: All diagrams diverge at sufficiently high-order in perturbation theory. (i.e. for a given N we can get a divergent diagram by

going to higher orders i.e. choose a number of Vertices such that $D > 0$)

In terms of the coefficients of V in D (call this y) these neatly becomes:
Super-renormaliable : $y < 0$
Renormalisability : y=0
Non-Renormalisable $y > 0$

We now consider the four superficially divergent cases for our scalar theory N=0,1,2,3,4. As mentioned above the number of fields must be even to get an amplitude so we are left with N=0,2,4. N=0 is just a vacuum correction. So we are left with N=2 and N=4. Explicit calculation for these terms is done in Appendix F1 for the moment we merely state that these two terms give us three divergent terms and that we hope to absorb these in someway by using the constants in our lagrangian.

We now need an expression for the physical mass defining :

$$1 = \mid \Omega\rangle\langle\Omega \mid + \sum_{\lambda} \int \frac{d^3p}{2\pi^3} \frac{1}{2E_{\vec{p}}} \mid \lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}} \mid \tag{6.3.16}$$

where $\mid \lambda_{\vec{p}}\rangle$ are boosts with momentum $\vec{p}$ of the zero-momentum eigenvalues of H λ_0 with energy m_λ and the sum runs over all these states and (p212 eq 7.2 [9]):

$$E_{\vec{p}}(\lambda) = \sqrt{\mid \vec{p}^2 \mid + m_\lambda^2} \tag{6.3.17}$$

Inserting this into the two-point correlator allows us to rewrite it as (p213 eq 7.5 [9]):

$$\langle\Omega \mid T\phi(x)\phi(y) \mid \Omega\rangle = \int_0^\infty \frac{dp^4}{(2\pi)^4} \sum_\lambda \frac{i}{p^2 - m_\lambda^2 + i\epsilon} \mid \langle\Omega\phi(0)\lambda\rangle \mid^2 \tag{6.3.18}$$

(see appendix) . Since in the energy spectrum the mass of the particle is isolated (if we have two or more particles we can have all energies $(2m)^2$ or higher using their relative momentums and bound states will give us a few states just below this energy) . Thus translating into momentum space and setting y=0 for simplicity:

$$\int d^4x \langle\Omega \mid T\phi(x)\phi(0) \mid \Omega\rangle = \frac{iZ}{p^2 - m^2 + i\epsilon} + (\text{ terms with poles away from } p^2 = m^2) \tag{6.3.19}$$

where Z is the value of :

$$\mid \langle\Omega \mid \phi(0) \mid \lambda\rangle \mid^2 \tag{6.3.20}$$

We redefine our field to eliminate Z i.e :

$$\phi = Z^{\frac{1}{2}} \phi_r \tag{6.3.21}$$

We relate our observable constants (m, λ) to our terms in the lagrangian (m_0, λ_0) via :

$$\delta_Z = Z - 1 \;\; \delta_m = m_0^2 Z - m^2, \;\; \delta\lambda = \lambda_0 Z^2 - \lambda \tag{6.3.22}$$

Thus our lagrangian goes:

$$\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4 \tag{6.3.23}$$

$$\frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4 \tag{6.3.24}$$

$$\frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!}\phi_r^4 + \frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_m \phi_r^2 - \frac{\delta\lambda}{4!}\phi_r^4 \tag{6.3.25}$$

We can now define our physical parameters in terms of our two remaining divergent diagrams. The N=2 divergent diagram has already been used to define m . Since in this space it is :

$$\begin{aligned} \frac{1}{Z} \times \int d^4x \langle \Omega \mid T\phi(x)\phi(0) \mid \Omega \rangle \;\; &= \;\; \frac{i}{p^2 - m^2} \\ &+ \;\; (\text{ terms with poles away from } p^2 = m^2) \end{aligned} \tag{6.3.26}$$

We choose to define λ in terms of our N =4 diagram taking the case where all particles have zero three-momentum and the same mass. The new terms give new Feynman rules. The counter-term vertex follows by direct analogue , the counter-term propagator is found by ....

$i(p^2\delta_Z - \delta_m)$ $\frac{i}{p^2 - m^2 + i\epsilon}$

Figure 6.3.1: The new Feynman propagators the one on the left is the counterterm propagator and has associated value $i(p^2\delta_Z - \delta_m)$, the one on the right is the normal propagator and by analogue with before has value $\frac{i}{p^2-m^2+i\epsilon}$ where m is now the physical mass

It can now be seen that we have three unknown terms that do not contribute to our physical calculations δ_Z δ_m, $\delta\lambda$ and at any order these can be chosen using our definitions of m and λ to cancel out the divergent sub-diagrams that will occur (see section 10.5 [9] for an example calculation)

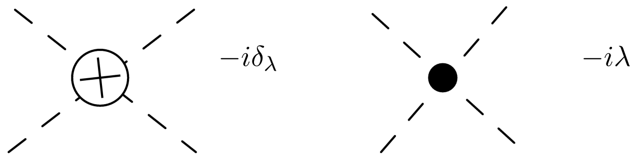


Figure 6.3.2: The new Feynman vertices , the one on the left is the counterterm vertex and has associated value $-i\delta_\lambda$, the one on the right is the normal vertex $-i\lambda$ where λ is now the physical coupling constant. the coefficients can be found by look at the coefficients of the two ϕ_r^4 terms

We now consider the QED case. Here the superficial degree of divergence D is calculated similarly. Again we only have one type of vertex which adds four powers of momentum for each loop however we have two types of propagators, the vector propagator which is proportional to p^{-2} and the fermion propagator which is proportional to $\frac{\not{p}}{p^2}$ or p^{-1} thus for L = number of loops ,P_f is the number of fermions and P_V is the number of vector fields.

$$D = 4L - P_f - 2P_V \tag{6.3.27}$$

By momentum conservation each propagator of either type adds a potentially undertermined momenta and each vertex imposes momentum conservation, one being required for total momentum conservation. The number of undetermined momenta is the number of loops so:

$$L = P_f + P_V - V + 1 \tag{6.3.28}$$

Each external vector particle goes to a vertex and each propagator goes to two vertices hence

$$V = 2P_V + N_V \tag{6.3.29}$$

N_V are the number of external vector fields since each vertex has two fermions attached

$$V = \frac{1}{2}(2P_f + N_f) \tag{6.3.30}$$

N_f is the number of external fermion fields by analogue. Substituting for L in D we have :

$$D = 4(P_f + P_V - V + 1) - P_f - 2P_V \tag{6.3.31}$$

Using:

$$P_f = V - \frac{1}{2}N_f \tag{6.3.32}$$

$$P_V = \frac{1}{2}(V - N_V) \tag{6.3.33}$$

we get :

$$\begin{aligned} D &= 4[(V - \frac{1}{2}N_f) + \frac{1}{2}(V - N_V) - V + 1] - 2\frac{1}{2}(V - N_V) - (V - \frac{1}{2}N_f) \\ &= 4 + 4[\frac{1}{2}(V - N_f - N_V)] - 2V - \frac{1}{2}N_f - N_V = 4 - N_V - \frac{3}{2}N_f \end{aligned} \tag{6.3.34}$$

Clearly now there are only a finite number of diagrams that might diverge, there are in fact seven corresponding to:
$N_f = 0$ which has four $N_V = 0, 1, 2, 3, 4$
$N_f = 1$ which has two $N_V = 0, 1, 2$
$N_f = 2$ which has one $N_V = 0, 1$
the $N_f = 1$ cases are excluded since $N_f = 2V$.
For convenience we show these diagrams below:

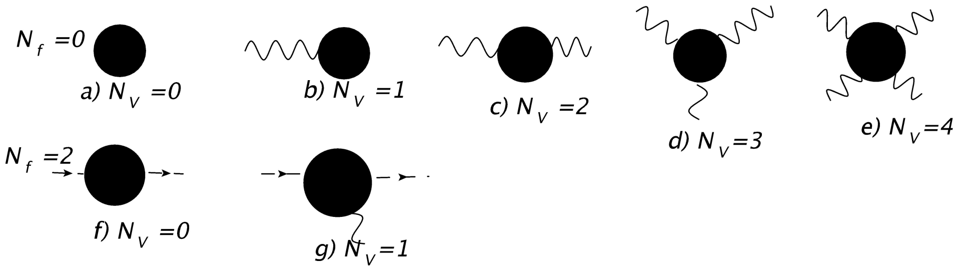


Figure 6.3.3: Diagrams that superficially diverge for the QED case

a) only effects the vacuum and thus we can ignore it . In the case of an odd photon vertex e.g. b) , d) , the incoming propagators can be ignored as they will just add a polarisation vector and thus for b) we get a term in position space of the form :

$$-ie \int d^4 x e^{-iq.x} \langle \Omega \mid T j_\mu(x) \mid \Omega \rangle \tag{6.3.35}$$

for photon of momentum q, where (p318 eq 10.5 [9]):

$$j^\mu = \overline{\psi} \gamma^\mu \psi \tag{6.3.36}$$

see our Feynman diagram vertices. We require charge conjugation to be a symmetry of the vacuum p318 l18 [9] :

$$C \mid \Omega \rangle = \mid \Omega \rangle \tag{6.3.37}$$

We know (see Appendix F2) :

$$C j^\mu(x) C^\dagger = -j^\mu(x) \tag{6.3.38}$$

and thus:

$$\langle\Omega \mid j_\mu(x) \mid \Omega\rangle = \langle\Omega \mid C^\dagger C j_\mu(x) C^\dagger C \mid \Omega\rangle = -\langle\Omega \mid j_\mu(x) \mid \Omega\rangle = 0 \quad (6.3.39)$$

using $C^\dagger C = I$ as C is unitary . So this term vanishes . In the three-vertex case we get two terms like this that both vanish

We now consider g) by substituting in $N_V = 1, N_f = 2$ we find D=0. We can expand in terms of the momenta of any of the propagators. The constants of this expansion will be successively higher order derivatives with respect to momentum of the propagator. Since both types of propagators are already at least inversely proportional to momentum their nth constant , for $n > 0$ n starting at 0, will be inversely proportional to p^{n+1}or higher and thus the term in the expansion will be inversely proportional momentum and thus converge. Thus only the constant term can diverge

A similar argument works on f) substituting $N_f = 2, N_V = 0$ we find D=1. Expanding in the propagator gives terms inversely proportional to momentum, this means we have one term that diverges linearly and one that diverges logarithmically. As the terms in the taylor expansion will be reduced by a power of p for each term and thus any terms after the first two will converge

Nothing we have done so far is explicitly dependent on the use of vector particles rather then scalar particles therefore this analysis applies equally well to yukawa theory.

For the remaining two diagrams c) and e) we make use of ward identities (useful the two examples we showed earlier). These allow us to exclude the first n powers of the expansion in terms of momentum and thus lower the divergence by N. substituting $N_V = 2, N_f = 0$ for c) we find D=2 originally and thus the term will only diverge logarithmically 2-2 =0 and substituting $N_V = 4, N_f = 0$, D=0 for e) means the term will not diverge D=-4. Hence we now have four divergent terms

By direct analogue with our scalar case.
The fermion interaction f) corresponds to:

$$\frac{iZ_2}{\not{p} - m} + \text{terms with poles away from } \not{p} = m \quad (6.3.40)$$

The vector interaction c) corresponds to (see section 10.1 [9]):

$$\frac{-iZ_3 g_{\mu\nu}}{q^2} + \text{terms with poles away from } q^2 = 0 \quad (6.3.41)$$

We define :

$$\psi = Z_2^{\frac{1}{2}} \psi_r \tag{6.3.42}$$

and

$$A^{\mu i} = Z_3^{\frac{1}{2}} A^{\mu} \tag{6.3.43}$$

using e_0 p330 l30 [9] as our bare coupling constant and m_0 as our bare mass our lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4} Z_3 (F_r^{\mu\nu})^2 + Z_2 \overline{\psi_r}(i \not{\partial} - m_0)\psi_r - e_0 Z_2 Z_3^{\frac{1}{2}} \overline{\psi}_r \gamma^{\mu} \psi_r A_{r\mu} \tag{6.3.44}$$

Defining the physical charge by :

$$eZ_1 = e_0 Z_2 Z_3^{\frac{1}{2}} \tag{6.3.45}$$

with:

$$\delta_3 = Z_3 - 1 \ \delta = Z_2 - 1 \tag{6.3.46}$$

$$\delta_m = Z_2 m_0 - m \text{ and } \delta_1 = Z_1 - 1 \tag{6.3.47}$$

we gain:

$$\begin{aligned} \mathcal{L} = & - \frac{1}{4}(F_r^{\mu\nu})^2 + \overline{\psi_r}(i \not{\partial} - m_0)\psi_r - e\overline{\psi}_r \gamma^{\mu} \psi_r A_{r\mu} \\ & - \frac{1}{4}\delta_3 (F_r^{\mu\nu})^2 + \overline{\psi_r}(i\delta_2 \partial - \delta_m)\psi_r - e\delta_1 \overline{\psi}_r \gamma^{\mu} \psi_r A_{r\mu} \end{aligned} \tag{6.3.48}$$

Our four divergent terms can be cancelled off by our four delta's
Following figure 10.4 [9] we simply quote the Feynman rules for QED with counter-terms the normal propagators, normal and counter-term vertex are obviously seen by analogue with our original case:

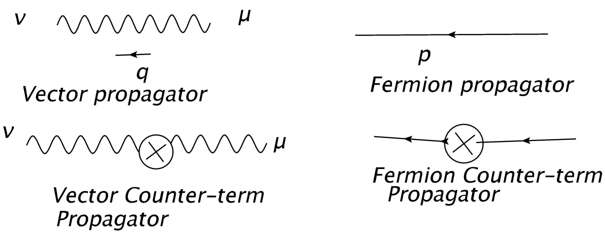


Figure 6.3.4: We associate with the fermion-propagator the factor $\frac{i}{\not{p}-m+i\epsilon}$ with the vector-propagator in the Feynman gauge as $\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$ with the counter-term fermion-propagator $i(\not{p}\delta_2 - \delta_m)$, and with the counter-term vector-propagator $-i(g^{\mu\nu}q^2 - q^{\mu}q^{\nu})\delta_3$

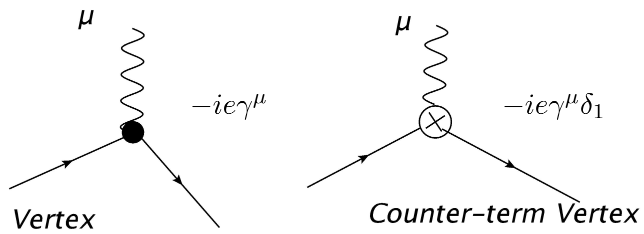


Figure 6.3.5: We associate with the normal vertex $-ie\gamma^\mu$ and with the counter-term vertex $-ie\gamma^\mu\delta_1$

By comparison with both QED and scalar fields we see that in Yukawa theory we get five delta's our extra one relative to QED coming from the mass of the scalar field. The two diagrams corresponding to c) and e) pick up an extra divergence from the location of the mass of the scalar field which we can cancel away with our extra delta

This section in no-way is a full treatment of Renormalisation, but is merely included to highlight the issue and the property of the theories that we will be using that they are renormalisable and we can thus get physical quantities that are non-divergent

6.4 The Quantum Action

We now define the idea of the quantum or effective action, it is analogous to the Gibbs free energy in magnetic system p366 l21-22 [9], this is useful as there are symmetries in magnetic systems that are often broken by the system being magnetised in a given way. Using our path integral formalism we have in scalar field theory:

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}[\phi]+J\phi} \tag{6.4.1}$$

where J is a source term .

$$\begin{aligned} \frac{\delta}{\delta J(x)}E[J] &= i\frac{\delta}{\delta J(x)}logZ = \frac{1}{Z}i\frac{\delta}{\delta J(x)}Z \\ &= -\frac{\int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}[\phi]+J\phi}\phi(x)}{\int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}[\phi]+J\phi}} \end{aligned} \tag{6.4.2}$$

We can write this as (p366 eq11.45 [9]) :

$$\frac{\delta}{\delta J(x)}E[J] = -\langle\Omega \mid \phi(x) \mid \Omega\rangle_J \tag{6.4.3}$$

which is the vacuum expectation value of the field in the presence of the source J(x) .
We define :

$$\phi_{cl}(x) = \langle\Omega \mid \phi(x) \mid \Omega\rangle_J \tag{6.4.4}$$

Now we can define the effective action as (p366 eq11.47 [9]):

$$\Gamma[\phi_{cl}] = -E[J] - \int d^4yJ(y)\phi_{cl}(y) \tag{6.4.5}$$

We can show that (see Appendix F3) :

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = -J(x) \tag{6.4.6}$$

and thus without sources :

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = 0 \tag{6.4.7}$$

.

6.5 Summary

Chapter 7

The Path to the Higgs mechanism

7.1 Spontaneous Symmetry Breaking

Spontaneous Symmetry Breaking is an interesting physical phenomena, that appears in both the classical and quantum world. A classical example of where it occurs is in a magnetic system where the aligning of magnetic fields generates a field with a preferred direction. Generally we get spontaneous symmetry breaking when we violate symmetries of the lagrangian

For later convenience we consider the Lagrangian of the complex scalar field with $(\phi\phi^\dagger)^2$ coupling :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^\dagger - \frac{1}{2}m^2\phi\phi^\dagger - \frac{\lambda}{4}(\phi\phi^\dagger)^2 \tag{7.1.1}$$

and specify that $m^2 < 0$ i.e. $\exists\mu^2 = -m^2 \ \mu \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^\dagger + \frac{1}{2}\mu^2\phi\phi^\dagger - \frac{\lambda}{4}(\phi\phi^\dagger)^2 \tag{7.1.2}$$

This lagrangian is clearly unchanged under the substitution :

$$\phi \to -\phi \qquad (\phi^\dagger = -\phi^\dagger) \tag{7.1.3}$$

Using the definition of H (2.1.13) :

$$H = \int d^3x[\pi\dot{\phi} - \mathcal{L}] \tag{7.1.4}$$

where π is the conjugate momentum in this case $\pi = \dot{\phi}^\dagger$ writing :

$$\partial_\mu\phi\partial^\mu\phi^\dagger = \dot{\phi}\dot{\phi}^\dagger - \partial_i\phi\partial_i\phi^\dagger =|\pi|^2 - |\nabla\phi|^2 \tag{7.1.5}$$

We get:

$$H = \int d^3x[\frac{1}{2} \mid \pi \mid^2 + \frac{1}{2} \mid \nabla\phi \mid^2 - \frac{1}{2}\mu^2\phi\phi^\dagger + \frac{\lambda}{4}(\phi\phi^\dagger)^2 \tag{7.1.6}$$

since $\mid \pi \mid^2, \mid \nabla\phi \mid^2 > 0 \ \forall \phi, \pi$ to minimise the hamiltonian and thus energy we need to minimise :

$$V(\phi) = -\frac{1}{2}\mu^2\phi\phi^\dagger + \frac{\lambda}{4}(\phi\phi^\dagger)^2 \tag{7.1.7}$$

i.e. we need to find:

$$\frac{\partial V(\phi)}{\partial \phi} = 0 \tag{7.1.8}$$

This gives :

$$-\frac{1}{2}\mu^2\phi^\dagger + \frac{\lambda}{2}(\phi\phi^\dagger)\phi^\dagger = 0 \tag{7.1.9}$$

which factorises to :

$$\phi^\dagger(-\frac{1}{2}\mu^2 + \frac{\lambda}{2}(\phi\phi^\dagger)) = 0 \tag{7.1.10}$$

which implies $\phi^\dagger = 0$ or $\phi\phi^\dagger = \frac{\mu^2}{\lambda}$

$$\frac{\partial^2 V(\phi)}{\partial \phi^2} = \frac{\lambda}{2}(\phi^\dagger)^2 = 0 \text{ if } \phi^\dagger = 0 \tag{7.1.11}$$

writing :

$$\phi = \phi_1 + i\phi_2 \tag{7.1.12}$$

and using:

$$\phi\phi^\dagger = \phi_1^2 + \phi_2^2 = \frac{\mu^2}{\lambda} \tag{7.1.13}$$

$$Re[(\phi^\dagger)^2] = Re[\phi_1^2 + \phi_2^2 - 2i\phi_1\phi_2] = \frac{\mu^2}{\lambda} \tag{7.1.14}$$

so

$$\phi\phi^\dagger = \frac{\mu^2}{\lambda} \tag{7.1.15}$$

is a minima
We choose our minima to lie in the real direction i.e. :

$$\phi = \phi_1' + v + i\phi_2' \tag{7.1.16}$$

where:

$$v = \sqrt{\frac{\mu^2}{\lambda}} \tag{7.1.17}$$

vacuum expectation value and ϕ_1', ϕ_2' have minimum value zero
This choice of v breaks the symmetry of the problem and is equivalent to

choosing a unique minima
We now simplify slightly by introducing:

$$\Phi = \frac{\phi}{\sqrt{2}} = \frac{(\phi_1' + v + i\phi_2')}{\sqrt{2}} \tag{7.1.18}$$

and thus our original Lagrangian becomes:

$$\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^\dagger + \mu^2 \Phi\Phi^\dagger - \lambda(\Phi\Phi^\dagger)^2 \tag{7.1.19}$$

since v is constant :

$$\partial_\mu \Phi \partial^\mu \Phi^\dagger = \frac{1}{2}\partial_\mu \phi_1' \partial^\mu {\phi_1'}^\dagger + \frac{1}{2}\partial_\mu \phi_2' \partial^\mu {\phi_2'}^\dagger \tag{7.1.20}$$

$$\begin{aligned}(\Phi\Phi^\dagger) &= \frac{(\phi_1' + v + i\phi_2')((\phi_1' + v - i\phi_2')}{2} = \frac{(\phi_1' + v)^2 + {\phi_2'}^2}{2} \\ &= \frac{{\phi_1'}^2 + 2v\phi_1' + v^2 + {\phi_2'}^2}{2}\end{aligned} \tag{7.1.21}$$

$$(\Phi\Phi^\dagger)^2 = [\frac{{\phi_1'}^2 + 2v\phi_1' + v^2 + {\phi_2'}^2}{2}]^2 \tag{7.1.22}$$

$$= \frac{{\phi_1'}^4 + 4v{\phi_1'}^3 + 2v^2{\phi_1'}^2 + 2{\phi_1'}^2{\phi_2'}^2 + 4v^2{\phi_1'}^2 + 4v^3\phi_1' + 4v\phi_1'{\phi_2'}^2 + v^4 + 2v^2{\phi_2'}^2 + {\phi_2'}^4}{4} \tag{7.1.23}$$

substituting and using $v = \sqrt{\frac{\mu^2}{\lambda}}$ we find (see Appendix G1) :

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1' \partial^\mu {\phi_1'}^\dagger + \frac{1}{2}\partial_\mu \phi_2' \partial^\mu {\phi_2'}^\dagger + \frac{\mu^4}{4\lambda} - \mu^2 {\phi_1'}^2 \tag{7.1.24}$$

If we ignore higher then quadratic powers of the fields (i.e. interaction-terms) . This can be seen to be the sum of the a complex scalar field ϕ_2 with zero -mass and a complex scalar field ϕ_1 with mass $m^2 = 2\mu^2$ as mass for a field ξ are of the form:

$$-\frac{1}{2}m^2\xi^2 \tag{7.1.25}$$

In our full Lagrangian:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1^\dagger + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2^\dagger + \frac{\mu^4}{4\lambda} - \mu^2 {\phi_1'}^2 - \frac{\lambda {\phi_1'}^4}{4} - \mu^2\lambda^{\frac{1}{2}}\phi_1'{\phi_2'}^2 \\ &- \lambda^{\frac{1}{2}}\mu{\phi_1'}^3 - \frac{\lambda{\phi_1'}^2{\phi_2'}^2}{2} - \frac{\lambda{\phi_2'}^4}{4}\end{aligned} \tag{7.1.26}$$

Our original symmetry $\phi_1 \to -\phi_1, \phi_2 \to -\phi_2$ is no longer evident

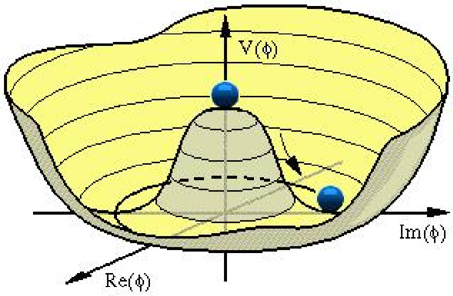


Figure 7.1.1: [10] An illustration of the Potential used. As calculated in the text there is a maximum on the axis and the set of minimas form a circle round the axis choosing one of these minima as the unique minima, breaks the symmetry.

7.2 The Goldstone's Boson and Goldstone's Theorem

Simply stated Goldstone's theorem states that every spontaneously broken symmetry leads to a massless Boson known as a a Goldstone Boson. As will be observed later these Bosons are not seen when considering the Higgs Mechanism, they are taken to be eaten by the other Bosons (more later). Thus here we prove Goldstone's theroem before showing why it can be disregarded for the Higg's mechanism, we will later demonstrate the role of the Goldstone Boson in the Higgs mechanism.

We consider here only the case of scalar fields. Obviously to have a broken symmetry our original system must have a symmetry and thus from our discussion above must have a conserved current $J_\mu(x)$. We define a current :

$$Q(t) = \int d^3x J_0(\vec{x}, t) \tag{7.2.1}$$

with the definition of a conserved current $\partial^\mu J_\mu = 0$ we obtain by 3-integration that :

$$\int d^3\vec{x}\vec{\nabla}.\vec{J}(\vec{x}, t) = \dot{Q} \tag{7.2.2}$$

We want the situation with conserved charge, but where the vacuum is not left invariant by the charge operator i.e :

$$Q \mid 0\rangle \neq \dot{Q} = 0 \tag{7.2.3}$$

We define (eq 2.27 [11]):

$$M_\mu^{ij}(k) = \int d^4x \mid [J_\mu^i(x), \phi_j(0)] \mid 0\rangle exp[i(kx)] \tag{7.2.4}$$

In the basis of eigenvectors of $P_\mu \quad | n\rangle$ this becomes:

$$\begin{aligned} M^{ij}_\mu(k) &= \sum_n \int d^4x\ exp[i(kx)][\langle 0 \mid J^i_\mu(x) \mid n\rangle\langle n \mid \phi_j(0) \mid 0\rangle \\ &- \langle 0 \mid \phi_j(0) \mid n\rangle\langle n \mid J^i_\mu(x) \mid 0\rangle] \qquad (7.2.5) \\ &= \sum_n \int d^4x\ exp[i(kx)][\langle 0 \mid e^{-iPx}J^i_\mu(0)e^{iPx} \mid n\rangle\langle n \mid \phi_j(0) \mid 0\rangle \\ &- \langle 0 \mid \phi_j(0) \mid n\rangle\langle n \mid e^{-iPx}J^i_\mu(0)e^{iPx} \mid 0\rangle] \qquad (7.2.6) \end{aligned}$$

using the transform between Heisenberg and Schrödinger representation:

$$\begin{aligned} M^{ij}_\mu(k) &= \sum_n [(\int d^4x exp[i(P_n+k)])\langle 0 \mid J^i_\mu(0) \mid n\rangle\langle n \mid \phi_j(0) \mid 0\rangle \\ &- (\int d^4x exp[-i(-P_n+k)])\langle 0 \mid J^i_\mu(0) \mid n\rangle\langle n \mid \phi_j(0) \mid 0\rangle] \qquad (7.2.7) \end{aligned}$$

using

$$\begin{aligned} e^{iPx} \mid 0\rangle &= \mid 0\rangle \\ \langle 0 \mid e^{iPx} &= \langle 0 \mid \\ e^{iPx} \mid n\rangle &= \mid n\rangle e^{iP_nx} \\ \langle n \mid e^{-iPx} &= e^{-iP_nx}\langle n \mid \qquad (7.2.8) \end{aligned}$$

$$\begin{aligned} M^{ij}_\mu(k) &= (2\pi)^4 \sum_n \delta^4(P_n+k)\langle 0 \mid J^i_\mu(0) \mid n\rangle\langle n \mid \phi_j(0) \mid 0\rangle \quad (7.2.9) \\ &- \delta^4(P_n-k)\langle 0 \mid \phi_j(0) \mid n\rangle\langle n \mid J^i_\mu(0) \mid 0\rangle \qquad (7.2.10) \end{aligned}$$

We make the assumption of"manifest covariance" this can be violated later to allow for the massive bosons of the Higgs mechanism. This condition is (eq 2.32 [12]) :

$$\langle 0 \mid J_\mu(0) \mid \vec{P}, E_p\rangle = P_\mu a(P^2) \qquad (7.2.11)$$

where a is an arbitrary function
We then obtain :

$$M^{ij}_\mu(k) = \epsilon(k_0)k_\mu\rho^{ij}_1(k^2) + k_\mu\rho^{ij}_2(k^2) \qquad \epsilon(k_0)\{ \begin{array}{c} 1\ k_0 \geq 1 \\ -1\ k_0 < 0 \end{array} \qquad (7.2.12)$$

where this form comes from the fact that each delta has a part proportional to $k^2 \times k_\mu$ by the postulate and that one delta contributes when $k_0 < 0$ and the other when$k_0 > 0$. ρ_1, ρ_2 are arbirtary
clearly

$$\frac{\partial}{\partial x^\mu} M^{ij}_\mu(k) = 0 \qquad (7.2.13)$$

since M is in k-space we know conservation of current:

$$\frac{\partial}{\partial x^{\mu}} J^{i}_{\mu}(x) = 0 \tag{7.2.14}$$

and

$$\frac{\partial}{\partial x^{\mu}} \phi(0) = 0 \tag{7.2.15}$$

since there is no spatial or time dependence. Thus the only difference between $M^{ij}_{\mu}(k)$ and $\frac{\partial}{\partial x^{\mu}} M^{ij}_{\mu}(k) = 0$ is the factor of k^{μ} which comes from the exponential
hence:

$$\epsilon(k_0) k^2 \rho^{ij}_1(k^2) + k^2 \rho^{ij}_2(k^2) = 0 \tag{7.2.16}$$

Hence we get two equations for ρ :

$$\begin{aligned} k_0 > 0 \;\; k^2[\rho^{ij}_1(k^2) + \rho^{ij}_2(k^2)] &= 0 \\ k_0 < 0 \;\; k^2[\rho^{ij}_1(k^2) - \rho^{ij}_2(k^2)] &= 0 \end{aligned} \tag{7.2.17}$$

The two solutions are thus:

$$\rho_1(k^2) = c_1 \delta(k^2) \tag{7.2.18}$$

$$\rho_2(k^2) = c_2 \delta(k^2) \tag{7.2.19}$$

as away from $k^2 = 0$ $\rho_1, \rho_2 = 0$ to satisfy this equation
If either $c_1, c_2 \neq 0$ then there exists a $P^2 = 0$ state i.e. a zero mass state p12 column 2 l26-30 [12] . This will apply for every symmetry hence a boson for each symmetry.
For a broken symmetry we assume Q does not annihilate the vacuum. Hence :

$$\langle 0 \mid Q_i(t), \phi_j(0) \mid 0 \rangle_{t=0} \neq 0 \tag{7.2.20}$$

From above we have :

$$\begin{aligned} M^{ij}_0(k) &= \epsilon(k_0) k_0 c_1 \delta(k^2) + k_0 c_2 \delta(k^2) \\ &= \int d^4x \; exp[i(kx)] \langle 0 \mid [J^i_0(x), \phi_j(0)] \mid 0 \rangle \end{aligned} \tag{7.2.21}$$

$$\int_{-\infty}^{\infty} dk_0 M^{ij}_0(k)_{\vec{k}=0} = \int d^4x 2\pi \delta(t) \langle 0 \mid [J^i_0(x), \phi_j(0)] 0 \rangle \tag{7.2.22}$$

using the definition of δ in the right-hand side :

$$\begin{aligned} \int_{-\infty}^{\infty} dk_0 M^{ij}_0(k)_{\vec{k}=0} &= \int dt 2\pi \delta(t) \langle 0 \mid Q(t), \phi_j(0)] \mid 0 \rangle \\ &= \langle 0 \mid Q(0), \phi_j(0)] \mid 0 \rangle \neq 0 \end{aligned} \tag{7.2.23}$$

using (see appendix G2) :

$$\delta(k^2) = \frac{\delta(\mid \vec{k} \mid -k_0) + \delta(\mid \vec{k} \mid +k_0)}{2 \mid \vec{k} \mid} \tag{7.2.24}$$

we obtain:

$$\begin{aligned} \int_{-\infty}^{\infty} dk_0 M_0^{ij}(k)_{\vec{k}=0} &= \int_{-\infty}^{\infty} dk_0(\epsilon(k_0)k_0c_1 + k_0c_2)\frac{\delta(\mid \vec{k} \mid -k_0) + \delta(\mid \vec{k} \mid +k_0)}{2 \mid \vec{k} \mid} \\ &= c_1 \end{aligned} \tag{7.2.25}$$

ϵ leads to summing on the c_1 term the difference of sign leads to cancellation of c_2

The analysis could be extended to other types of fields by considering states with spins.

7.3 Gauge Theory

This follows [9] Section 15.1
Although we have briefly introduced the idea of gauge theories above, with the concept of a vector field and its relation to the concept of gauge in electro-magnetism, we clarify this here. We initially consider only Abelien Gauge theories and discuss non-abelien theories in the next chapter.

In analogue with our discussion on symmetries we consider a theory of fields that obey the local U(1) symmetry $\alpha(x)$ i.e. they are invarient under the transform

$$\psi \to e^{i\alpha(x)}\psi \tag{7.3.1}$$

spinor fields are used here so we can compare with electomagnetism and because the single derivative makes them marginally easier then scalar fields. The process is the same for scalar fields however.
If we consider the standard Dirac Lagrangian (2.3.8):

$$\overline{\psi}(i\partial\!\!\!/ - m)\psi \tag{7.3.2}$$

then:

$$\overline{\psi} \to \overline{\psi}e^{-i\alpha(x)} \tag{7.3.3}$$

since

$$e^{i\alpha(x)^\dagger} = e^{-i\alpha(x)} \tag{7.3.4}$$

which commutes with γ^0 being a scalar
hence $m\overline{\psi}\psi$ is invariant

We consider the definition of the derivative in the direction of vector n^μ (p482 eq 15.2 [9]):

$$n^\mu \partial_\mu = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\psi(x + \epsilon n) - \psi(x)] \tag{7.3.5}$$

n inside the brackets is also a four-vector
This is clearly badly defined since as α is a function of the space-time coordinates the transform of the two halves of this derivative will be different
We now define:

$$U(x, y) \to e^{i\alpha(y)} U(x, y) e^{-i\alpha(x)} \tag{7.3.6}$$

with U(x,x) =1,which we can write as (p482 eq 15.3 [9])

$$U(y, x) = exp[i\phi(x, y)] \tag{7.3.7}$$

Thus:

$$U(y, x)\psi(x) \to e^{i\alpha(y)} U(y, x)\psi(x) \tag{7.3.8}$$

which is the same transformation as for $\psi(y)$
We now define the covariant gauge derivative as in the direction n^μ as (eq 15.4 p483 [9]):

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x)] \tag{7.3.9}$$

where both sides now hopefully transform in the same way .
As we have chosen U to be a phase transform we can expand it in terms of ϵ like a power series to give:

$$U(x + \epsilon n, x) = 1 - i\epsilon e n^\mu A_\mu(x) + O(\epsilon^2) \tag{7.3.10}$$

where e is an arbitrary variable and the space-time dependence comes in $A_\mu(x)$ which is required to be a four-vector as U is scalar and n^μ is a four-vector
Substituting this into $n^\mu D_\mu$ gives:

$$\begin{aligned} n^\mu D_\mu \psi(x) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}([\psi(x + \epsilon n) - \psi(x)] \\ &+ i\epsilon e n^\mu A_\mu(x)\psi(x) - O(\epsilon^2)\psi(x)) \end{aligned} \tag{7.3.11}$$

which gives :

$$D_\mu \psi(x) = \partial_\mu(x) + ieA_\mu(x) \tag{7.3.12}$$

canceling the $n^\mu{}'s$ on both sides
Writing the infinitesimal transform of $e^{i\alpha(x+\epsilon n)}$ as:

$$1 + i\partial_\mu \alpha(x + \epsilon n^\mu) \tag{7.3.13}$$

we gain for the infinitesimal transform of U :

$$1-ie\epsilon n^\mu A_\mu(x) \to (1+i\partial_\mu\alpha(x+\epsilon n^\mu))(1-ie\epsilon n^\mu A_\mu(x))(1-i\partial_\mu\alpha(x)) \quad (7.3.14)$$

which canceling the ones and taking only linear terms gives :

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \quad (7.3.15)$$

This is the same as the gauge invariance we had for electromagnetism the zeroth component of A is altered by an arbitrary time derivative and the spatial components are altered by an arbitrary divergence
Thus :

$$\begin{aligned} D_\mu\psi(x) &= \partial_\mu\psi(x) + ieA_\mu(x)\psi(x) \\ &\to [\partial_\mu(x) + ie(A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x))]e^{i\alpha(x)}\psi(x) \end{aligned} \quad (7.3.16)$$

passing the $e^{i\alpha(x)}$ to the left picks up an extra factor $\partial_\mu\alpha(x)$ from the derivative which cancels with the $\partial_\mu\alpha(x)$ from the transformation of the field
Thus

$$[\partial_\mu(x)+ie(A_\mu(x)-\frac{1}{e}\partial_\mu\alpha(x))]e^{i\alpha(x)}\psi(x) = e^{i\alpha(x)}[\partial_\mu+ieA_\mu(x)]\psi(x) \quad (7.3.17)$$

So:

$$D_\mu\psi(x) = e^{i\alpha(x)}D_\mu\psi(x) \quad (7.3.18)$$

Thus the Lagrangian :

$$\overline{\psi}(x)(i\not{D} - m)\psi \quad (7.3.19)$$

is invariant under this transform the slash notation makes no difference since $e^{i\alpha}$ is scalar so passes through the slashed
since $D_\mu\psi$ has the same transformation law as ψ we know $D_\mu D_\nu\psi$ will also have the same transformation law
Thus:

$$[D_\mu, D_\nu]\psi(x) \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi(x) \quad (7.3.20)$$

using before the transform :

$$D_\mu = \partial_\mu + ieA_\mu(x) \quad (7.3.21)$$

$$[D_\mu, D_\nu] = [\partial_\mu(x) + ieA_\mu(x).\partial_\nu(x) + ieA_\nu(x)] \quad (7.3.22)$$

$$[D_\mu, D_\nu]\psi(x) = [\partial_\mu, \partial_\nu]\psi + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi - e^2[A_\mu, A_\nu]\psi \quad (7.3.23)$$

since partial derivatives commute and their is no summation :

$$[\partial_\mu, \partial_\nu] = [A_\mu, A_\nu] = 0 \quad (7.3.24)$$

Thus:

$$[D_\mu, D_\nu]\psi(x) = ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi \tag{7.3.25}$$

using that the derivatives apply to everything on the right:

$$\begin{aligned}[\partial_\mu, A_\nu]\psi &= \partial_\mu A_\nu \psi + A_\nu \partial_\mu \psi - A_\nu \partial_\mu \psi \\ &= \partial_\nu A_\mu \psi \end{aligned} \tag{7.3.26}$$

Thus :

$$[D_\mu, D_\nu]\psi(x) = ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ieF_{\mu\nu}\psi \tag{7.3.27}$$

or:

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \tag{7.3.28}$$

since $[D_\mu, D_\nu]\psi$ transforms like ψ , $[D_\mu, D_\nu] = ieF_{\mu\nu}$ must be invariant
Using that $F_{\mu\nu}$ has dimension 2 i.e. as our lagrangian has dimension four our f terms should be squares. Requiring invariance and that the theory is renormalisable we gain the most general Lagrangian to be :

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 - c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{7.3.29}$$

The final term can be ignored as it violates time and parity symmetry (see Appendix G3)
There is nothing to stop us identifying the A_μ with the A_μ of electromagnetism,especially since the Lagrangian is correct

7.4 Non-Abelien Gauge Theory

We now consider a non-Abelien gauge theory. Our field multiplet transforms as :

$$\psi \to exp(i\alpha^i(x)t^i) \tag{7.4.1}$$

and non-Abelien implies that

$$[t^i, t^j] \neq 0 \tag{7.4.2}$$

For later convenience we will only consider the three-dimensional rotation group O(3) which is equivalent to the group SU(2). Since we are now in two dimensions we start with a doublet of Dirac fields (p486 eq 15.19 [9]):

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{7.4.3}$$

In this case the transformation is (p486 eq15.21 [9]):

$$\psi(x) \to V(x)\psi(x) \tag{7.4.4}$$

where:

$$V(x) = exp[i\alpha^i(x)\frac{\sigma^i}{2}] \tag{7.4.5}$$

where σ^i are the Pauli-spin matrices
Again we are looking for a gauge co-variant derivative and thus we want an analogous U this time it must be a 2-dimensional matrix such that :

$$U(y,x) \to V(y)U(y,x)V^\dagger(x) \tag{7.4.6}$$

where (p488 l3 [9]):

$$V^\dagger(x) = exp[-i\alpha^i(x)\frac{\sigma^i}{2}] \tag{7.4.7}$$

U(y,y) =1
We can now do an expansion in small ϵ:

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu B^i_\mu \frac{\sigma^i}{2} + O(\epsilon^2) \tag{7.4.8}$$

where g is an arbitrary constant and now we have to associate a vector field with each spin matrix i.e. there are three of them.
Hence with a completely analogous definition of the gauge co-variant derivative:

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)] \tag{7.4.9}$$

separating out the two parts we get :

$$D_\mu = \partial_\mu - igB^i_\mu\frac{\sigma^i}{2} \tag{7.4.10}$$

We want the transformation of the fields B^i_μ. We consider the infinitesimal transformation of U:

$$1 + ig\epsilon n^\mu B^i_\mu\frac{\sigma^i}{2} \to V(x+\epsilon n)(1 + ig\epsilon n^\mu B^i_\mu\frac{\sigma^i}{2})V^\dagger(x) \tag{7.4.11}$$

We consider $V(x+\epsilon n)V^\dagger(x)$
taylor expanding the first term we gain:

$$V(x+\epsilon n)V^\dagger(x) = [(1+\epsilon n^\mu\partial_\mu + O(\epsilon^2))V(x)]V^\dagger(x) \tag{7.4.12}$$

assuming $V(x)V^\dagger(x) = 1$ V is a unitary transform and dropping the ϵ term :

$$V(x+\epsilon n)V^\dagger(x) = 1 + \epsilon n^\mu(\partial_\mu V(x))V^\dagger(x) \tag{7.4.13}$$

We use the identity:

$$0 = \partial_\mu(1) = \partial_\mu(V(x)V^\dagger(x)) = (\partial_\mu V(x))V^\dagger(x) + \partial_\mu(V^\dagger(x))V(x) \tag{7.4.14}$$

thus:

$$(\partial_\mu V(x))V^\dagger(x) = -\partial_\mu(V^\dagger(x))V(x) \quad (7.4.15)$$

Thus :

$$V(x+\epsilon n)V^\dagger(x) = 1 - \epsilon n^\mu(\partial_\mu V^\dagger(x))V(x) \quad (7.4.16)$$

(7.4.11) becomes:

$$1 + ig\epsilon n^\mu B^i_\mu \frac{\sigma^i}{2} \to 1 - \epsilon n^\mu(\partial_\mu V^\dagger(x))V(x) + ig\epsilon n^\mu V(x)B^i_\mu \frac{\sigma^i}{2}V^\dagger(x) \quad (7.4.17)$$

canceling the 1's and dividing through by $ig\epsilon n^\mu$ we get:

$$B^i_\mu(x)\frac{\sigma^i}{2} \to V(x)(B^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu)V^\dagger(x) \quad (7.4.18)$$

The infinitesimal transform becomes using the infinitesimal form of (7.4.5) and (7.4.7):

$$B^i_\mu(x)\frac{\sigma^i}{2} \to (1 + i\frac{\alpha^i\sigma^i}{2})(B^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu)(1 - i\frac{\alpha^i\sigma^i}{2}) \quad (7.4.19)$$

which taking only linear terms in α^i :

$$B^i_\mu(x)\frac{\sigma^i}{2} \to B^i_\mu(x)\frac{\sigma^i}{2} + \frac{1}{g}(\partial_\mu\alpha^i)\frac{\sigma^i}{2} + i[\alpha^i\frac{\sigma^i}{2}, B^j_\mu\frac{\sigma^j}{2}] \quad (7.4.20)$$

We can now show that the infinitesimal transforms for both ψ and $D_\mu\psi$ are the same:

$$D_\mu\psi \to (\partial_\mu - igB^i_\mu\frac{\sigma^i}{2} - i(\partial_\mu\alpha^i)\frac{\sigma^i}{2} + g[\alpha^i\frac{\sigma^i}{2}, B^j_\mu\frac{\sigma^j}{2}])(1 + i\alpha^k\frac{\sigma^k}{2})\psi \quad (7.4.21)$$

using the transformation of B^i_μ (7.4.20) and that the infinitesimal transform of ψ is :

$$\psi \to (1 + i\alpha^k\frac{\sigma^k}{2})\psi \quad (7.4.22)$$

$$\begin{aligned}&(\partial_\mu - igB^i_\mu\frac{\sigma^i}{2} - i(\partial_\mu\alpha^i)\frac{\sigma^i}{2} + g[\alpha^i\frac{\sigma^i}{2}, B^j_\mu\frac{\sigma^j}{2}])(1 + i\alpha^k\frac{\sigma^k}{2}) \quad \psi \\ &= (1 + i\alpha^k\frac{\sigma^k}{2})(\partial_\mu - igB^i_\mu\frac{\sigma^i}{2})\psi + [\partial_\mu(i\alpha^k\frac{\sigma^k}{2}) - i(\partial_\mu\alpha^i)\frac{\sigma^i}{2}] \quad \psi\end{aligned} \quad (7.4.23)$$

where all other terms vanish as they are higher then linear powers in α. So:

$$D_\mu\psi \to (1 + i\alpha^i\frac{\sigma^i}{2})D_\mu\psi \quad (7.4.24)$$

as required.
By comparison with above we consider $[D_\mu, D_\nu]\psi(x)$
which as $D_\mu\psi$ transforms like ψ transforms as:

$$[D_\mu, D_\nu]\psi(x) \to V(x)[D_\mu, D_\nu]\psi(x) \tag{7.4.25}$$

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - igB^i_\mu\frac{\sigma^i}{2}, \partial_\nu - igB^i_\nu\frac{\sigma^i}{2}] \qquad (7.4.26) \\ &= [\partial_\mu, \partial_\nu] - ig([\partial_\mu, B^i_\nu\frac{\sigma^i}{2}] + [B^i_\mu\frac{\sigma^i}{2}, \partial_\nu]) - g^2[B^i_\mu\frac{\sigma^i}{2}, B^i_\nu\frac{\sigma^i}{2}] \end{aligned}$$

the first term vanishes since partial derivatives commute, the middle term gives :

$$-ig(\partial_\mu B^i_\nu\frac{\sigma^i}{2} - \partial_\nu B^i_\mu\frac{\sigma^i}{2}) \tag{7.4.27}$$

by direct comparison with the abelian case (7.3.27) Thus :

$$[D_\mu, D_\nu] = -ig(\partial_\mu B^i_\nu\frac{\sigma^i}{2} - \partial_\nu B^i_\mu\frac{\sigma^i}{2}) - g^2[B^i_\mu\frac{\sigma^i}{2}, B^i_\nu\frac{\sigma^i}{2}] \tag{7.4.28}$$

and thus we can define:

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\frac{\sigma^i}{2} \tag{7.4.29}$$

where :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} = (\partial_\mu B^i_\nu\frac{\sigma^i}{2} - \partial_\nu B^i_\mu\frac{\sigma^i}{2}) - ig[B^i_\mu\frac{\sigma^i}{2}, B^j_\nu\frac{\sigma^j}{2}] \tag{7.4.30}$$

For convenience we rewrite :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} = (\partial_\mu B^i_\nu\frac{\sigma^i}{2} - \partial_\nu B^i_\mu\frac{\sigma^i}{2}) - ig[B^j_\mu\frac{\sigma^j}{2}, B^k_\nu\frac{\sigma^k}{2}] \tag{7.4.31}$$

$$[B^j_\mu\frac{\sigma^j}{2}, B^k_\nu\frac{\sigma^k}{2}] = B^j_\mu B^k_\nu[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}] \tag{7.4.32}$$

as the B's commute with each other and the sigma matrices (for each μ, i they are essentially coefficients.

$$[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}] = \frac{1}{4}\sigma^j\sigma^k - \frac{1}{4}\sigma^k\sigma^j = \frac{1}{4}(\sigma^j\sigma^k - \sigma^k\sigma^j) \tag{7.4.33}$$

using the commutator relation for the Pauli spin matrices (1.5.7)

$$\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k \tag{7.4.34}$$

we gain $\sigma^j\sigma^k = i\epsilon^{jki}\sigma^i$ and $\sigma^k\sigma^j = i\epsilon^{kji}\sigma^i = -i\epsilon^{jki}\sigma^i$ by the anti-symmetry of epsilon
so:

$$[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}] = \frac{1}{4}(2i\epsilon^{jki}\sigma^i) = \frac{1}{2}i\epsilon^{ijk}\sigma^i \quad (7.4.35)$$

again using the anti-symmetry of epsilon. Thus (7.4.31) :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} = (\partial_\mu B^i_\nu\frac{\sigma^i}{2} - \partial_\nu B^i_\mu\frac{\sigma^i}{2}) + gB^j_\mu B^k_\nu\epsilon^{ijk}\frac{\sigma^i}{2} \quad (7.4.36)$$

So :

$$F^i_{\mu\nu} = (\partial_\mu B^i_\nu - \partial_\nu B^i_\mu) + gB^j_\mu B^k_\nu\epsilon^{ijk} \quad (7.4.37)$$

From above the transformation law for F must be :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} \to V(x)F^i_{\mu\nu}\frac{\sigma^i}{2}V^\dagger(x) \quad (7.4.38)$$

infinitesimally this becomes :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} \to (1 + i\alpha^i\frac{\sigma^i}{2})F^i_{\mu\nu}\frac{\sigma^i}{2}(1 - i\alpha^i\frac{\sigma^i}{2}) \quad (7.4.39)$$

and thus :

$$F^i_{\mu\nu}\frac{\sigma^i}{2} \to F^i_{\mu\nu}\frac{\sigma^i}{2} + [i\alpha\frac{\sigma^i}{2}, F^i_{\mu\nu}\frac{\sigma^i}{2}] \quad (7.4.40)$$

ignoring terms higher then linear in alpha
Thus $F^i_{\mu\nu}\frac{\sigma^i}{2}$ is not gauge invariant
However :

$$(F^i_{\mu\nu})^2 = 2tr[(F^i_{\mu\nu}\frac{\sigma^i}{2})^2] \quad (7.4.41)$$

is. Consider the infinitesimal transform:

$$\begin{aligned}(F^i_\mu\frac{\sigma^i}{2})^2 &\to V(F^j_\mu\frac{\sigma^j}{2})V^\dagger VF^j_\mu\frac{\sigma^j}{2})V^\dagger \\ &\to V(F^j_\mu\frac{\sigma^j}{2})^2V^\dagger \end{aligned} \quad (7.4.42)$$

which becomes infinitesimally

$$\begin{aligned}(F^i_\mu\frac{\sigma^i}{2})^2 &\to (1 + \alpha^i\frac{\sigma^i}{2})(F^j_\mu\frac{\sigma^j}{2})^2(1 - \alpha^i\frac{\sigma^i}{2}) \quad (7.4.43) \\ &\to (F^j_\mu\frac{\sigma^j}{2})^2 + [(F^j_\mu\frac{\sigma^j}{2})^2, \sigma^i] \quad (7.4.44)\end{aligned}$$

since $(\sigma^i)^2$ is the identity (see Appendix B5) we can take out $(F^j_\mu\frac{\sigma^j}{2})^2$ of the commutator and thus the commutator vanishes as it is now $[1, \sigma^i]$
The trace is required as we need the Lagrangian to be scalar . Thus

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^i_{\mu\nu})^2 \quad (7.4.45)$$

is invariant . This assumes that $exp(i\alpha^i \frac{\sigma^i}{2})$ commutes with the γ_μ which is true as these matrices can be considered as a constant scalars in the basis of the multiplet
For Non-Abelien Gauge theory we obtain different Feynman diagrams this will be considered latter

7.5 A Brief Summary of QED

Previously we have mentioned QED, we now summarise the salient points of this theory. QED is an abbreviation of Quantum Electrodynamics, it is amongst the first developed Quantum Field Theories and is a "prototype for new theories " , "the jewel of physics our proudest possession" (l32, l30-31 p8[13]). QED describes the interaction between charged particles its Lagrangian is given by (5.1.1):

$$\mathcal{L} = -\frac{1}{4}(F^{\mu\nu})^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu \qquad (7.5.1)$$

Photons are represented by the vector fields and can be seen to be massless. The massive fermions are represented by the Dirac spinor fields. In this Lagrangian we only include type of fermion field which we will take to represent the electron. It is obvious that we could include different field types for any charged particle, we like. We see that the Lagrangian is the sum of the Dirac Lagrangian describing the propagation of the fermion fields and the Maxwell- Lagrangian describing the propagation of the vector field, the final term $(\overline{\psi}\gamma^\mu\psi A_\mu)$ is an interaction term between the fermion fields and the vector fields. The presence of only one interaction term means that our Feynman diagrams have only one vertex type. The Feynman rules for QED given above are repeated here for convenience.A more through analysis of Feynman diagrams is given in section 4.5 .

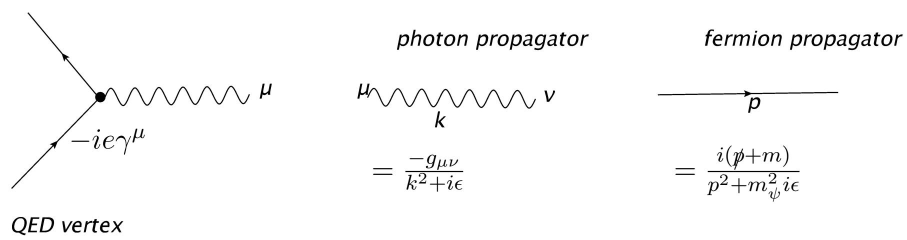


Figure 7.5.1: Propagators and Vertex for QED

We refer the reader to section 6.3 for the discussion on the renormalisation of QED. We can also see that the gauge invariant form of the Dirac lagrangian combined with the Maxwell lagrangian.

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(F^{\mu\nu})^2 + \overline{\psi}(i\not{D} - m)\psi \\
&= -\frac{1}{4}(F^{\mu\nu})^2 + \overline{\psi}(i[\not{\partial} + ie\gamma^\mu A_\mu(x)] - m)\psi \\
&= -\frac{1}{4}(F^{\mu\nu})^2 + \overline{\psi}(i\not{\partial} - m)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu
\end{aligned}
\tag{7.5.2}
$$

is exactly the same as our QED lagrangian and thus QED is gauge invariant with invariance under the symmetry group U(1)

7.6 Left and Right-handed Fermions

We now introduce the idea of handiness of fermions. The experimentally verifiable results that we will require are that no right-handed neutrinos have been detected, thus we assume they do not exists, and that left-handed fermions occur in doublets and right-handed fermions occur in singlets

We can choose to write any fermion field as :

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{7.6.1}$$

The justification for this is that the generators that act on spinor fields and thus define them (see section 2.3) are in block diagonal form (see appendix B6) , thus each block can be though to act on a different component. Remembering the Dirac equation (2.3.8) $(i\not{\partial} - m)\psi$ we can show that this can be rewritten as (see appendix G4):

$$\begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{7.6.2}$$

ignoring the mass terms we see that (see appendix G5) :

$$\overline{\psi} i\not{\partial}\psi = \overline{\psi}_L i\not{\partial}\psi_L + \overline{\psi}_R i\not{\partial}\psi_R \tag{7.6.3}$$

Notation is chosen to be dimensionally correct i.e $\psi_L = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ and similarly for ψ_R

The left-hand side describes the lagrangian fro the propagation of a massless fermion and the right-hand side describes the propagation of it massless left and right-handed parts. Since the two parts are separated there is no reason why they have to obey the same gauge symmetry and thus can have different gauge co-variant derivatives. Obviously there should be an experimental reason for doing this

At this level all fermions will now have to be considered as massless, we will use the Higgs mechanism in the next section to restore their masses

It remains to justify that Left-handed fermions obey SU(2) and U(1) group symmetry and thus are in a doublet and that Right-handed fermions only obey U(1) symmetry and are thus in a singlet. The famous Cobalt-60 experiment of Wu [14] showed that there is a fundamental difference between left-handed and right-handed particles, as the neutrinos admitted where always left handed. if right-handed neutrinos do not exist it is impossible for these right-handed particles to form a doublet with electrons and thus the electrons must be in a singlet. We assume all fermions can be formulated in the same way

7.7 The Higgs Mechanism

We consider here only the GWS model of electro-weak interaction as an example of the Higgs mechanism this is used as it is part of the standard model and explains electro-weak interactions, as- well as giving masses to all standard model particles.

In deriving the Higgs mechanism we consider a massless Dirac Lagrangaian where the left-handed fields are invariant under SU(2) and U(1) symmetries and the right-handed particles are invariant under the U(1) group. We add on the terms for the propagation of the vector Bosons and then describe the vacuum with a U(1) SU(2) invariant complex scalar field with a negative minimum and self-interactions and U(1) charge $\frac{1}{2}$. The vacuum boson interacts with the fermions through yakuwa like terms and with the bosons through the vector-scalar interaction terms from the gauge invariant derivative :

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{4}(F^i_{\mu\nu})^2 + \mid D^1_\mu\phi \mid^2 - m^2 \mid \phi \mid^2 - \lambda \mid \phi \mid^2 \\
&+ i\overline{E}^i_L \not{D}^1 E^i_L + i\overline{e}^i_R \not{D}^2 e^i_R - \lambda^{ij}_E \overline{E}^i_L \phi e^j_R - (\lambda^{ij}_E \overline{E}^i_L \phi e^j_R)^\dagger \\
&+ i\overline{Q}^i_L \not{D}^1 Q^i_L + i\overline{u}^i_R \not{D}^2 u^i_R - i\overline{d}^i_R \not{D}^2 d^i_R - \lambda^{ij}_d \overline{Q}^i_L \phi d^j_R \\
&- (\lambda^{ij}_d \overline{Q}^i_L \phi d^j_R)^\dagger - \lambda^{ij}_u \epsilon^{ab} \overline{Q}^i_{La} \phi^\dagger_b u^j_R - (\lambda^{ij}_u \overline{Q}^i_{La} \phi^\dagger_b u^j_R)^\dagger
\end{aligned} \tag{7.7.1}$$

where:

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{7.7.2}$$

$$D^1_\mu = \partial_\mu - igYa_\mu - ig'T^iB^i_\mu \tag{7.7.3}$$

which is required for the doublets as they respect both SU(2) and U(1) symmetry groups

$$D_\mu^2 = \partial_\mu - igYa_\mu \tag{7.7.4}$$

which is required for the singlets as they respect only the U(1) symmetry group

The sum on i and j is over the three generations of fermions

E_L is a vector with components :

$$\begin{aligned} E_L^1 &= \begin{pmatrix} e^- \\ \nu^e \end{pmatrix}_L \\ E_L^2 &= \begin{pmatrix} \mu \\ \nu^\mu \end{pmatrix}_L \\ E_L^3 &= \begin{pmatrix} \tau \\ \nu^\tau \end{pmatrix}_L \end{aligned} \tag{7.7.5}$$

Q_L is a vector with components :

$$\begin{aligned} Q_L^1 &= \begin{pmatrix} u \\ d \end{pmatrix}_L \\ Q_L^2 &= \begin{pmatrix} c \\ s \end{pmatrix}_L \\ Q_L^3 &= \begin{pmatrix} t \\ b \end{pmatrix}_L \end{aligned} \tag{7.7.6}$$

$$(F_{\mu\nu}^i)^2 = 2tr[(F_{\mu\nu}^i \frac{\sigma^i}{2})^2] \tag{7.7.7}$$

$$F_{\mu\nu}^i \frac{\sigma^i}{2} = (\partial_\mu B_\nu^i \frac{\sigma^i}{2} - \partial_\nu B_\mu^i \frac{\sigma^i}{2}) - ig''[B_\mu^j \frac{\sigma^j}{2}, B_\nu^k \frac{\sigma^k}{2}] \tag{7.7.8}$$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{7.7.9}$$

λ, Y, g, g', g'' are coupling constants

$\lambda_E, \lambda_u, \lambda_d$ are matrices of coupling constants

$$\tau^i = \frac{\sigma^i}{2} \tag{7.7.10}$$

a_μ are the bosons associated with U(1) symmetry

B_μ^i are the bosons associated with SU(2) symmetry

$$m^2 = -\mu^2 \;\; \mu = \mathbb{R} \tag{7.7.11}$$

The gauge covariant derivative is constructed by adding together the non-derivative parts of the U(1) gauge co-variant derivative $D_\mu = \partial_\mu - igYA_\mu$

and the SU(2) derivative $\partial_\mu - ig'T^iB^i_\mu$ our arbitrary constants are different from those used above (sections 7.3 and 7.4) :

$$T^i = \frac{\sigma^i}{2} \tag{7.7.12}$$

$$\epsilon^{ab} = 1 \; \epsilon^{ab} = -\epsilon^{ba} \tag{7.7.13}$$

We have not included the right-hand neutrino as there is no experimental evidence it exists
We have included the quarks as they acquire mass by this mechanism, we are not however going to include the strong interactions and we ignore gluons as they are massless.

For simplicity we consider the Boson and Fermion parts separately
Initially consider the ϕ field . We follow the section on spontaneous symmetry breaking (section 7.1). We want to find the minimum of ϕ , $\phi_0$ignoring the coupling to the vector fields as above $\phi_0\phi_0^\dagger = \frac{v}{2}$ .We can then write:

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{7.7.14}$$

up to phase changes in SU(2) and U(1) which we ignore (we simplify notation at the expense of clarity by donating the fields that followed which are transformed by this phase shift with the same notation as the untransformed fields) as this part of the theory is invariant under these changes . We can then rewrite :

$$\phi^0 = \frac{1}{\sqrt{2}}(v + a(x) + i\chi(x)) \tag{7.7.15}$$

where :

$$\langle a(x)\rangle = \langle \chi(x)\rangle = \langle \phi^+\rangle = 0 \tag{7.7.16}$$

rewriting :

$$\phi(x) = \phi'(x) + \phi_0(x) \tag{7.7.17}$$

where :

$$\phi'(x) = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(a(x) + i\chi(x)) \end{pmatrix} \tag{7.7.18}$$

again using an SU(2)XU(1) transform (U(x)) we can write :

$$\phi'(x) = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h(x) \end{pmatrix} \; \langle h(x)\rangle = 0 \tag{7.7.19}$$

This allows us to write:

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} \tag{7.7.20}$$

the so called unitary gauge.
Considering the potential term :

$$\mathcal{L}_V = -m^2 \mid \phi \mid^2 - \lambda \mid \phi \mid^2 \tag{7.7.21}$$

We know:

$$\mid \phi \mid^2 = \phi\phi^\dagger = \frac{1}{2}(h(x)+v)^2 \quad -m^2 = \mu^2 \tag{7.7.22}$$

and finding the minimum as in (section 7.1) as :

$$v = \sqrt{\frac{\mu^2}{\lambda}} \tag{7.7.23}$$

We write:

$$\begin{aligned} \mathcal{L}_V &= \frac{1}{2}\mu^2(h(x)+\sqrt{\frac{\mu^2}{\lambda}})^2 - \frac{1}{4}\lambda(h(x)+\sqrt{\frac{\mu^2}{\lambda}})^4 \\ &= \frac{\mu^2}{\lambda} - \mu^2 h^2 - \lambda v h^3 - \frac{1}{4}\lambda h^4 \end{aligned} \tag{7.7.24}$$

(This is similar to section 7.1) where calculations are done more explicitly If we consider $h(x)$ as the field that describes the Higgs boson. then remembering that scalar fields have mass terms of the form $-\frac{1}{2}m^2\phi^2$ in the lagrangian we see that h has a mass:

$$\sqrt{2}\mu = \sqrt{2\lambda}v \tag{7.7.25}$$

(compare $-\frac{1}{2}m^2\phi^2$ to $-\mu^2 h^2$)
We now consider the kinetic lagrangian:

$$\mathcal{L}_K = \mid D_\mu \phi \mid^2 \tag{7.7.26}$$

writing :

$$\phi = \phi' + \phi(0) \tag{7.7.27}$$

ignoring for the moment the switch to unitary gauge and recalling (7.7.3)

$$D_\mu = \partial_\mu - igYa_\mu - ig'T^i B^i_\mu \tag{7.7.28}$$

We realise that there will be terms that involve only gauge fields and constants since ϕ_0 is a constant vector coming from $\mid D_\mu \phi_0 \mid^2$
since $\partial_\mu \phi_0 = 0$ since ϕ_0 is a constant vector :

$$\mid D_\mu \phi_0 \mid^2 = (D_\mu \phi)^\dagger D^\mu \phi \tag{7.7.29}$$

We gain:

$$\Delta\mathcal{L} = (-igYa_\mu - ig'T^i B^i_\mu)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix})^\dagger(-igYa^\mu - ig'T^i B^{\mu i})\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{7.7.30}$$

using the definition of ϕ_0 (7.7.14):

$$\Delta\mathcal{L} = \frac{1}{2}(0,v)(gYa_\mu + g'T^iB_\mu{}^i)(gYa^\mu + g'T^iB^{\mu i})\begin{pmatrix}0\\v\end{pmatrix} \tag{7.7.31}$$

taking out a factor of -1 from each of the gauge terms and combining the two $\frac{1}{\sqrt{2}}$ factors from the ϕ_0 and recalling that the spin matrices are hermitian (this can be seen by recalling their form (1.3.14))and $ii^\dagger = 1$.
We can then write this in component form as (see Appendix G6) as :

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}[g'^2(B_\mu^1)^2 + g'^2(B_\mu^2)^2 + 4(-\frac{g'}{2}B_\mu^3 + gYa^\mu)^2] \tag{7.7.32}$$

We can define :

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(B_\mu^1 \mp iB_\mu^2) \tag{7.7.33}$$

Thus :

$$W_\mu^+W^{\mu-} = \frac{1}{\sqrt{2}}(B_\mu^1 - iB_\mu^2)\frac{1}{\sqrt{2}}(B^{1\mu} + iB^{2\mu}) = \frac{1}{2}[(B_\mu^1)^2 + (B_\mu^2)^2] \tag{7.7.34}$$

using properties of four-vectors (1.6.4)
We note (see Appendix G7) that the Y is the U(1) charge associated with the scalar field $\frac{1}{2}$ in this case so that:

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}[g'^2W_\mu^+W^{\mu-} + (-g'B_\mu^3 + ga^\mu)^2] \tag{7.7.35}$$

Thus we define:

$$Z_\mu^0 = \frac{1}{\sqrt{g^2 + g'^2}}(g'B_\mu{}^3 - ga_\mu) \tag{7.7.36}$$

where we have normalised the field
and its its orthorgonal companion

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'B_\mu{}^3 + ga_\mu) \tag{7.7.37}$$

Now:

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}[g'^2 2W_\mu^+W^{\mu-} + (g^2 + g'^2)(Z_\mu^0)^2] \tag{7.7.38}$$

remembering that Boson mass terms are of the form $\frac{1}{2}m^2(\phi_\mu)^2$ (section 2.4) we see that :

$$m_{W^\pm} = g'\frac{v}{2} \tag{7.7.39}$$

(compare $\frac{1}{2}m_{W^\pm}^2$ and $\frac{1}{2}\frac{v^2}{4}g'^2$)

$$m_{Z^0} = \sqrt{g^2 + g'^2}\frac{v}{2} \tag{7.7.40}$$

(compare $\frac{1}{2}m_{Z^0}^2$ and $\frac{1}{2}\frac{v^2}{4}(g'^2+g^2)$)

$$m_A = 0 \qquad (7.7.41)$$

as it does not appear
Thus

$$\Delta\mathcal{L} = m_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2}m_Z^2(Z_\mu^0)^2 \qquad (7.7.42)$$

We can write the relation between Z^0 A and B^3, a in terms of a mixing angle θ_W :

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} cos\theta_W & -sin\theta_W \\ sin\theta_W & cos\theta_W \end{pmatrix} \begin{pmatrix} B^3 \\ a \end{pmatrix} \qquad (7.7.43)$$

where :

$$cos\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad (7.7.44)$$

$$sin\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \qquad (7.7.45)$$

The identities:

$$Q = T^3 + Y \qquad (7.7.46)$$

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} \qquad (7.7.47)$$

and :

$$T^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) \qquad (7.7.48)$$

We can rewrite the co-variant derivative in terms of the fields and the two variables e , θ_W (See appendix G7) :

$$\begin{aligned} D_\mu = \partial_\mu \quad & - \quad i\frac{g'}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) \\ & - \quad i\frac{g'}{cos\theta_W} Z_\mu(T^3 - sin^2\theta_W Q) - ieA_\mu Q \end{aligned} \qquad (7.7.49)$$

where :

$$g' = \frac{e}{sin\theta_W} \qquad (7.7.50)$$

Given that $cos\theta_W$ by definition is the ratio between the mass of the W's and the mass of Z the two are related by:

$$m_W = cos\theta_W m_Z \qquad (7.7.51)$$

In our unitary gauge version of ϕ (7.7.20)

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h(x)+v \end{pmatrix} \qquad (7.7.52)$$

the derivative of v vanishes since v is constant:

$$| D_\mu \phi |^2 = \partial_\mu \phi^\dagger \partial^\mu \phi + \phi^\dagger (g' B^a_\mu T^a + gYa_\mu)(g' B^{a\mu} T^a + gYa^\mu)\phi \tag{7.7.53}$$

$$\partial_\mu \phi = \partial_\mu \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu h(x) \end{pmatrix} \tag{7.7.54}$$

so :

$$\partial_\mu \phi^\dagger \partial^\mu \phi = \frac{1}{2} (\partial_\mu h)^2 \tag{7.7.55}$$

The vector-part of our gauge co-variant derivative transforms the same way for h(x) +v as it did above for v^2. since we know the transformation for v^2 we can divide through by v^2 and multiply by $(h(x) + v)^2$ to give (following (7.7.42)) :

$$\Delta \mathcal{L} = \frac{1}{v^2} [m_W^2 W^+_\mu W^{\mu -} + \frac{1}{2} m_Z^2 (Z^0_\mu)^2] (h(x) + v)^2 \tag{7.7.56}$$

and thus in unitary gauge,following (7.7.42)

$$\mathcal{L} = \frac{1}{2} (\partial_\mu h)^2 + [m_W^2 W^+_\mu W^{\mu -} + \frac{1}{2} m_Z^2 (Z^0_\mu)^2] (\frac{h(x)}{v} + 1)^2 \tag{7.7.57}$$

We now consider the fermion terms.
Considering the quark terms first.
i.e.:

$$-\lambda_d^{ij} \overline{Q}^i_L \phi d^j_R - (\lambda_d^{ij} \overline{Q}^i_L \phi d^j_R)^\dagger - \lambda_u^{ij} \epsilon^{ab} \overline{Q}^i_{La} \phi^\dagger_b u^j_R - (\lambda_d^{ij} \overline{Q}^i_{La} \phi^\dagger_b d^j_R)^\dagger \tag{7.7.58}$$

P transforms swap left and right-handed fermions, C transforms charge plus to negative charges. Thus if we impose CP as a symmetry of the lagrangian for the fermion -scalar interaction terms , the terms after the coefficients are interchanged with their hermitian conjugates l11-12 p722 [9] . Since in the lagrangian the hermitian conjugates already appear the transform is just equivalent to:

$$\lambda_d^{ij} \rightarrow (\lambda_d^{ij})^* \qquad \lambda_u^{ij} \rightarrow (\lambda_u^{ij})^* \tag{7.7.59}$$

If we define two unitary matrices U_u and W_u by:

$$\lambda_u \lambda_u^\dagger = U_u D_u^2 U_u^\dagger \tag{7.7.60}$$

$$\lambda_u^\dagger \lambda_u = W_u D_u^2 W_u^\dagger \tag{7.7.61}$$

D_u^2 is a diagonal matrix with positive eigenvalues
Thus (eq 20.137 eq722 [9]):

$$\lambda_u = U_u D_u W_u^\dagger \tag{7.7.62}$$

by exact analogue :

$$\lambda_d = U_d D_d W_d^\dagger \tag{7.7.63}$$

where U_d, D_d are defined analogously
We now make the transforms:

$$u_R^i \to W_u^{ij} u_R^j \ d_R^i \to W_u^{ij} d_R^j \tag{7.7.64}$$

and

$$u_L^i \to U_u^{ij} u_L^j \ d_L^i \to U_u^{ij} d_L^j \tag{7.7.65}$$

which we can do by the gauge freedom for the U(1) group
in the unitary gauge (7.7.20)

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} \tag{7.7.66}$$

Writing:

$$-(\lambda_u^{ij} \epsilon^{ab} \overline{Q}_L^i \phi u_R^j)^\dagger - \lambda_d^{ij} \overline{Q}_{La}^i \phi_b^\dagger d_R^j - (\lambda_d^{ij} \overline{Q}_{La}^i \phi_b^\dagger d_R^j)^\dagger \tag{7.7.67}$$

as :

$$-\lambda_d^{ij} \overline{Q}_L^i \phi d_R^j - \lambda_u^{ij} \epsilon^{ab} \overline{Q}_{La}^i \phi_b^\dagger u_R^j + h.c. \tag{7.7.68}$$

where h.c. denotes the hermitian conjugate :

$$-\lambda_d^{ij} \overline{Q}_L^i \phi d_R^j - \lambda_u^{ij} \epsilon^{ab} \overline{Q}_{La}^i \phi_b^\dagger u_R^j + h.c. \tag{7.7.69}$$

remembering: $\epsilon^{11} = \epsilon^{22} = 0$

$$-\lambda_d^{ij} ({u_L^i}^\dagger, {d_L^i}^\dagger) \gamma^0 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} d_R^j - \lambda_u^{ij} \epsilon^{ab} (u^{i\dagger}, d^{i\dagger})_{La} \gamma^0 (0, v)_b u_R^j + h.c. \tag{7.7.70}$$

since the 1 component of (0,v) is 0 we need a=1, b=2 :

$$-\lambda_d^{ij} (\overline{u}_L^i, \overline{d}_L^i) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} d_R^j - \lambda_u^{ij} \epsilon^{12} (u^{i\dagger}, d^{i\dagger})_{L1} \gamma^0 \frac{1}{\sqrt{2}} (0, h(x)+v)_2 u_R^j + h.c. \tag{7.7.71}$$

which becomes:

$$-\lambda_d^{ij} \overline{d}_L^i d_R^j \frac{h(x) + v}{\sqrt{2}} - \lambda_u^{ij} \overline{u}_L^i u_R^j \frac{h(x) + v}{\sqrt{2}} + h.c. \tag{7.7.72}$$

clearly we now have a coupling between the higgs field and the fermions and constant terms multiplying two vector like terms.
in preparation for later we can then write this as:

$$-\lambda_d^{ij} \frac{v}{\sqrt{2}} \overline{d}_L^i d_R^j (\frac{h(x)}{v} + 1) - \lambda_u^{ij} \frac{v}{\sqrt{2}} \overline{u}_L^i u_R^j (\frac{h(x)}{v} + 1) + h.c. \tag{7.7.73}$$

The transform $\overline{u}_L$ will give a $U_u^\dagger$ term and the transform $\overline{d}_L$ will give a $U_d^\dagger$
These transforms mean that :

$$\sum_i(\overline{u}_i(i\not{D}^2)u_R^i + \overline{d}_i(i\not{D}^2)d_R^i) \rightarrow \sum_i(\overline{u}_i(i\not{D}^2)u_R^i + \overline{d}_i(i\not{D}^2)d_R^i) \qquad (7.7.74)$$

and

$$\begin{aligned} - & \quad \lambda_d^{ij}\frac{v}{\sqrt{2}}\overline{d}_L^i d_R^j(\frac{h(x)}{v}+1) - \lambda_u^{ij}\frac{v}{\sqrt{2}}\overline{u}_L^i u_R^j(\frac{h(x)}{v}+1) + h.c. \\ \rightarrow & \quad -D_d^{ij}\frac{v}{\sqrt{2}}\overline{d}_L^i d_R^j(\frac{h(x)}{v}+1) - D_u^{ij}\frac{v}{\sqrt{2}}\overline{u}_L^i u_R^j(\frac{h(x)}{v}+1) + h.c. \end{aligned} \qquad (7.7.75)$$

since :

$$D_d^{ij} = D_u^{ij} = 0 \; j \neq i \qquad (7.7.76)$$

we have the only contributing terms being :

$$-D_d^{ii}\frac{v}{\sqrt{2}}\overline{d}_L^i d_R^i(\frac{h(x)}{v}+1) - D_u^{ii}\frac{v}{\sqrt{2}}\overline{u}_L^i u_R^i(\frac{h(x)}{v}+1) + h.c. \qquad (7.7.77)$$

which we can rewrite with mass terms :

$$-m_d^i\overline{d}_L^i d_R^i(\frac{h(x)}{v}+1) - m_u^i\overline{u}_L^i u_R^i(\frac{h(x)}{v}+1) + h.c. \qquad (7.7.78)$$

where

$$m_d^i = D_d^{ii}\frac{v}{\sqrt{2}} \qquad (7.7.79)$$

and :

$$m_u^i = D_u^{ii}\frac{v}{\sqrt{2}} \qquad (7.7.80)$$

as normal fermion mass terms are of the of the form $m\overline{\psi}\psi$
Obviously we can completely analogously for the lepton sector define

$$\lambda_l = U_l D_l W_l^\dagger \qquad (7.7.81)$$

the transforms:

$$e_L^i \rightarrow U_l^{ij}e_L^j \qquad \nu_L^i \rightarrow U_{l1}^{ij}\nu_L^j \qquad e_R^i \rightarrow U_l^{ij}e_R^j \qquad (7.7.82)$$

and thus our electron and neutron propagation terms are again unaffected and the mass term becomes:

$$-m_l^i\overline{e}_L^i e_R^i(\frac{h(x)}{v}+1) + h.c. \qquad (7.7.83)$$

where :

$$m_l^i = D_l^{ii}\frac{v}{\sqrt{2}} \qquad (7.7.84)$$

We now consider the operators $Q = T^3 + Y$. It is shown in appendix G7 that T^3 is a conserved charge combining this with the conservation of Y from above we deduce Q is conserved. if we associate Q with the electric charge p702 l19 [9] then the coupling on A_μ suggests that it should be identified with the photon if e is the electron charge, which now has the correct charge and mass and through the $\not{D}$ terms couples to all the fermions with strength Q (We ignore the difference between D^1 and D^2 as this is equivalent to $T^1, T^2, T^3 = 0$ the last of which we will use to get the charges for these). The right-handed fields have T^3 =0 as they are only invariant under the U(1) group, their charges are then given by assigning the value Y for each right-handed field Y=-1 for e^i_R , $Y = \frac{2}{3}$ u^i_R, $Y = -\frac{1}{3}$ d^i_R. For all our-right-handed fields $T^3 = \pm\frac{1}{2}$ for the top and bottom components respectively (as $T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) since Q= -1 for e^i_L then$Y = -\frac{1}{2}$ for the doublet (as it is a U(1) symmetry charge it should be the same for both components), this means the neutrino is chargeless as required. The choice $Y = \frac{1}{6}$ gives the correct charge for the quark components i.e. $u^i_L, Q = \frac{2}{3}$ $d^i_L, Q = -\frac{1}{3}$

Obviously we want this theory to be renormalisable, our consideration of the non-gauged constituents of the theory in section 6.3 leads us to suggest that is. it has been proved that this is indeed the case but we will not dwell on it here.The article [15] looks into proving the renomalisability of this and other similar theories.

We illustrate below the Feynman diagrams for the vertices:

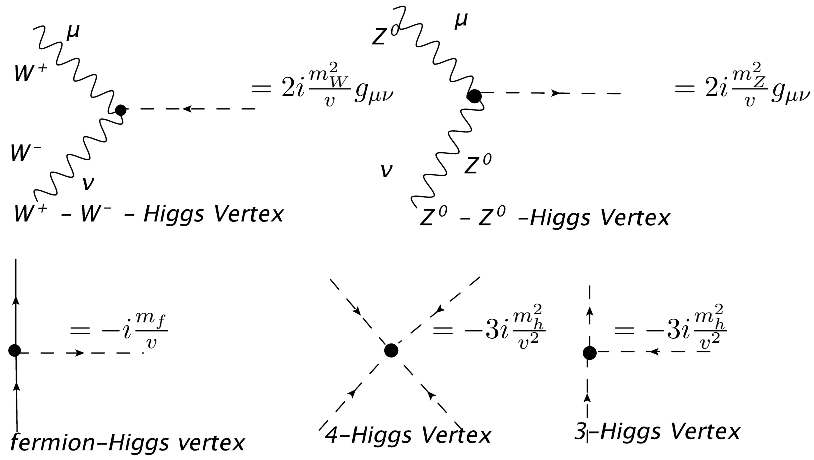


Figure 7.7.1: All possible couplings with the Higgs-boson in the unitary gauge. through the kinetic term it can couple to itself either cubicaly (with coefficient $\sqrt{\frac{\lambda}{2}}\frac{m_h}{v} = \frac{m_h^2}{2v}$, $m_h = \sqrt{2\lambda}v$)or quartically (with coefficient $\frac{\lambda}{4} = \frac{m_h^2}{8v^2}$) . It couples to the fermion fields with coefficient $\frac{m_f}{v}$ .It couples the Z_μ^0 field to the Z_ν^0 field with coefficient $\frac{m_Z^2}{v}g^{\mu\nu}$. Finally it couples the W_μ^- field to the W_ν^+ field with coefficient $2\frac{m_W^2}{v}g^{\mu\nu}$

We will consider the Feynman rules in more detail when we come to doing calculations for the Higgs Boson in the next chapter

7.8 Summary

In this section we have introduced the idea of spontaneously broken symmetries , and then introduced the goldstone theorem as a way of associating bosons with broken symmetries. We then introduced the idea of gauge theories in both their Abelian and non-Abelian forms , giving a summary of QED as an example of Abelian Gauge-invariant theory. Finally we demonstrate the Higgs mechanism in GWS theory

Chapter 8

Detecting the Higgs Boson

8.1 Higgs Boson Detection Paths

At the time of writing, the Higgs boson predicted above has not been detected. The Boson itself will be short-lived because there are lower energy particles the lagrangian allows it to decay into, it is also uncharged and therefore will not interact with electro-magnetic fields. This means that the best way of detecting the Higgs is by waiting for it to decay into more detectable products and detecting these products which will have a given energy, momentum and angular distribution dependent on the Higgs. The immediate products of the decay of the Higgs may still be difficult to detect particles e.g. the top quark and thus we may have to under-go a series of decays to obtain measurable products. This is called the decay path

There are several different ways the Higgs can decay, in assessing which decay path to use there are obviously two key properties that it must have:
1) the decay products must be distinguishable from the background , i.e. we must our measurable products must occur at a much greater frequency then can be be expected from similar processes (i.e. the angular range or momentum or energy range is indistinguishable) natural background radiation or other processes involved in the experiment and the products
2) The decay products must occur relatively frequently, due to the inherent error in experimental processes and probabilistic nature of quantum mechanics we want a statistically significant sample of decay events and thus they must occur relatively frequently

As can be seen different Higgs decay channels produce different results for different Higgs masses. We need to select a decay channel that gives measurable results.

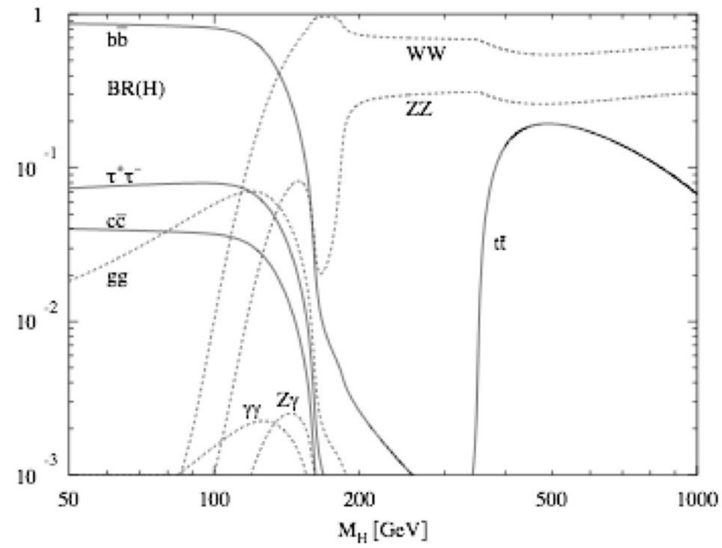


Figure 2: *Total decay width $\Gamma(H)$ in GeV and the main branching ratios $BR(H)$ of Standard Model Higgs decay channels, using the inputs of Tab. 2.*

Figure 8.1.1: (Figure 2 [16] , the rest of the article details the program used for producing this model) Diagram showing different decay widths for the main branching ratios of the standard Model Higgs decay channels at different Higgs masses.

The branching ratio for a given decay channel is defined as the ratio between two decay widths in this case the branching ratio for Higgs decaying to ZZ would be the ratio between the decay width for $H \rightarrow ZZ$ and the decay width for all possible decays of the Higgs (see p 392 eq12.151 [20] for the $Z \rightarrow \mu^+\mu^-$) :

$$B(ZZ) = \frac{\Gamma(H \rightarrow ZZ)}{\Gamma(H \rightarrow all)} \tag{8.1.1}$$

where Γ is the indicated decay width
The decay width for a given decay channel is defined as the inverse of the mean lifetime (the time required for the number of original particles to reduce to $\frac{1}{e}$ of its original value . The differential decay width can be obtained by taking the probability of particles with given momentum propagating to particles with other given momentum (the amplitude squared) divided by the transition time and then multiplied by the number of momentum states in momentum space for each momentum (a simple example is given in Appendix:Decay rate and cross-section p139 [8])

8.2 The LHC and Detecting the Higgs Boson

There already exist experimental bounds on the possible mass of the Higgs Boson. Initially we have a have a lower bound for the Higgs mass of 114.4 $\frac{GeV}{c^2}$ at 95% certainty level given by LEP experiments [17], we then have an upper bound given by 185$\frac{GeV}{c^2}$ by analysis of other known paramaters (see figure below and section 5 of [18] for details) , finally experiments at Tevatron have excluded Higgs masses in the range 158 $\frac{GeV}{c^2}$ - 175$\frac{GeV}{c^2}$ at 95% certainity level (l9-11 [19]).

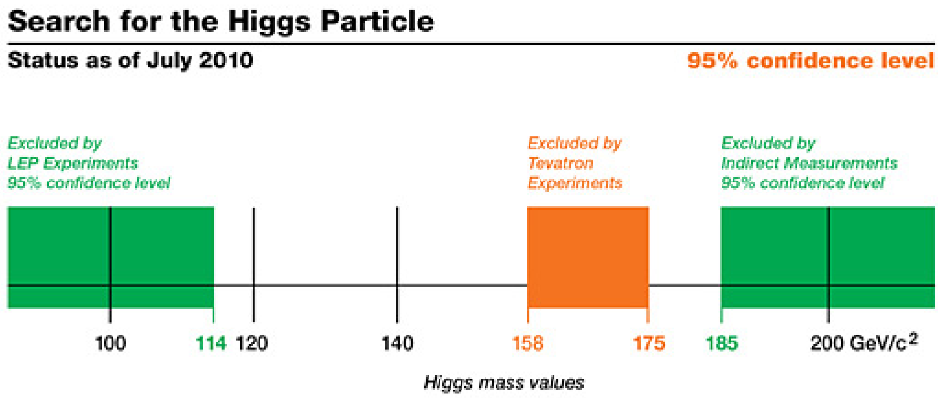


Figure 8.2.1: (top Figure) [19])This graph shows the possible range for the Higgs-mass, it shows the excluded regions and those yet to be probed

The LHC (Large Hadron Collider) is an experimental apparatus, which has detecting the Higgs Boson as one of its key aims. Physically it is a torus of circumference 27km(line 7 [21])which is cryogenically cooled to a temperature of 1.9K(line 12 [21]) it collides together two counter-rotating beams of protons with a centre of mass energy up to 14TeV (twice the beam energy l10 [21]) . The collisions produce the excess energy which is needed considering energ-mass conservation to make the Higgs Boson.These collisions are arranged so that they take place at one of four particle detectors positioned around the ring of these ATLAS and CMS are the ones relevant to detecting the Higgs Boson [22] .

The LHC should be able to discover or exclude all remaining possible masses for the standard model Higgs Boson and thus prove or disprove its existence. (See [22] section 2 l2-3)

Although the LHC uses protons and therefore the interactions are affected by the strong force which we have not considered we will take two useful properties of the LHC:

1) $H \to ZZ^* \to 4\mu$ is a"gold-plated" channel for the LHC , i.e is amongst the cleanest ways to get a clear signal of the presence of a Higgs Boson (see the section on intermediate and high mass of section 1.2.1.2 [23], section 1.2.1.1 and section 1.2.1.2 [23] contains a nice analysis of how different decays occur at different energies , "golden " is used in this reference rather then the more common"gold-plated ")
2) The outer detection layers of CMS is made up of muon detectors (see figure 1.8 [23]) making muons good candidate for this detector

8.3 Grassmann numbers

Grassmann numbers are numbers that anti-commute i.e. for two Grassmann numbers θ and η (eq9.62 p299 [9]):

$$\theta\eta = -\eta\theta \tag{8.3.1}$$

This means that :

$$\theta^2 = -\theta^2 \tag{8.3.2}$$

which gives:

$$2\theta^2 = 0 \tag{8.3.3}$$

i.e.:

$$\theta^2 = 0 \tag{8.3.4}$$

This means that the taylor expansion is curtailed after the linear term and so any Grassmann function can be taylor expanded to be written as:

$$f(\theta) = A + B\theta \tag{8.3.5}$$

A and B normal constant numbers which commute with Grassmann numbers. We consider the integral :

$$\int d\theta f(\theta) = \int d\theta A + B\theta \tag{8.3.6}$$

if this is invariant under a shift :

$$\theta \to \theta + \eta \tag{8.3.7}$$

η independent of θ
we gain:

$$\int d\theta A + B\theta = \int d\theta (A + B\eta) + B\theta \tag{8.3.8}$$

assuming normal integration:

$$\int d\theta A + B\theta = A\theta + B\frac{\theta^2}{2} + C \tag{8.3.9}$$

where C is a constant
The $\theta^2 = 0$ as this is a general property of Grassmann numbers (8.3.4). The integration gives different linear terms on each side so it can not be linear and therefore the integration is given by a constant
choosing :

$$\int d\theta\theta = 1 \tag{8.3.10}$$

We gain (eq 9.63 p300 [9]):

$$\int d\theta(A + B\theta) = B \tag{8.3.11}$$

We also define(eq9.64 p300 [9]):

$$\int d\theta d\eta\eta\theta = +1 \tag{8.3.12}$$

We can treat θ^* and θ as different complex Grassmann numbers
We consider:

$$\int d\theta^* d\theta e^{-\theta^* b\theta} \tag{8.3.13}$$

using the taylor expansion the exponential :

$$e^{-\theta^* b\theta} = 1 - \theta^* b\theta \tag{8.3.14}$$

again curtailing after the squared term
hence:

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = \int d\theta^* d\theta(1 - \theta^* b\theta) \tag{8.3.15}$$

as :

$$\theta^* b\theta = -b\theta\theta^* \tag{8.3.16}$$

so :

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = \int d\theta^* d\theta(1 + b\theta\theta^*) \tag{8.3.17}$$

as we found that :

$$\int d\theta 1 = 0 \tag{8.3.18}$$

and (8.3.12):

$$\int d\theta d\eta\eta\theta = +1 \tag{8.3.19}$$

$$\int d\theta^* d\theta(1 + b\theta\theta^*) = b \tag{8.3.20}$$

Thus :

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = b \tag{8.3.21}$$

we consider the n-dimensional Grassmann vector which has components ϕ_i i=1,...,n
We consider a unitary transform of ϕ :

$$\phi'_i = U_{ij}\phi_j \tag{8.3.22}$$

$$\sum_i \theta'_i = \frac{1}{n!}\epsilon^{ij...l}\theta'_i\theta'_j....\theta'_l \tag{8.3.23}$$

where permuting the θ 's picks up a factor of -1 as they are Grassmann numbers which are cancelled by permuting the variables in the epsilon. Thus all the permutations are equivalent and thus equivalent $\sum_i \theta'_i$ to there are n! possible permutations which explains the n!
We use the transform (8.3.22) $\phi'_i = U_{ij}\phi_j$, to give :

$$\sum_i \theta'_i = \frac{1}{n!}\epsilon^{ij...l}Uii'\theta_{i'}Ujj'\theta_{j'}....Ull'\theta_{l'} \tag{8.3.24}$$

$$(\epsilon^{i'j'k'...l'})^2 = 1 \tag{8.3.25}$$

without implicit summation if the i' j' k' ...l' are all different

$$(\epsilon^{i'j'k'...l'})^2 = n! \tag{8.3.26}$$

with implicit summation as there are n! ways of permuting i' j' k' ...l' all different and if they are not all different $\epsilon^{i'j'k'...l'} = 0$
So :

$$1 = \frac{(\epsilon^{i'j'k'...l'})^2}{n!} \tag{8.3.27}$$

So:

$$\begin{aligned}\sum_i \theta'_i &= \frac{1}{n!}\epsilon^{ij...l}U_{ii'}U_{jj'}....U_{ll'}\theta_{i'}\theta_{j'}....\theta_{l'}\frac{(\epsilon^{i'j'k'...l'})^2}{n!} \\ &= \frac{1}{n!}\epsilon^{ij...l}\epsilon^{i'j'k'...l'}Uii'Ujj'....Ull'\sum_i \theta_i \end{aligned}\tag{8.3.28}$$

using an analogous definition for $\sum_i \theta'_i$ as for $\sum_i \theta_i$ Using:

$$detU = \frac{1}{n!}\epsilon^{ij...l}\epsilon^{i'j'k'...l'}Uii'Ujj'....Ull'[] \tag{8.3.29}$$

Thus :

$$\sum_i \theta'_i = detU\sum_i \theta_i \tag{8.3.30}$$

We consider the hermitian matrix B .
It is known that a hermitian matrix can be written :

$$B = UDU^\dagger \tag{8.3.31}$$

where D is the diagonal matrix of eigenvalues [].
We then consider the analogue of the Gaussian integral in Grassmann numbers:

$$(\prod_i \int d\theta_i^* d\theta_i) e^{-\theta_i^* B_{ij} \theta_j} \tag{8.3.32}$$

We transform θ to $U\theta$. The surviving factor in each exponential is $\phi\phi^*$ the product over these transforms to pick up a factor of $det(U)(detU)^*$.
This is also true of the derivative so we can use the transformed variables instead
Thus:

$$\theta^T B\theta = \theta^T UDU^\dagger \theta \tag{8.3.33}$$

using the appropriate unitary transform on θ gives :

$$\theta^T B\theta = \sum_i \theta_i^* b_i \theta_i \tag{8.3.34}$$

where b_i are the eigenvalues of B
Thus (8.3.32) becomes:

$$(\prod_i \int d\theta_i^* d\theta_i) e^{-\sum_i \theta_i^* b_i \theta_i} \tag{8.3.35}$$

using the property of exponentials we can bring the exponential inside to give:

$$\prod_i \int d\theta_i^* d\theta_i e^{-\theta_i^* b_i \theta_i} \tag{8.3.36}$$

since each:

$$\int d\theta_i^* d\theta_i e^{-\theta_i^* b_i \theta_i} = b_i \tag{8.3.37}$$

using the normal exponential (8.3.21) :

$$(\prod_i \int d\theta_i^* d\theta_i) e^{-\sum_i \theta_i^* b_i \theta_i} = \prod_i b_i = det(B) \tag{8.3.38}$$

since the determinant of any matrix is the product of its eigenvalues

The continuum analogue of this is:

$$det(B) = \int \mathcal{D}c\mathcal{D}\bar{c} exp[-i \int d^4 x \bar{c} B c] \tag{8.3.39}$$

as c^* is equivalent to $\bar{c}$ as these are dummy variables
and $\mathcal{D}\bar{c}$ and $\mathcal{D}c^*$ are unitarily equivalent l3 p302 [9]
Knowing we can write the Dirac fields(since they anti-commute as (eq 9.71 p301 [9]):

$$\psi(x) = \sum_i \psi_i \phi_i(x) \tag{8.3.40}$$

where ψ_i are Grassmann numbers and the ϕ_i are a basis of four-component spinors

8.4 Faddeev-Popov Ghosts

Similarly to before (Section 7.4) we define :

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(\phi^1 - i\phi^2) \\ v + (h + i\phi^3) \end{pmatrix} \tag{8.4.1}$$

and also define (eq 21.35 p739 [9]) :

$$\phi_i(x) = \phi_{i0} + \chi_i(x) \tag{8.4.2}$$

with $A^a_\mu = B^a_\mu$ a=1,2,3 $A^Y = a_\mu$
we now consider :

$$Z = \int \mathcal{D}A\chi e^{i\int \mathcal{L}[A,\chi]} \tag{8.4.3}$$

We introduce the definition (eq 16.23 p512 [9]) :

$$1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \tag{8.4.4}$$

which is just the continuum analogue of the definition of delta with a switch of variables from G to α
G is taken to be a gauge-fixing condition i.e. some condition such that G=0 fixes the gauge
Assuming G is linear in α and $det(\frac{\delta G(A^\alpha)}{\delta\alpha})$ is independent of A_μ we can write :

$$Z = det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \int \mathcal{D}\alpha \int \mathcal{D}A\chi e^{i\int \mathcal{L}[A,\chi]}\delta(G(A^\alpha)) \tag{8.4.5}$$

Obviously we can write $\mathcal{L}[A,\chi]$ by changing the gauge , our transformed vector is just our original boson plus a linear shift and unitary rotations l9-13 p513 [9] . This means the measure is preserved so we can write :

$$Z = det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \int \mathcal{D}\alpha \int \mathcal{D}A^\alpha\chi e^{i\int \mathcal{L}[A^\alpha,\chi]}\delta(G(A^\alpha)) \tag{8.4.6}$$

and then eliminate the α 's by changing the dummy variable from A^α to A :

$$Z = det(\frac{\delta G(A^\alpha)}{\delta\alpha})(\int \mathcal{D}\alpha)\int \mathcal{D}A\chi e^{i\int\mathcal{L}[A,\chi]}\delta(G(A)) \tag{8.4.7}$$

Thus $(\int \mathcal{D}\alpha)$ just becomes part of the normalisation
We now think of a gaussian weighting function centred on w=0 , width $\sqrt{\epsilon}$ and rewrite the delta as:

$$\delta(G - \sqrt{\epsilon}w) \tag{8.4.8}$$

we get :

$$\begin{aligned} Z &= det(\frac{\delta G(A^\alpha)}{\delta\alpha})C\int \mathcal{D}wexp[-i\int dx^4\frac{w^2}{2\epsilon}] \\ &\times \int \mathcal{D}A\chi e^{i\int\mathcal{L}[A,\chi]}\delta(G(A) - \sqrt{\epsilon}w(x)) \end{aligned} \tag{8.4.9}$$

C a normalisation constant
using the δ to eliminate the w integral we get :

$$Z = det(\frac{\delta G(A^\alpha)}{\delta\alpha})C\int \mathcal{D}A\chi e^{i\int dx^4\mathcal{L}[A,\chi]-\frac{G^2}{2\epsilon}} \tag{8.4.10}$$

We define the R_ϵ gauge (eq 21.45 p741[9]) :

$$G^a = \frac{1}{\sqrt{\epsilon}}(\partial_\mu A^{a\mu} - \epsilon g F^a{}_i\chi_i) \tag{8.4.11}$$

where (eq21.36 p739 [9]):

$$F^a{}_i = T^a_{ij}\phi_{0j} \tag{8.4.12}$$

where (l16 p740 [9]):

$$T^a = -i\frac{\sigma^a}{2}\ T^Y = -iY = -i\frac{1}{2} \tag{8.4.13}$$

where $Y = \frac{1}{2}$ as we are looking at the scalar doublet
We know the transform of the gauge fields is (eq 21.31 p739 [9]):

$$\delta A^a_\mu = \frac{1}{g}(D_\mu\alpha)^a \tag{8.4.14}$$

comparing with our transforms above (7.3.15) , (7.4.20) comparing the co-variant derivatives above (7.3.21) , (7.4.10) we have (eq21.33 p739 [9]):

$$D_\mu\phi_i = \partial_\mu\phi_i + gA^a_\mu T^a_{ij}\phi_j \tag{8.4.15}$$

using our χ we get:

$$\begin{aligned} D_\mu\phi_i &= \partial_\mu(\chi_i + \phi_{0i}) + gA^a_\mu T^a_{ij}(\chi_j + \phi_{0j}) \\ &= \partial_\mu(\chi_i) + gA^a_\mu T^a_{ij}\chi_j + gA^a_\mu F^a_i \end{aligned} \tag{8.4.16}$$

using our definition of F (8.4.12)
Our general lagrangian as in eq 21.32 p729 [9] is:

$$\mathcal{L} = \frac{-1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D\phi)^2 - V(\phi) \tag{8.4.17}$$

So considering only the quadratic terms in the Lagrangian as these give the mass terms and as shown before the propagtaors, higher order terms give interactions
Thus our squared terms are :

$$-\frac{1}{4}(F^a_{\mu\nu})^2 \tag{8.4.18}$$

$$\mid D_\mu\phi \mid^2 \tag{8.4.19}$$

$$-\frac{1}{2}M_{ij}\chi_i\chi_j \tag{8.4.20}$$

which is the mass squared term coming from the potential term
as our mass terms are the $\chi_i\chi_j$ terms in the potential and our potential is:

$$V(\phi) = \frac{1}{2}m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{8.4.21}$$

we pick up the required χ terms from the ϕ^2 coefficients in the m^2 term and from the $\lambda(\phi^\dagger\phi)^2$ which we can write as $\lambda((\chi+\phi_0)^\dagger(\chi+\phi_0))^2$ and thus there will be a term of the form $\lambda\phi_0^\dagger\phi_0\chi^\dagger\chi$
Thus (p740 eq 21.41 [9]):

$$M_{ij} = \frac{\partial^2}{\partial\phi_i\partial\phi_j}V(\phi) \mid_{\phi_0} \tag{8.4.22}$$

$$\begin{aligned} \mid D_\mu\phi \mid^2 &= \frac{1}{2}(\partial_\mu(\chi_i) + gA^a_\mu T^a_{ij}\chi_j + gA^a_\mu F^a_i) \\ &\times \ (\partial^\mu(\chi_j) + gA^{a\mu}T^a_{ji}\chi_i + gA^{a\mu}F^a_j) \end{aligned} \tag{8.4.23}$$

which has squared terms:

$$\mid D_\mu\phi \mid^2 = \frac{1}{2}(\partial_\mu\chi)^2 + g\partial^\mu\chi^i A^a_\mu F^a_i + \frac{1}{2}g^2F^a_jF^b_jA^a_\mu A^{\mu b} \tag{8.4.24}$$

considering $A^a_\mu A^\mu b$ as afermion term which requires a mass coefficient we can identify the mass:

$$m^2_{ab} = g^2F^a_jF^b_j \tag{8.4.25}$$

$$-\frac{1}{4}(F^a_{\mu\nu})^2 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \tag{8.4.26}$$

$\partial_\mu A^a_\nu\partial^\mu A^{\nu a}$ is equivalent to $\partial_\nu A^a_\mu\partial^\nu A^{\mu a}$ by changing the names of the summation variables

$\partial_\mu A^a_\nu \partial^\nu A^{\mu a}$ is equivalent to $\partial_\nu A^a_\mu \partial^\mu A^{\nu a}$ by changing the names of the summation variables

$$\partial_\mu A^a_\nu \partial^\mu A^{\nu a} \tag{8.4.27}$$

is equivalent to :

$$-A^a_\nu \partial_\mu \partial^\mu A^{\nu a} \tag{8.4.28}$$

by integration by parts and ignoring the boundary terms as A goes to 0 at the boundaries wich is

$$-A^a_\nu \partial^2 A^{\nu a} \tag{8.4.29}$$

using the definition from ∂^2 (the vector ∂ and the (1.6.4))
we know the identity is considering (1.6.1) :

$$g^{\mu\nu} g_{\mu\nu} = I \tag{8.4.30}$$

Thus our term becomes:

$$-A^a_\nu \partial^2 g^{\mu\nu} g_{\mu\nu} A^{\nu a} \tag{8.4.31}$$

which is :

$$-A^a_\nu \partial^2 g^{\mu\nu} A_\mu{}^a \tag{8.4.32}$$

$$\partial_\mu A^a_\nu \partial^\nu A^{\mu a} \tag{8.4.33}$$

integrates by parts to :

$$-A^a_\nu \partial_\mu \partial^\nu A^{\mu a} \tag{8.4.34}$$

again ignoring the boundary terms:

$$-A^a_\nu \partial^\nu \partial_\mu A^{\mu a} \tag{8.4.35}$$

as the partial derivatives commute .
using the properties of four-vectors that $a^\mu b_\mu = a_\mu b^\mu$ (easily seen from our definitions in Section 1.6)this becomes:

$$-A^a_\nu \partial^\nu \partial^\mu A_\mu{}^a \tag{8.4.36}$$

So in:

$$-\frac{1}{4}(F^a_{\mu\nu})^2 \tag{8.4.37}$$

The Lagrangian square terms becomes:

$$\begin{aligned} - & \frac{1}{2} A^a_\mu (-g^{\mu\nu}\partial^2 + \partial^\mu \partial^\nu) A^a_\nu + \frac{1}{2}(\partial_\mu \xi)^2 + g\partial^\mu \xi_i + g\partial^\mu \chi_i A^a_\mu F^a_i \\ + & \frac{1}{2}(m_A)^{2ab} A^a_\mu A^{\mu b} - \frac{1}{2} M_{ij}\chi_i \chi_j \end{aligned} \tag{8.4.38}$$

Recall (8.4.11)

$$G^a = \frac{1}{\sqrt{\epsilon}}(\partial_\mu A^{a\mu} - \epsilon g F^a{}_i \chi_i) \tag{8.4.39}$$

Thus we get:

$$\begin{aligned} \frac{-1}{2}G^2 &= \frac{1}{\sqrt{\xi}}(\partial_\mu A^{a\mu} - \xi g F_i^a \chi_i)\frac{1}{\sqrt{\xi}}(\partial_\nu A^{a\nu} - \xi g F_i^a \chi_i) \\ &= -\frac{1}{2\xi}(\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} - 2\partial_\mu A^{a\mu} xig F_i^a \chi_i + \xi^2 g^2 [F_i^a \chi_i]^2) \end{aligned} \tag{8.4.40}$$

which is:

$$\frac{-1}{2}G^2 = -\frac{1}{2\xi}(\partial_\mu A^{a\mu}\partial_\nu A^{a\nu}) + g\partial_\mu A^{a\mu} F_i^a \chi_i - \frac{1}{2}\xi g^2 [F_i^a \chi_i]^2 \tag{8.4.41}$$

using the property of four vectors that $a^\mu b_\mu = a_\mu b^\mu$ this becomes:

$$\frac{-1}{2}G^2 = -\frac{1}{2\xi}(\partial_\mu A^{a\mu}\partial^\nu A^a_\nu) + g\partial_\mu A^{a\mu} F_i^a \chi_i - \frac{1}{2}\xi g^2 [F_i^a \chi_i]^2) \tag{8.4.42}$$

if this is inside the integral we can we can integrate by parts and ignore the boundary terms to get:

$$\frac{-1}{2}G^2 = \frac{1}{2\xi}(A^{a\mu}\partial_\mu \partial^\nu A^a_\nu) + g\partial_\mu A^{a\mu} F_i^a \chi_i - \frac{1}{2}\xi g^2 [F_i^a \chi_i]^2 \tag{8.4.43}$$

since from our definition (8.4.12) F_i^a is constant
we can write this as :

$$\frac{-1}{2}G^2 = \frac{1}{2\xi}(A^{a\mu}\partial_\mu \partial^\nu A^a_\nu) - g A^{a\mu} F_i^a \partial_\mu \chi_i - \frac{1}{2}\xi g^2 [F_i^a \chi_i]^2 \tag{8.4.44}$$

the square terms in our constrained lagrangian $\mathcal{L} - \frac{1}{2}G^2$ are:

$$\begin{aligned} &- \frac{1}{2}A^a_\mu(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu)A^a_\nu + \frac{1}{2}(\partial_\mu \xi)^2 + g\partial^\mu \xi_i + g\partial^\mu \chi_i A^a_\mu F^a_i \\ &+ \frac{1}{2}(m_A)^{2ab} A^a_\mu A^{\mu b} - \frac{1}{2}M_{ij}\chi_i\chi_j + \frac{1}{2\xi}(A^{a\mu}\partial_\mu\partial^\nu A^a_\nu) - g A^{a\mu} F^a_i \partial_\mu \chi_i - \frac{1}{2}\xi g^2 [F^a_i \chi_i]^2 \end{aligned} \tag{8.4.45}$$

which is :

$$\begin{aligned} &- \frac{1}{2}A^a_\mu([-g^{\mu\nu}\partial^2 + (1 - \frac{1}{\xi})\partial^\mu\partial^\nu]\delta^{ab} - g^2 F^a_i F^b_i g^{\mu\nu})A^b_\nu \\ &+ \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}(M_{ij} + \epsilon g^2 F^a_i F^a_j)\chi_i\chi_j \end{aligned} \tag{8.4.46}$$

where we have rewritten :

$$\frac{1}{2}(m_A^2)^{ab} A^a_\mu A^{a\mu} \tag{8.4.47}$$

as:

$$A^a_\mu \frac{1}{2} g^2 F^a_i F^b_i g^{\mu\nu} A^b_\nu \tag{8.4.48}$$

We now consider the term:

$$\frac{\delta G^a}{\delta \alpha^b} \tag{8.4.49}$$

our scalar fields have gauge transform(eq 21.30 p739 [9]):

$$\delta\phi_i = -\alpha^a(x) T^a_{ij}\phi_j \tag{8.4.50}$$

as the variation of the field has to cancel the variation of the gauge transformatiom multiplying the field
as (8.4.11):

$$G^a = \frac{1}{\sqrt{\xi}}(\partial_\mu A^{a\mu} - \epsilon g F^a_i \chi_i) \tag{8.4.51}$$

The variation is then :

$$\delta G^a = \frac{1}{\sqrt{\xi}}(\partial_\mu \delta A^{a\mu} - \epsilon g F^a_i \delta\chi_i) \tag{8.4.52}$$

χ and ϕ differ by only a constant so :

$$\delta\chi_i = -\alpha^a(x) T^a_{ij}\phi_j \tag{8.4.53}$$

as stated above (**??**):

$$\delta A^{a\mu} = \frac{1}{g}(D_\mu \alpha)^a \tag{8.4.54}$$

so:

$$\delta G^a = \frac{1}{\sqrt{\xi}}(\partial_\mu \frac{1}{g}(D_\mu \alpha)^a - \epsilon g F^a_i(-\alpha^b(x) T^b_{ij}\phi_j)) \tag{8.4.55}$$

which is:

$$\delta G^a = \frac{1}{\sqrt{\xi}}(\partial_\mu \frac{1}{g}(D_\mu \alpha)^a + \epsilon g (T^a\phi_0)\alpha^b(x) T^b(\chi + \phi_0)) \tag{8.4.56}$$

replacing the F in terms of T. So:

$$\frac{\delta G^a}{\delta \alpha^b} = \frac{1}{\sqrt{\xi}}(\partial_\mu \frac{1}{g}(D_\mu)^{ab} + \epsilon g (T^a\phi_0).T^b(\xi + \phi_0)) \tag{8.4.57}$$

we showed above for the hermitian matrix B (**??**):

$$det(B) = \int \mathcal{D}c\mathcal{D}\bar{c} exp[-\int d^4 x i\bar{c}Bc] \tag{8.4.58}$$

$\frac{\delta G^a}{\delta a^b}$ is hermitian so following Fadeev and Poppov we get:

$$det(\frac{\delta G^a}{\delta \alpha^b}) = \int \mathcal{D}c\bar{]} exp[i \int d^4 x \mathcal{L}_{ghost}] \tag{8.4.59}$$

where c and $\bar{c}$ are anti-commuting ghost fields and thus this term leads to a new set of Feynman diagrams for the ghost fields
our :

$$\mathcal{L}_{ghost} = -\bar{c}^a g\sqrt{\xi}\frac{\delta G^a}{\delta \alpha^b} c^b \tag{8.4.60}$$

where the $g\sqrt{\xi}$ is chosen through our normalisation of the fields to give the succinct expression
so:

$$\mathcal{L}_{ghost} = -\bar{c}^a(\partial_\mu(D_\mu\alpha)^a + \epsilon g^2(T^a\phi_0)\alpha^b(x)T^b(\xi+\phi_0))c^b \tag{8.4.61}$$

8.5 Feynman Diagrams

We consider here only a few of the feynman diagrams of GWS theory, this is for the sake of clarity the entire set of feynman diagrams is given in Appendix B pp509-512 [20]. We consider first the scalar fields from our overall expression (8.4.46) for the squared terms in the constrained Lagrangian it is clear that the only part involving the scalar fields is :

$$\frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}(M_{ij} + \epsilon g^2 F_i^a F_j^a)\chi_i\chi_j \tag{8.5.1}$$

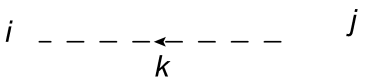
which considering (2.2.1) is just a complex scalar field with mass term:

$$M_{ij} + \epsilon g^2 F_i^a F_j^a \tag{8.5.2}$$

we showed in (8.5.3) that for a normal complex scalar field the propagator is:

$$G_F = \frac{-i\hbar}{-k_\mu k^\mu + m^2 - i\epsilon} \tag{8.5.3}$$

Thus we have the Feynman diagram eq 21.53 p742 [9]:



$$= \left(\frac{i}{k^2 - \xi g^2 F^T F - M^2}\right)_{ij}$$

Figure 8.5.1: Propagator for the scalars associated with the constrained Lagrangian

We now conside the Vector fields we have in (8.4.46) the squared terms :

$$\frac{1}{2}A_\mu^a([-g^{\mu\nu}\partial^2 + (1-\frac{1}{\xi})\partial^\mu\partial^\nu]\delta^{ab} - g^2 F_i^a F_i^b g^{\mu\nu})A_\nu^b \tag{8.5.4}$$

which is the propagation term for a vector boson with mass squared term :

$$g^2 F_i^a F_i^b g^{\mu\nu} \tag{8.5.5}$$

We found (D.10.49) in that the expression for the massless vector boson is:

$$\tilde{D}_{F\nu\rho}(k) = \frac{-i}{k^2 + i\epsilon}(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2}) \tag{8.5.6}$$

We thefore expect an analogous expression which is a Green's function of :

$$-([-g^{\mu\nu}\partial^2 + (1 - \frac{1}{\xi})\partial^\mu\partial^\nu]\delta^{ab} - g^2 F_i^a F_i^b g^{\mu\nu}) \tag{8.5.7}$$

we postulate that our propagator with down indices in momentum space is :

$$\left(\tfrac{-i}{k^2 - g^2 FF^T}(g_{\nu\rho} - (1-\xi)\tfrac{k_\nu k_\rho}{k^2 - \xi g^2 FF^T})\right)^{ab} \tag{8.5.8}$$

substiuting for a=b and changing the derivatives to momentum vectors as we are in momentum space we have (which also picks up a minus sign for each pair of k's):

$$-([g^{\mu\nu}k^2 - (1 - \frac{1}{\xi})k^\mu k^\nu] - g^2 FF^T g^{\mu\nu})\left(\tfrac{-i}{k^2 - g^2 FF^T}(g_{\nu\rho} - (1-\xi)\tfrac{k_\nu k_\rho}{k^2 - \xi g^2 FF^T})\right)$$

multiplying out we get:

$$\begin{aligned} i[g^{\mu\nu}g_{\nu\rho} &+ \frac{1}{k^2 - g^2 FF^T}[((k^2 - g^2 FF^T)g^{\mu\nu} + (1 - \frac{1}{\xi})k^\mu k^\nu)\frac{(1-\xi)k_\nu k_\rho}{k^2 - \xi g^2 FF^T} \\ &- g_{\nu\rho}(1 - \frac{1}{\xi})k^\mu k^\nu] \end{aligned} \tag{8.5.9}$$

which becomes using (1.6.4) :

$$\begin{aligned} i[g^{\mu\nu}g_{\nu\rho} &+ \frac{1}{k^2 - g^2 FF^T}[(-(k^2 - g^2 FF^T)k^\mu k_\rho + (1 - \frac{1}{\xi})k^\mu k_\rho k^2)\frac{(1-\xi)}{k^2 - \xi g^2 FF^T} \\ &- (1 - \frac{1}{\xi})k^\mu k_\rho] \end{aligned} \tag{8.5.10}$$

which factorises to:

$$\begin{aligned} i[g^{\mu\nu}g_{\nu\rho} &+ \frac{k^\mu k_\rho}{k^2 - g^2 FF^T}[(-(k^2 - g^2 FF^T) + (1 - \frac{1}{\xi})k^2)\frac{(1-\xi)}{k^2 - \xi g^2 FF^T} \\ &- (1 - \frac{1}{\xi})] \end{aligned} \tag{8.5.11}$$

cancelling k^2

$$\begin{aligned} i[g^{\mu\nu}g_{\nu\rho} &+ \frac{k^\mu k_\rho}{k^2 - g^2 FF^T}[-(-g^2 FF^T) - \frac{1}{\xi}k^2)\frac{(1-\xi)}{k^2 - \xi g^2 FF^T} \\ &- (1 - \frac{1}{\xi})] \end{aligned} \tag{8.5.12}$$

which is :

$$i[g^{\mu\nu}g_{\nu\rho} \quad + \quad \frac{k^{\mu}k_{\rho}}{k^2 - g^2FF^T}[-(\frac{1}{\xi} - 1)$$
$$- \quad (1 - \frac{1}{\xi})] \tag{8.5.13}$$

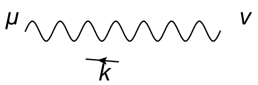
i.e.

$$ig^{\mu\nu}g_{\nu\rho} \tag{8.5.14}$$

which just re-writing the product of g's is:

$$\delta^{\mu}{}_{\rho} \tag{8.5.15}$$

Thus we can denote our vector bosons in GWs theory by the Feynman diagrams (eq21.54 p743 [9])



$$= \tfrac{-i}{k^2 - m^2}[g^{\mu\nu} - \tfrac{k^{\mu}k^{\nu}}{k^2 - \xi m^2}(1 - \xi)$$

Figure 8.5.2: Propagator for the vector bosons associated with the constrained Lagrangian with have reintroduced tha mass i.e.$m = M_Z for Z_0, = m_{W^{\pm}}$ for $W^{\pm}$ and $M_A = 0$ for A

To specialise we only consider the muons coupling. we find their general coupling to the Z^0 boson and then find their coupling to the Higgs in the unitary gauge and the coupling of one Higgs boson to two Z^0 bososns in the unitary gauge.
Consider (7.7.1). The term involving muons is:

$$i\overline{\mu}_L \not{D} \mu_L + i\overline{\mu}_R \not{D} \mu_R \tag{8.5.16}$$

where from (**??**):

$$D_{\mu} = \partial_{\mu} \quad - \quad i\frac{g'}{\sqrt{2}}(W^{+}_{\mu}T^{+} + W^{-}_{\mu}T^{-})$$
$$- \quad i\frac{g'}{cos\theta_W}Z_{\mu}(T^3 - sin^2\theta_W Q) - ieA_{\mu}Q \tag{8.5.17}$$

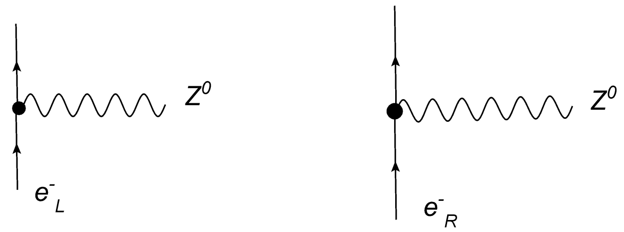
clearly then for both left and right-handed muons interacting with Z_{μ} the vertex has two muons going in and one Z_{μ} coming out and the coefficient associated with the vertex is recalling (2.3.24) nd that in going from the Lagrangian to the vertex we pick up a factor -i:

$$\gamma^{\mu}i\frac{g'}{cos\theta_W}(T^3 - sin^2\theta_W Q) \tag{8.5.18}$$

We know Q the charge is -1 for both fields and that T^3 =0 for the right-handed field as it is a singlet and $-\frac{1}{2}$ for left-handed muon as it is the bottom component of the doublet. Recall (7.7.50) to give:

$$\frac{i\gamma^{\mu}e}{cos\theta_W sin\theta_W}(T^3 + sin^2\theta_W) \tag{8.5.19}$$

Thus our required Feynman diagrams are:



$=\frac{i\gamma^{\mu}e}{cos\theta_W sin\theta_W}(-\frac{1}{2} + sin^2\theta_W)$ $\qquad$ $=\frac{i\gamma^{\mu}e}{cos\theta_W sin\theta_W}(sin^2\theta_W)$

Figure 8.5.3: Vertex for Left-handed ($T^3 = -\frac{1}{2}$) and right-handed fermions ($T^3 = 0$) meeting the Z^0 boson

Recall the term that links Higgs to the Z^0 (7.7.57) in the unitary gauge:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + [m_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2}m_Z^2(Z_\mu^0)^2](\frac{h(x)}{v} + 1)^2 \tag{8.5.20}$$

As we are only considering one Higgs boson coming out the term we want is:

$$\frac{1}{2}m_Z^2(Z_\mu^0)^2[2\frac{h(x)}{v}] \tag{8.5.21}$$

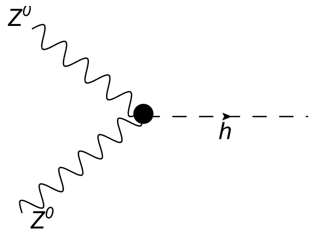
so for two incoming Z^0 with four-vector indices μ and ν we have a coefficient for our vertex proportional to:

$$-i\gamma^{\mu\nu}\frac{m_Z^2}{v} \tag{8.5.22}$$

where the $\gamma^{\mu\nu}$ switches the indices : the exact term is(see Figure 20.6 p716 [9]):

$$2i\gamma^{\mu\nu}\frac{m_Z^2}{v} \tag{8.5.23}$$

Thus our relevant Feynan diagram is:



$$=2i\frac{m_Z^2}{v}g^{\mu\nu}$$

Figure 8.5.4: Vertex for two Z^0 's annihilating to make a Higgs boson

Recall the term that links Higgs to the fermions (7.7.83) in the unitary gauge:

$$-m_l^i \overline{e}_L^i e_R^i (\frac{h(x)}{v} + 1) + h.c \tag{8.5.24}$$

We will also assume that their is only one generation of leptons so that we do not have tor worry about their mass mixing matrices so the term we want is:

$$-m_f \overline{f}_L f_R [\frac{h(x)}{v}] + h.c. \tag{8.5.25}$$

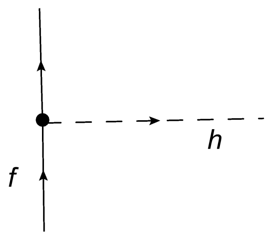
so for two incoming Z^0 with four-vector indices μ and ν we have a coefficient for our vertex proportional to:

$$i\frac{m_f}{v} \tag{8.5.26}$$

the exact term is see Figure 20.6 p716 [9]:

$$-i\gamma^{\mu\nu}\frac{m_f}{v} \tag{8.5.27}$$

Thus our relevant Feynan diagram is:



$$=-i\frac{m_f}{v}$$

Figure 8.5.5: Vertex for two fermions (left or right-handed muons) annihilating to make a Higgs boson

We know the Higgs boson is a scalar with mass m_h in the unitary gauge therefore it has a propagator given by:

$$\frac{i}{k^2 - m_h^2 - i\epsilon} \tag{8.5.28}$$

8.6 Calculations for Different Higgs Decay Paths

For simplicity we here ignore the counter term vertices and take diagrams at one loop level. We ignore the Fadeev-Poppov ghosts and work in the unitary gauge. We will also assume that their is only one generation of leptons so that we do not have tor worry about their mass mixing matrices.The unitary gauge is $\xi = \infty$ limit of the R_ϵ gauge (8.4.11)(p738 [9] ll27). The explicit calculation we will do is the terms associated with the Feynman diagrams for two paths that involve Higgs boson production. The two paths we choose have the both start and finish with muon They are : Two right-handed muons become by interacting to make a Higgs Boson two left-handed

$$\mu_R^- \mu_R^- \to h \to \mu_L^- \mu_L^- \tag{8.6.1}$$

which gives the Feynman diagram :

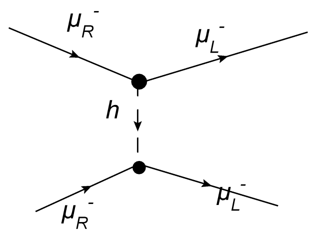


Figure 8.6.1: A two right-handed muons exchange a Higgs Boson and emerge as two right-handed muons

and
two right-handed muons and anti-muons annihilate to each make a Z^0 which annihilate to make a Higgs which decays into two Z^0 that then decay into left-handed muons and anti-muons

$$\mu_R^- \mu_R^- \mu_R^+ \mu_R^+ \to Z^0 Z^0 \to h \to Z^0 Z^0 \to \mu_L^- \mu_L^- \mu_L^+ \mu_L^+ \tag{8.6.2}$$

which gives the Feynman diagram

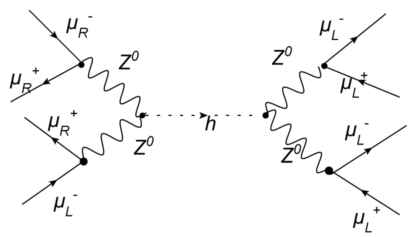


Figure 8.6.2: A two right-handed muons annihilate with their anti-particles to produce two Z^0 's which annihilate two produce a Higgs Boson which decays into two Z_0's which decay to two left-handed muons and their anti-particles

Considering figure 8.5.4 in the unitary gauge we see the propagator for the Z^0 boson is :

$$\frac{-i}{k^2 - m_Z^2} g^{\mu\nu} \tag{8.6.3}$$

Thus for our first process our by the Feynman rules we get recall section 4.5 we get a term of the form:

$$u^s_{\mu_L}(p)u^s_{\mu_L}(p)\frac{i}{k^2 - m_h^2 - i\epsilon}v^s_{\mu_R}v^s_{\mu_R} \tag{8.6.4}$$

and our second process gets a term of the form:

$$\begin{aligned} &= u^s_{\mu_R}(p)u^s_{\mu_R}\overline{u}^s_{\mu_R}(p)\overline{u}^s_{\mu_R}(p)(p)\frac{-i}{k^2 - m_Z^2}g^{\mu\nu} \qquad (8.6.5) \\ &\times \frac{-i}{k^2 - m_Z^2}g^{\mu\nu}\frac{i}{k^2 - m_h^2 - i\epsilon}\frac{-i}{k^2 - m_Z^2}g^{\mu\nu}\frac{-i}{k^2 - m_Z^2}g^{\mu\nu}v^s_{\mu_L}(p)v^s_{\mu_L}\overline{v}^s_{\mu_L}(p)\overline{v}^s_{\mu_L}(p) \end{aligned}$$

These can then be manipulated to get the transition amplitude

8.7 Summary

In this chapter we have considered the Decay paths of the Higgs as the way in which it it most likely to be detected and considered its possible detectetion at the LHC. we have then introduced Grassman numbers and used these to introduce the concept of Fadeev-Poppov ghosts which are needed for gauge invariance. Finally we have introduced some of the Feynman diagrams in unitary gauge and used these to do a sample calculation

Chapter 9

Conclusion

In this report we have studied the Higgs mechanism as an important aspect of quantum field theory and an integral part of the standard model. Although the main justification for studying the Higgs mechanism is that it provides a way of giving the vector bosons that carry the weak force masses (a phenomena seen in experiments) whilst allowing the theory to be renormalisable, it is also heavily involved in the formulation of the GWS model which is believed to deacribe the weak force.

Initially we begin by introducing the basics of a quantum field theory. Specifically we introduce the three typesof fields needed for our model (scalar, vector and Dirac fields) . We then introduce the two most common ways of quantising a theory path integral formalism and the path integral formalism. Finally we introduce the idea of an interacting field as a pertubation on the exact solutions we have found for our fields using the LSZ formula to calculate the relation between the perturbed and unperturbed fields and Introdusing the Feynman diagrams as a diagramatic way of working out terms in our expansion.We also introduced conserved currents which are important as many theories are now formulated by taking all possible interactions that are allowed by gauge invariance and the conservation of certain quantities.

We then gave a basic view of renormalisation, this is important as a theory being renormalisable means that we can draw Feynman diagrams with counter-term vertices in such a way that they are nomn-divergent and thus give physically sensible results at all orders. Renormalisability is also a constraint that we apply to our lagrangian to exclude explicit vector boson mass terms which is the reason for the Higgs mechanism.

We then described spontaneous symmetry breaking as a prelude to the symmetry breaking required for the Higgs mechanism. We then discussed the idea of a gauge invariant theory giving both an abelian U(1) and non-abelian

SU(2) example and highlighting QED as an example of a gauge theory which is known to give accurate real world predictions. Finally we defined left and right-handed fermions which is necessary due to nature's asymmetry with handiness.Upto this point the material is completely general in that it is chosen for its use for describing the Higgs mechanism but is fundamental for any understanding of quantum field theory.

We then introduced the Higgs mechanism in the context of GWS theory. we specifically showed how if we take the QED lagrangian and add on a negative minimum self-interacting propagating scalar field then invariance under SU(2) and U(1) transforms allows us to re-write the theory in terms of combined bosons which acquire mass and can be identified with the weak force gauge bosons. We also showed how manipulation of this lagrangian leads to a mass term for the scalar field which in unitary gauge becomes the Higgs boson and can give mass to the fermions. The mass terms that we acquired are all related to our vacuum expectation value.

We then attempted to link the hypothetical Higgs boson to obtainable experimental results.We discussed the idea of higgs boson decay and identified decay paths as a way of identifying its existence we then linked this with the experimental efforts at the LHC.

Our final part was to ltry and formulate Feynman rules that would allow us to make calculations. We defined Grassmann numbers and used these to introduce gauge fixing by the Fadeev-Poppov method another technique that has general applications. Having found the gauge constrained Lagrangians we gave an example of how to find the propagators for the scalars and vector bososns in any gauge along with one of their vertices as well as giving Higgs boson vertices and propagators in the unitary gauge. We concluded with a sample caluculation.

In conclusion although this project focuses on the Higgs mechanism it is useful in general as it covers many general field theory techniques and introduces most of the methods neededed to cover the standard model up to the strong force. This project could be extended by inclusion of the strong force or by looking beyond the standard model. We also note that the single Higgs boson is just the simplest standard model way of producing the Higgs mechanism and there could in fact be a Higgs sector see p717 [9]

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Appendix A

Appendix A : Chapter 1

A.1 det(AB) =det(A) det (B)

We consider only 2×2 matrices let :

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{A.1.1}$$

so :

$$det(A) = a_1 a_4 - a_2 a_3 \tag{A.1.2}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{A.1.3}$$

so :

$$det(B) = b_1 b_4 - b_2 b_3 \tag{A.1.4}$$

$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{pmatrix} \tag{A.1.5}$$

so

$$\begin{aligned} det(AB) &= (a_1 b_1 + a_2 b_3)(a_3 b_2 + a_4 b_4) - (a_1 b_2 + a_2 b_4)(a_3 b_1 + a_4 b_3) \\ &= a_1 a_3 b_1 b_2 + a_1 b_1 a_4 b_4 + a_2 a_3 b_3 b_2 + a_2 a_4 b_3 b_4 \\ &- a_1 a_3 b_1 b_2 - a_1 a_4 b_2 b_3 - a_2 a_3 b_1 b_4 - a_2 a_4 b_3 b_4 \\ &= a_1 b_1 a_4 b_4 + a_2 a_3 b_3 b_2 - a_1 a_4 b_2 b_3 - a_2 a_3 b_1 b_4 \end{aligned} \tag{A.1.6}$$

$$\begin{aligned} det(A)det(B) &= (a_1 a_4 - a_2 a_3)(b_1 b_4 - b_2 b_3) \\ &= a_1 a_4 b_1 b_4 + a_2 a_3 b_2 b_3 - a_2 a_3 b_1 b_4 - a_1 a_4 b_2 b_3 \end{aligned} \tag{A.1.7}$$

so :

$$det(AB) = det(A)det(B) \tag{A.1.8}$$

A.2 elements of R are real

Recall the generic transform (1.3.9):

$$\begin{pmatrix} -z' & x'+iy' \\ x'-iy' & z' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} -z & x+iy \\ x-iy & z \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \tag{A.2.1}$$

$$\begin{pmatrix} -z' & x'+iy' \\ x'-iy' & z' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} -za^*+b^*(x+iy) & zb+a(x+iy) \\ b^*z+a^*(x-iy) & -b(x-iy)az \end{pmatrix}$$

We get four equations:
1)$-z' = -z(aa^* - bb^* + x(ab^* + ba^*) + yi(ab^* - ba^*)$
2)$x' + iy' = 2abz + (a^2 - b^2)x + i(a^2 + b^2)y$
3)$x' - iy' = 2a^*b^*z + (-b^{*2} + a^{*2})x + -i(b^{*2} + a^{*2})y$
4)$z' - z(aa^* - bb^* - x(ab^* + ba^*) + yi(-ab^* + ba^*)$

summing 2) and 3) and division by 2 gives:

$$x' = (ab+a^*b^*)z + \frac{(a^2+a^{*2}-b^2-b^{*2})}{2}x + i\frac{(a^2-a^{*2}+b^2-b^{*2})}{2}y \tag{A.2.2}$$

summing i 3) and -i 2) and division by 2 gives:

$$\begin{aligned} y' &= -i(ab-a^*b^*)z + -i\frac{(a^2-a^{*2}-b^2+b^{*2})}{2}x \\ &+ \frac{(a^2+a^{*2}+b^2+b^{*2})}{2}y \end{aligned} \tag{A.2.3}$$

This implies the matrix equation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{(a^2+a^{*2}-b^2-b^{*2})}{2} & i\frac{(a^2-a^{*2}+b^2-b^{*2})}{2} & (ab+a^*b^*) \\ -i\frac{(a^2-a^{*2}-b^2+b^{*2})}{2} & \frac{(a^2+a^{*2}+b^2+b^{*2})}{2} & -i(ab-a^*b^*) \\ (-b^*a-a^*b) & i(ba^*-ab^*) & (-bb^*+aa^*) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{A.2.4}$$

So generally:

$$R = \begin{pmatrix} \frac{(a^2+a^{*2}-b^2-b^{*2})}{2} & i\frac{(a^2-a^{*2}+b^2-b^{*2})}{2} & (ab+a^*b^*) \\ -i\frac{(a^2-a^{*2}-b^2+b^{*2})}{2} & \frac{(a^2+a^{*2}+b^2+b^{*2})}{2} & -i(ab-a^*b^*) \\ (-b^*a-a^*b) & i(ba^*-ab^*) & (-bb^*+aa^*) \end{pmatrix} \tag{A.2.5}$$

writing :

$$\begin{aligned} a &= a_1 + ia_2 \\ a^* &= a_1 - ia_2 \\ b &= b_1 + ib_2 \\ b^* &= b_1 - ib_2 \end{aligned} \tag{A.2.6}$$

with $a_1, a_2, b_1, b_2 \in \mathbb{R}$

$$\begin{aligned} a^2 &= a_1^2 - a_2^2 + 2ia_1a_2 \\ a^{*2} &= a_1^2 - a_2^2 - 2ia_1a_2 \end{aligned} \tag{A.2.7}$$

This means :

$$a^2 + a^{*2} = 2(a_1^2 - a_2^2) \in \mathbb{R} \tag{A.2.8}$$

$$a^2 - a^{*2} = 4ia_1a_2 \tag{A.2.9}$$

which is purely imaginary and similarly for b
Thus:

$$\frac{(a^2 + a^{*2} - b^2 - b^{*2})}{2}, \frac{(a^2 + a^{*2} + b^2 + b^{*2})}{2} \in \mathbb{R} \tag{A.2.10}$$

$$\frac{(a^2 - a^{*2} + b^2 - b^{*2})}{2} \tag{A.2.11}$$

is purely imaginary so:

$$i\frac{(a^2 - a^{*2} + b^2 - b^{*2})}{2} \in \mathbb{R} \tag{A.2.12}$$

for any complex number c :

$$cc^* = \mid c \mid^2 \in \mathbb{R} \tag{A.2.13}$$

so :

$$(-bb^* + aa^*) \in \mathbb{R} \tag{A.2.14}$$

$$ab = (a^*b^*) \tag{A.2.15}$$

$$ab^* = (ab^*)^* \tag{A.2.16}$$

for a generic complex number:

$$c = c_1 + ic_2 \qquad c_1, c_2 \in \mathbb{R} \tag{A.2.17}$$

with conjugate:

$$c^* = c_1 - ic_2 \tag{A.2.18}$$

$$c + c^* = 2c_1 \in \mathbb{R} \tag{A.2.19}$$

$$c - c^* = 2ic_2 \tag{A.2.20}$$

is imaginary
So

$$(ab + a^*b^*), (-b^*a - a^*b) \in \mathbb{R} \tag{A.2.21}$$

and :

$$(ab - a^*b^*), (ba^* - ab^*) \tag{A.2.22}$$

are imaginary
Thus:

$$i(ab - a^*b^*), i(ba^* - ab^*) \in \mathbb{R} \tag{A.2.23}$$

Thus all elements of R are real

A.3 Properties of Hermitian Operators

let $\hat{A}$ be hermitian and $\mid n_1\rangle, \mid n_2\rangle$ be eigenstates with eigenvalues n_1, n_2 respectively

$$(\langle n_2 \mid \hat{A} \mid n_1\rangle)^\dagger = \langle n_1 \mid \hat{A}^\dagger \mid n_2\rangle = \langle n_1 \mid \hat{A} \mid n_2\rangle \tag{A.3.1}$$

$$\begin{aligned} 0 &= (\langle n_2 \mid \hat{A} \mid n_1\rangle)^\dagger - \langle n_1 \mid \hat{A} \mid n_2\rangle \\ &= (n_1\langle n_2 \mid n_1\rangle)^\dagger - n_2\langle n_1 \mid n_2\rangle \\ &= (n_1^\dagger - n_2)\langle n_1 \mid n_2\rangle \end{aligned} \tag{A.3.2}$$

the case:

$$\mid n_2\rangle = \mid n_1\rangle \tag{A.3.3}$$

gives:

$$n_1 - n_1^\dagger = 0 \tag{A.3.4}$$

i.e n_1 real .
If $n_1 \neq n_2$

$$\langle n_1 \mid n_2\rangle = 0 \tag{A.3.5}$$

so the two eigenvectors are orthogonal
The Proof of linear independence means that we can a space with the same number of dimensions that we have degenerate eigenvectors

A.4 The Harmonic Oscillator in Quantum Mechanics

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{Q} \tag{A.4.1}$$

$\hat{H}$ is the hamiltonian operator
$\hat{P}$ is the momentum operator
$\hat{Q}$ is the position operator

$$\hat{H} \mid n\rangle = E_n \mid n\rangle \tag{A.4.2}$$

where E_n is the energy eigenvalue associated with the eigenvector $\mid n\rangle$

the annihilation operator is defined as:

$$\hat{a} = \frac{1}{\lambda_0\sqrt{2}}\hat{Q} + i\frac{\lambda_0}{\hbar\sqrt{2}}\hat{P} \tag{A.4.3}$$

with $\lambda_0 = \sqrt{\frac{\hbar}{m\omega}}$

the creation operator is defined as:

$$\hat{a}^\dagger = \frac{1}{\lambda_0\sqrt{2}}\hat{Q} - i\frac{\lambda_0}{\hbar\sqrt{2}}\hat{P} \tag{A.4.4}$$

with $\lambda_0 = \sqrt{\frac{\hbar}{m\omega}}$
define:

$$\begin{aligned}
\hat{N} &= \hat{a}^\dagger\hat{a} && \text{(A.4.5)}\\
&= \frac{1}{2}(\frac{Q^2}{\lambda_0^2} - i\frac{\lambda_0}{\hbar}\hat{P})(\frac{\hat{Q}}{\lambda_0} + i\frac{\lambda_0}{\hbar}\hat{P}) \\
&= \frac{1}{2}(\frac{Q^2}{\lambda_0^2} + \frac{\lambda_0^2}{\hbar^2}\hat{P}^2) + \frac{i}{2\hbar}(\hat{Q}\hat{P} - \hat{P}\hat{Q}) \\
&= \frac{1}{2}(\frac{Q^2}{\lambda_0^2} + \frac{\lambda_0^2}{\hbar^2}\hat{P}^2) + \frac{i}{2\hbar}[\hat{Q},\hat{P}] && \text{(A.4.6)}
\end{aligned}$$

since (1.5.6) :

$$[\hat{Q},\hat{P}] = i\hbar\hat{I} \tag{A.4.7}$$

$$\begin{aligned}
\hat{N} &= \frac{1}{2}(\frac{Q^2}{\lambda_0^2} + \frac{\lambda_0^2}{\hbar^2}\hat{P}^2) + \frac{i}{2\hbar}(i\hbar)\hat{I} \\
&= \frac{1}{2}(\frac{Q^2}{\lambda_0^2} + \frac{\lambda_0^2}{\hbar^2}\hat{P}^2 - \hat{I}) \\
&= \frac{1}{\hbar\omega}(\frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\hat{Q}^2) - \frac{1}{2}\hat{I} \\
&= \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\hat{I} && \text{(A.4.8)}
\end{aligned}$$

rearranging gives:

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}\hat{I}) \tag{A.4.9}$$

this implies that any eigenvalue of $\hat{N}$ is automatically an eigenvalue of $\hat{H}$ since $\hat{I}$ merely adds a constant to the eigenvalue of $\hat{N}$, as the identity matrix takes any vector to itself

We want to show:

$$[\hat{a}, \hat{a}^\dagger] = \hat{I} \tag{A.4.10}$$

This is:

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I} \tag{A.4.11}$$

which with the appropriate subsitutions becomes:

$$\frac{1}{2}(\frac{\hat{Q}}{\lambda_0} + i\frac{\lambda_0}{\hbar}\hat{P})(\frac{\hat{Q}}{\lambda_0} - i\frac{\lambda_0}{\hbar}\hat{P}) - \frac{1}{2}(\frac{\hat{Q}}{\lambda_0} - i\frac{\lambda_0}{\hbar}\hat{P})(\frac{\hat{Q}}{\lambda_0} + i\frac{\lambda_0}{\hbar}\hat{P}) = \hat{I} \tag{A.4.12}$$

using the definition of $\lambda_0 = \sqrt{\frac{\hbar}{m\omega}}$ This becomes:

$$\begin{aligned} \frac{i}{\hbar}(\hat{P}\hat{Q} - \hat{P}\hat{Q}) &= \hat{I} \\ \frac{i}{\hbar}[\hat{P}, \hat{Q}] &= \hat{I} \\ -\frac{i}{\hbar}[\hat{Q}, \hat{P}] &= \hat{I} \end{aligned} \tag{A.4.13}$$

which is true since by (1.5.6):

$$[\hat{Q}, \hat{P}] = i\hbar\hat{I} \tag{A.4.14}$$

We want to show $\hat{a} \mid n\rangle$ is an eigenvector of $\hat{N}$

$$\hat{N}\hat{a} \mid n\rangle = \hat{a}\hat{a}^\dagger\hat{a} \mid n\rangle \tag{A.4.15}$$

as $[\hat{a}, \hat{a}^\dagger] = \hat{I}$ (A.4.10) we get:

$$\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - \hat{I} \tag{A.4.16}$$

This means (A.4.15) becomes:

$$\begin{aligned} \hat{a}(\hat{a}\hat{a}^\dagger - \hat{I}) \mid n\rangle &= \hat{a}(\hat{N} - \hat{I}) \mid n\rangle \\ &= \hat{a}(n-1) \mid n\rangle \end{aligned} \tag{A.4.17}$$

as $\hat{N} \mid n\rangle = n \mid n\rangle$ by definition in the last line, rearrangement then gives:

$$\hat{N}\hat{a} \mid n\rangle = (n-1)\hat{a} \mid n\rangle \tag{A.4.18}$$

therefore $\hat{a} \mid n\rangle$ is an eigenvalue of $\hat{N}$ with eigenvalue n-1 since the eigenvalues of $\hat{N}$ are unique by its construction as a hermitian matrix:

$$\hat{a} \mid n\rangle \propto \mid \hat{n} - 1\rangle \tag{A.4.19}$$

the constant of proportionality is equal to the modulus of the vector i.e .:

$$\sqrt{\langle n \mid \hat{a}^{\dagger}\hat{a} \mid n\rangle} \tag{A.4.20}$$

$$\langle n \mid \hat{a}^{\dagger}\hat{a} \mid n\rangle = \langle n \mid \hat{a}\hat{a}^{\dagger} - \hat{I} \mid n\rangle \tag{A.4.21}$$

c.f above (A.4.16) Using the definition of N (A.4.5)

$$\begin{aligned} \langle n \mid \hat{a}^{\dagger}\hat{a} \mid n\rangle &= \langle n \mid \hat{N} - \hat{I} \mid n\rangle = \langle n \mid n-1 \mid n\rangle = (n-1)\langle n \mid n\rangle \\ &= n-1 \end{aligned} \tag{A.4.22}$$

as $\langle n \mid n\rangle = 1$ as $\mid n\rangle$ are orthonormal
thus:

$$\hat{a} \mid n\rangle = \sqrt{n-1} \mid n-1\rangle \tag{A.4.23}$$

We want to show $\hat{a}^{\dagger} \mid n\rangle$ is an eigenvector of $\hat{N}$ using the definition of N (A.4.5) we get:

$$\hat{N}\hat{a}^{\dagger} \mid n\rangle = \hat{a}\hat{a}^{\dagger}\hat{a}^{\dagger} \mid n\rangle \tag{A.4.24}$$

(A.4.10)$[\hat{a}, \hat{a}^{\dagger}] = \hat{I}$ gives:

$$\hat{a}\hat{a}^{\dagger} = \hat{a}^{\dagger}\hat{a} + \hat{I} \tag{A.4.25}$$

Thus (A.4.24) gives:

$$\begin{aligned} (\hat{a}^{\dagger}\hat{a} + \hat{I})\hat{a}^{\dagger} \mid n\rangle &= \hat{a}^{\dagger}(\hat{N} + \hat{I}) \mid n\rangle \\ &= \hat{a}^{\dagger}(n+1) \mid n\rangle = (n+1)\hat{a}^{\dagger} \mid n\rangle \end{aligned} \tag{A.4.26}$$

where we use the definition $\hat{N} \mid n\rangle = n \mid n\rangle$ and that operators commute with constants in the last line. therefore $\hat{a}^{\dagger} \mid n\rangle$ is an eigenvalue of $\hat{N}$ with eigenvalue n+1 since the eigenvalues of $\hat{N}$ are unique by its construction as a hermitian matrix:

$$\hat{a}^{\dagger} \mid n\rangle \propto \mid \hat{n} + 1\rangle \tag{A.4.27}$$

the constant of proportionality is equal to the modulus of the vector i.e :

$$\sqrt{\langle n \mid \hat{a}\hat{a}^{\dagger} \mid n\rangle} \tag{A.4.28}$$

$$\begin{aligned} \langle n \mid \hat{a}\hat{a}^{\dagger} \mid n\rangle &= \langle n \mid \hat{N} \mid n\rangle \\ &= \langle n \mid \hat{N} \mid n\rangle = \langle n \mid n \mid n\rangle = n\langle n \mid n\rangle \\ &= n \end{aligned} \tag{A.4.29}$$

as $\langle n \mid n\rangle = 1$ as $\mid n\rangle$ are orthonormal
thus:

$$\hat{a}^{\dagger} \mid n\rangle = \sqrt{n} \mid n+1\rangle \tag{A.4.30}$$

A.5 Pauli Spin Matrices

We want to show (1.5.7)

$$\sigma^i \sigma^j = i\epsilon^{ijk}\sigma^k \tag{A.5.1}$$

$$\sigma^1\sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5.2}$$

which is $\sigma^1\sigma^2 = i\sigma^3$ as $\epsilon^{123} = 1$

$$\sigma^1\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.5.3}$$

which is $\sigma^1\sigma^3 = -i\sigma^2$ as $\epsilon^{132} = -1$

$$\sigma^2\sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5.4}$$

which is $\sigma^2\sigma^1 = -i\sigma^3$ as $\epsilon^{213} = -1$

$$\sigma^2\sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.5.5}$$

which is $\sigma^2\sigma^3 = i\sigma^1$ as $\epsilon^{231} = 1$

$$\sigma^3\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.5.6}$$

which is $\sigma^3\sigma^1 = i\sigma^2$ as $\epsilon^{312} = 1$

$$\sigma^3\sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.5.7}$$

which is $\sigma^3\sigma^2 = -i\sigma^1$ as $\epsilon^{321} = -1$

so as required $\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k$

$$\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = i\sigma^k(\epsilon^{ijk} + \epsilon^{jik}) = 0 \tag{A.5.8}$$

as $\epsilon^{ijk} = -\epsilon^{jik}$ by the anti-symmetry of epsilon

We want to show (1.5.10):

$$\sigma^\mu\sigma^\mu = \sigma^0 \tag{A.5.9}$$

$$\sigma^0\sigma^0 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{A.5.10}$$

$$\sigma^1\sigma^1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{A.5.11}$$

$$\sigma^2\sigma^2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{A.5.12}$$

$$\sigma^3\sigma^3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{A.5.13}$$

so $\sigma^\mu\sigma^\mu = \sigma^0$

$$\begin{aligned}\frac{a}{2}(\sigma^0+\sigma^3) &= \frac{a}{2}[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}] = \frac{a}{2}[\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}]\\ &= \begin{pmatrix}a & 0\\0 & 0\end{pmatrix}\end{aligned} \tag{A.5.14}$$

$$\begin{aligned}\frac{d}{2}(\sigma^0-\sigma^3) &= \frac{d}{2}[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}] = \frac{d}{2}[\begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}]\\ &= \begin{pmatrix}0 & 0\\0 & d\end{pmatrix}\end{aligned} \tag{A.5.15}$$

$$i\sigma^2 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \tag{A.5.16}$$

$$\begin{aligned}\frac{b}{2}(\sigma^1+i\sigma^2) &= \frac{b}{2}[\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}] = \frac{b}{2}[\begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}]\\ &= \begin{pmatrix}0 & b\\0 & 0\end{pmatrix}\end{aligned} \tag{A.5.17}$$

$$\begin{aligned}\frac{c}{2}(\sigma^1-i\sigma^2) &= \frac{c}{2}[\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}] = \frac{c}{2}[\begin{pmatrix}0 & 0\\2 & 0\end{pmatrix}]\\ &= \begin{pmatrix}0 & 0\\c & 0\end{pmatrix}\end{aligned} \tag{A.5.18}$$

so a general 2 dimensional matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ can be written as :

$$\frac{1}{2}((a+d)\sigma^0 + (b+c)\sigma^1 + i(b-c)\sigma^2 + (a-d)\sigma^3) \tag{A.5.19}$$

A.6 Proof that $\vec{\nabla} \times \vec{\nabla} a = 0$

for any potential a

$$\vec{\nabla} a = (a_x, a_y, a_z) \tag{A.6.1}$$

$$\vec{\nabla} \times \vec{\nabla} a = (a_{zy} - a_{yz}, a_{xz} - a_{zx}, a_{yx} - a_{xy}) = 0 \tag{A.6.2}$$

if all partial derivatives commute

A.7 Proof that $\vec{\nabla}.\vec{\nabla} \times \vec{C} = 0$

for any vector $\vec{C}$
$\vec{C} = (c_1, c_2, c_3)$

$$\vec{\nabla} \times \vec{C} = (c_{3y} - c_{2z}, c_{1z} - c_{3x}, c_{2x} - c_{1y}) \tag{A.7.1}$$

$$\vec{\nabla}.\vec{\nabla} \times \vec{C} = c_{3yx} - c_{2zx} + c_{1zy} - c_{3xy} + c_{2xz} - c_{1yz} = 0 \tag{A.7.2}$$

if all partial derivatives commute

A.8 Proof that $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\vec{\nabla}.\vec{A} - \vec{\nabla}^2 \vec{A}$

for any vector $\vec{A} = (a_1, a_2, a_3)$

$$\vec{\nabla}^2 \vec{A} = (a_{1xx} + a_{1yy} + a_{1zz}, a_{2xx} + a_{2yy} + a_{2zz}, a_{3xx} + a_{3yy} + a_{3zz}) \tag{A.8.1}$$

$$\vec{\nabla}.\vec{A} = a_{1x} + a_{2y} + a_{3y} \tag{A.8.2}$$

$$\vec{\nabla}\vec{\nabla}.\vec{A} = (a_{1xx} + a_{2yx} + a_{3zx}, a_{1xy} + a_{2yy} + a_{3zy}, a_{1xz} + a_{2yz} + a_{3zz}) \tag{A.8.3}$$

$$\begin{aligned} \vec{\nabla} \quad & \vec{\nabla}.\vec{A} - \vec{\nabla}^2 \vec{A} = \qquad \text{(A.8.4)} \\ (\quad & a_{2yx} + a_{3zx} - a_{1yy} - a_{1zz}, a_{1xy} + a_{3zy} - a_{2xx} - a_{2zz}, a_{1xz} + a_{2yz} - a_{3xx} - a_{3yy}) \end{aligned}$$

$$\vec{\nabla} \times \vec{A} = (a_{3y} - a_{2z}, a_{1z} - a_{3x}, a_{2x} - a_{1y}) \tag{A.8.5}$$

$$\begin{aligned} \vec{\nabla} \quad & \times \vec{\nabla} \times \vec{A} \qquad \text{(A.8.6)} \\ = \quad & (a_{2xy} - a_{1yy} - a_{1zz} + a_{3xz}, a_{3yz} - a_{2zz} - a_{2xx} + a_{1yx}, a_{1zx} - a_{3xx} - a_{3yy} + a_{2zy}) \end{aligned}$$

so $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\vec{\nabla}.\vec{A} - \vec{\nabla}^2 \vec{A}$ if partial derivatives commute

A.9 Converting Maxwell's Equations into Vector , Potential Form

Recall (1.7.1) and (1.7.4)

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} \tag{A.9.1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{A.9.2}$$

Use:

$$\vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{A.9.3}$$

(1.7.1) becomes:

$$\begin{aligned} \vec{\nabla}.(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}) &= \frac{\rho}{\epsilon_0} \\ -\vec{\nabla}^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{A.9.4}$$

which by rearrangment is:

$$\vec{\nabla}^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho}{\epsilon_0} \tag{A.9.5}$$

(1.7.4) becomes:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t})}{\partial t} \tag{A.9.6}$$

Recall

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \tag{A.9.7}$$

and from Appendix A8 that:
$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\vec{\nabla}.\vec{A} - \vec{\nabla}^2 \vec{A}$
We get the equation:

$$\vec{\nabla}\vec{\nabla}.\vec{A} - \vec{\nabla}^2\vec{A} = \mu_0 \vec{J} - \frac{1}{c^2}(\vec{\nabla}\frac{\partial \phi}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2}) \tag{A.9.8}$$

Rearranging gives:

$$(\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}) - \vec{\nabla}(\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}) = -\mu_0 \vec{J} \tag{A.9.9}$$

A.10 $\Box A^\mu = \mu_0 J^\mu$

In Lorenz gauge :

$$\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{A.10.1}$$

the two Maxwell's equations (1.7.7) , (1.7.8) become:

$$\vec{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{A.10.2}$$

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{A.10.3}$$

(A.10.2) becomes :

$$\vec{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\mu_0 c^2\rho \tag{A.10.4}$$

using:$\mu_0\epsilon_0 = \frac{1}{c^2}$
dividing by c gives:

$$\vec{\nabla}^2\frac{\phi}{c} - \frac{1}{c^2}\frac{\partial^2\frac{\phi}{c}}{\partial t^2} = -\mu_0 c\rho \tag{A.10.5}$$

combining with (A.10.3) becomes :

$$(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})(\frac{\phi}{c}, \vec{A}) = -\mu_0(\rho c, \vec{J}) \tag{A.10.6}$$

with :

$$\Box = [\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2] \qquad A^\mu = (\frac{\phi}{c}, \vec{A}),\ J^\mu = (\rho c, \vec{J}) \tag{A.10.7}$$

and thus we get:

$$\Box A^\mu = \mu_0 J^\mu \tag{A.10.8}$$

A.11 Components of $F_{\alpha\beta}$

Recall (1.7.13):

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{A.11.1}$$

interchanging indices we get:

$$F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta = -F_{\alpha\beta} \tag{A.11.2}$$

this means we need only consider F's where $\alpha \leq \beta$ as the cases $\alpha \geq \beta$ will just be the negatives of these. If both the indices are the same:

$$F_{\alpha\alpha} = \partial_\alpha A_\alpha - \partial_\alpha A_\alpha = 0 \tag{A.11.3}$$

so it remains to find the cases with $\mu < \nu$ i.e.:
$F_{01}, F_{02}, F_{03}, F_{12}, F_{13}, F_{23}$
Our first component is:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 \tag{A.11.4}$$

this becomes using the four derivatives and four vector form of A:

$$F_{01} = -\frac{1}{c}\frac{\partial}{\partial t}A_x - \frac{\partial}{\partial x}(\frac{\phi}{c}) \tag{A.11.5}$$

using $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$ we get:

$$F_{01} = \frac{E_x}{c} \tag{A.11.6}$$

Our second component is :

$$F_{02} = \partial_0 A_2 - \partial_2 A_0 \tag{A.11.7}$$

this becomes using the four derivatives and four vector form of A:

$$F_{02} = -\frac{1}{c}\frac{\partial}{\partial t}A_y - \frac{\partial}{\partial y}(\frac{\phi}{c}) \tag{A.11.8}$$

using $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$ we get:

$$F_{02} = \frac{E_y}{c} \tag{A.11.9}$$

Our third component is :

$$F_{03} = \partial_0 A_3 - \partial_3 A_0 \tag{A.11.10}$$

this becomes using the four derivatives and four vector form of A:

$$F_{03} = -\frac{1}{c}\frac{\partial}{\partial t}A_z - \frac{\partial}{\partial z}(\frac{\phi}{c}) \tag{A.11.11}$$

using $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$ we get:

$$F_{03} = \frac{E_z}{c} \tag{A.11.12}$$

Our fourth component is:

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 \tag{A.11.13}$$

this becomes using the four derivatives and four vector form of A:

$$F_{12} = -\frac{\partial}{\partial x}A_y + \frac{\partial}{\partial y}A_x \tag{A.11.14}$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$ we get:

$$F_{12} = -B_z \tag{A.11.15}$$

Our fifth component is:

$$F_{13} = \partial_1 A_3 - \partial_3 A_1 \tag{A.11.16}$$

this becomes using the four derivatives and four vector form of A:

$$F_{13} = -\frac{\partial}{\partial x}A_z + \frac{\partial}{\partial z}A_x \tag{A.11.17}$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$ we get:

$$F_{13} = B_y \tag{A.11.18}$$

Our sixth component is:

$$F_{23} = \partial_2 A_3 - \partial_3 A_2 \tag{A.11.19}$$

this becomes using the four derivatives and four vector form of A:

$$F_{23} = -\frac{\partial}{\partial y}A_z + \frac{\partial}{\partial z}A_y \tag{A.11.20}$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$ we get:

$$F_{23} = -B_x \tag{A.11.21}$$

Thus we get :

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{A.11.22}$$

A.12 $F^{\mu\nu}$ from $F_{\alpha\beta}$

$$\begin{aligned}
F^{\mu\nu} &= g^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu} \\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c}\\ -\frac{E_x}{c} & 0 & -B_z & B_y\\ -\frac{E_y}{c} & B_z & 0 & -B_x\\ -\frac{E_z}{c} & -B_y & B_x & 0\end{pmatrix} \\
&\times \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c}\\ -\frac{E_x}{c} & 0 & B_z & -B_y\\ -\frac{E_y}{c} & -B_z & 0 & B_x\\ -\frac{E_z}{c} & B_y & -B_x & 0\end{pmatrix} \\
&= \begin{pmatrix}0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c}\\ \frac{E_x}{c} & 0 & -B_z & B_y\\ \frac{E_y}{c} & B_z & 0 & -B_x\\ \frac{E_z}{c} & -B_y & B_x & 0\end{pmatrix} \qquad \text{(A.12.1)}
\end{aligned}$$

A.13 $\tilde{F}^{\mu\nu}$

$$\epsilon^{\mu\mu\nu\rho} = -\epsilon^{\mu\mu\nu\rho} \qquad \text{(A.13.1)}$$

by swapping $\mu's$. This implies:

$$2\epsilon^{\mu\mu\nu\rho} = 0 \qquad \text{(A.13.2)}$$

by analogue if any two indices of ϵ are the same it vanishes hence :

$$\tilde{F}^{\mu\mu} = \frac{1}{2}\epsilon^{\mu\mu\nu\rho}F_{\nu\rho} = 0 \qquad \text{(A.13.3)}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = -\frac{1}{2}\epsilon^{\nu\mu\rho\sigma}F_{\rho\sigma} = -\tilde{F}^{\nu\mu} \qquad \text{(A.13.4)}$$

as

$$F_{\rho\sigma} = -F_{\sigma\rho} \qquad \text{(A.13.5)}$$

ignoring summation:

$$\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho} \qquad \text{(A.13.6)}$$

So:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad \text{(A.13.7)}$$

with summation gives :

$$\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad \text{(A.13.8)}$$

without summation. This is because with μ and ν fixed because of the *epsilon* there are only two terms in the summation which are permutations of ρ and σ and since permutating the epsilon picks up one minus sign and permutintg the F picks up another minus sign and thus the two permutations are the same . This means that we need only consider six forms of $\tilde{F}^{\mu\nu}$ i.e. : $\tilde{F}^{01}, \tilde{F}^{02}, \tilde{F}^{03}, \tilde{F}^{12}, \tilde{F}^{13}, \tilde{F}^{23}$ We recall the components of $F^{\mu\nu}$ from Appendix A11:

$$\tilde{F}^{01} = \epsilon^{0123}F_{23} = -F_{23} = B_x \quad \text{(A.13.9)}$$
$$\tilde{F}^{02} = \epsilon^{0213}F_{13} = F_{13} = B_y \quad \text{(A.13.10)}$$
$$\tilde{F}^{03} = \epsilon^{0312}F_{12} = -F_{12} = B_z \quad \text{(A.13.11)}$$
$$\tilde{F}^{12} = \epsilon^{1203}F_{03} = -F_{03} = -\frac{E_z}{c} \quad \text{(A.13.12)}$$
$$\tilde{F}^{13} = \epsilon^{1302}F_{02} = F_{02} = \frac{E_y}{c} \quad \text{(A.13.13)}$$
$$\tilde{F}^{23} = \epsilon^{2301}F_{01} = -F_{01} = -\frac{E_x}{c} \quad \text{(A.13.14)}$$

which gives:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix} \quad \text{(A.13.15)}$$

A.14 Maxwell's equation in $F^{\alpha\beta}, \tilde{F}^{\mu\nu}$ Notation

We consider (1.7.22):

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu_0 J^{\beta} \quad \text{(A.14.1)}$$

Consider the case

$\beta = 0$

(1.7.22) becomes:

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = \mu_0 J^0 \quad \text{(A.14.2)}$$

from above Appendix A12 $F^{00} = 0, F^{10} = \frac{E_x}{c}, F^{20} = \frac{E_y}{c}, F^{30} = \frac{E_z}{c}$ and the time component of $J^0 = c\rho$ so we get:

$$\frac{\partial}{\partial x}(\frac{E_x}{c}) + \frac{\partial}{\partial y}(\frac{E_y}{c}) + \frac{\partial}{\partial z}(\frac{E_z}{c}) = \mu_0 c\rho \quad \text{(A.14.3)}$$

which becomes:

$$\vec{\nabla}.\vec{E} = c^2\mu_0\rho \tag{A.14.4}$$

using $c^2 = \frac{1}{\epsilon_0\mu_0}$ which is:

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} \tag{A.14.5}$$

Consider the case
$\underline{\beta = 1}$
(1.7.22) becomes:

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = \mu_0 J^1 \tag{A.14.6}$$

from above Appendix A12 $F^{01} = -\frac{E_x}{c}, F^{11} = 0, F^{21} = B_z, F^{31} = -B_y$
which means

$$-\frac{1}{c}\frac{\partial}{\partial t}\frac{E_x}{c} + \frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y = \mu_0 J_x \tag{A.14.7}$$

This is:

$$(\vec{\nabla} \times \vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t})_x \tag{A.14.8}$$

Consider the case: $\underline{\beta = 2}$
(1.7.22) becomes:

$$\frac{\partial F^{02}}{\partial x^0} + \frac{\partial F^{12}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{32}}{\partial x^3} = \mu_0 J^2 \tag{A.14.9}$$

from above Appendix A12 $F^{02} = -\frac{E_y}{c}, F^{12} = -B_z, F^{22} = 0, F^{32} = B_x$
using this we have:

$$-\frac{1}{c}\frac{\partial}{\partial t}\frac{E_y}{c} - \frac{\partial}{\partial x}B_z + \frac{\partial}{\partial z}B_x = \mu_0 J_y \tag{A.14.10}$$

which is:

$$(\vec{\nabla} \times \vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t})_y \tag{A.14.11}$$

Consider the case: $\underline{\beta = 3}$
(1.7.22) becomes:

$$\frac{\partial F^{03}}{\partial x^0} + \frac{\partial F^{13}}{\partial x^1} + \frac{\partial F^{23}}{\partial x^2} + \frac{\partial F^{33}}{\partial x^3} = \mu_0 J^3 \tag{A.14.12}$$

from above Appendix A12 :$F^{03} = -\frac{E_z}{c}, F^{13} = B_y, F^{23} = -B_x, F^{33} = 0$
using this we have:

$$-\frac{1}{c}\frac{\partial}{\partial t}\frac{E_z}{c} + \frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x = \mu_0 J_z \tag{A.14.13}$$

This is:

$$(\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t})_z \tag{A.14.14}$$

Thus we have the first two maxwell equation (1.7.1) and (1.7.4)
Recall (1.7.20):

$$\frac{\partial \tilde{F}^{\alpha\beta}}{\partial x^\alpha} = 0 \tag{A.14.15}$$

Consider the case:
$\beta = 0$
(1.7.20)becomes:

$$\frac{\partial \tilde{F}^{00}}{\partial x^0} + \frac{\partial \tilde{F}^{10}}{\partial x^1} + \frac{\partial \tilde{F}^{20}}{\partial x^2} + \frac{\partial \tilde{F}^{30}}{\partial x^3} = 0 \tag{A.14.16}$$

from above Appendix A12 $\tilde{F}^{00} = 0, \tilde{F}^{10} = -B_x, \tilde{F}^{20} = -B_y, \tilde{F}^{30} = B_z$
Using this we get:

$$\vec{\nabla}.\vec{B} = 0 \tag{A.14.17}$$

Consider the case:
$\beta = 1$
(1.7.20)becomes:

$$\frac{\partial \tilde{F}^{01}}{\partial x^0} + \frac{\partial \tilde{F}^{11}}{\partial x^1} + \frac{\partial \tilde{F}^{21}}{\partial x^2} + \frac{\partial \tilde{F}^{31}}{\partial x^3} = 0 \tag{A.14.18}$$

from above Appendix A12 $\tilde{F}^{01} = B_x, \tilde{F}^{11} = 0, \tilde{F}^{21} = \frac{E_z}{c}, \tilde{F}^{31} = -\frac{E_y}{c}$
Thus we get:

$$\frac{\partial}{\partial t}(B_x) + \frac{\partial}{\partial y}\frac{E_z}{c} - \frac{\partial}{\partial z}\frac{E_y}{c} \tag{A.14.19}$$

which implies:

$$(\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})_x \tag{A.14.20}$$

Consider the case:
$\beta = 2$
(1.7.20)becomes:

$$\frac{\partial \tilde{F}^{02}}{\partial x^0} + \frac{\partial \tilde{F}^{12}}{\partial x^1} + \frac{\partial \tilde{F}^{22}}{\partial x^2} + \frac{\partial \tilde{F}^{32}}{\partial x^3} = 0 \tag{A.14.21}$$

from above Appendix A12 $\tilde{F}^{02} = B_y, \tilde{F}^{12} = -\frac{E_z}{c}, \tilde{F}^{22} = 0, \tilde{F}^{32} = \frac{E_x}{c}$

$$\frac{\partial}{\partial t}(B_y) - \frac{\partial}{\partial x}\frac{E_z}{c} + \frac{\partial}{\partial z}\frac{E_x}{c} \tag{A.14.22}$$

$$(\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})_y \tag{A.14.23}$$

Consider the case:
$\underline{\beta = 3}$
(1.7.20)becomes:

$$\frac{\partial \tilde{F}^{03}}{\partial x^0} + \frac{\partial \tilde{F}^{13}}{\partial x^1} + \frac{\partial \tilde{F}^{23}}{\partial x^2} + \frac{\partial \tilde{F}^{33}}{\partial x^3} = 0 \tag{A.14.24}$$

from above Appendix A12 $\tilde{F}^{03} = B_z, \tilde{F}^{13} = \frac{E_y}{c}, \tilde{F}^{23} = -\frac{E_x}{c}, \tilde{F}^{33} = 0$
we get:

$$\frac{\partial}{\partial t}(B_z) + \frac{\partial}{\partial x}\frac{E_y}{c} + \frac{\partial}{\partial y}\frac{E_x}{c} \tag{A.14.25}$$

$$(\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})_z \tag{A.14.26}$$

This gives the other two Maxwell equations (1.7.2) , (1.7.3)

A.15 Showing The Maxwell Lagrangian gives the Equation of Motion

Recall (1.7.21) :

$$\mathcal{L} = \frac{-1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_\alpha J^\alpha \tag{A.15.1}$$

Euler-lagrange equation gives:

$$\partial_\beta [\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}] - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0 \tag{A.15.2}$$

This becomes

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = -J^\alpha \tag{A.15.3}$$

We also have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= \frac{-1}{4\mu_0}\frac{\partial(F_{\mu\nu}g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma})}{\partial(\partial_\beta A_\alpha)} \\
&= \frac{-1}{4\mu_0}g^{\mu\lambda}g^{\nu\sigma}(F_{\mu\nu}\frac{\partial F_{\lambda\sigma}}{\partial(\partial_\beta A_\alpha)} + \frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)}F_{\lambda\sigma}) \\
&= \frac{-1}{4\mu_0}g^{\mu\lambda}g^{\nu\sigma}(F_{\mu\nu}[\frac{\partial(\partial_\lambda A_\sigma)}{\partial(\partial_\beta A_\alpha)} - \frac{\partial(\partial_\sigma A_\lambda)}{\partial(\partial_\beta A_\alpha)}] + [\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\beta A_\alpha)}]F_{\lambda\sigma}) \\
&= -\frac{1}{4\mu_0}g^{\mu\lambda}g^{\nu\sigma}(F_{\lambda\sigma}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu) + F_{\mu\nu}(\delta^\beta_\lambda\delta^\alpha_\sigma - \delta^\beta_\sigma\delta^\alpha_\lambda)) \\
&= -\frac{1}{4\mu_0}[F_{\lambda\sigma}(g^{\beta\lambda}g^{\alpha\sigma} - g^{\alpha\lambda}g^{\beta\sigma}) + F_{\mu\nu}(g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta})] \\
&= -\frac{2F^{\beta\alpha} - 2F^{\alpha\beta}}{4\mu_0} \\
&= -\frac{F^{\beta\alpha}}{\mu_0} \qquad \text{(A.15.4)}
\end{aligned}$$

(A.15.3) rearranges to:

$$\partial_\beta[\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}] = \frac{\partial \mathcal{L}}{\partial A_\alpha} \qquad \text{(A.15.5)}$$

which means:

$$\partial_\beta[-\frac{F^{\beta\alpha}}{\mu_0}] = -J^\alpha \qquad \text{(A.15.6)}$$

i.e. :

$$\frac{\partial F^{\beta\alpha}}{\partial x^\beta} = \mu_0 J^\alpha \qquad \text{(A.15.7)}$$

which is the equation of motion above (1.7.22)

Appendix B

Appendix B : Chapter 2

B.1 Complex Scalar Field Obeys Klein-Gordon equation

Consider the complex scalar field Lagrangian (2.2.1)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^*(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x) \tag{B.1.1}$$

The action is the four integral of the Lagrangian density (2.1.5) :

$$S = \int d^4x \mathcal{L} \tag{B.1.2}$$

We know the variation of the action is 0 i.e.:

$$\delta S = 0 \tag{B.1.3}$$

Considering the two fields (ϕ, ϕ^*) as different and varying with respect to these two fields:

$$\delta S = \int d^4x [\frac{\delta \mathcal{L}}{\delta \phi}\delta\phi(y) + \frac{\delta \mathcal{L}}{\delta \phi^*}\delta\phi^*(y)] \tag{B.1.4}$$

by analogue with the real scalar case (2.1.7) we get :

$$\begin{aligned} \delta S &= \frac{1}{2}\int d^4x [(\frac{\partial \delta^4(x-y)}{\partial x_\mu}\partial_\mu \phi^*(x) - m^2\phi^*(x)\delta(x-y))\delta\phi(y) \\ &+ (\frac{\partial \delta^4(x-y)}{\partial x^\mu}\partial^\mu \phi(x) - m^2\phi(x)\delta(x-y))\delta\phi^*(y)] \end{aligned} \tag{B.1.5}$$

We consider only the second part as we want the equation for ϕ the two terms can be considered as both equal to zero as ϕ and ϕ^* are treated as different terms.
so we have:

$$\int d^4x (\frac{\partial \delta^4(x-y)}{\partial x^\mu}\partial^\mu \phi(x) - m^2\phi(x)\delta(x-y))\delta\phi^*(y) = 0 \tag{B.1.6}$$

where we have cancelled the $\frac{1}{2}$ as both sides are equal to zero
We integrate the first term by parts and ignore the constant term as $\partial^\mu \phi(x)$ should go to zero at the boundaries.
This gives:

$$\int d^4x(-\delta^4(x-y)\partial_\mu\partial^\mu\phi(x) - m^2\phi(x)\delta(x-y))\delta\phi^*(y) = 0 \tag{B.1.7}$$

We take out:

$$-\delta^4(x-y) \tag{B.1.8}$$

to give:

$$\int d^4x(\partial_\mu\partial^\mu\phi(x) + m^2\phi(x)) - \delta(x-y)\delta\phi^*(y) = 0 \tag{B.1.9}$$

using the delta we get :

$$(\partial_\mu\partial^\mu\phi(y) + m^2\phi(y)) - \delta\phi^*(y) = 0 \tag{B.1.10}$$

ignoring the factor $-\delta\phi^*(y)$ as both sides are zero we get:

$$\partial_\mu\partial^\mu\phi(y) + m^2\phi(y) = 0 \tag{B.1.11}$$

which is the Klein-Gordan equation (2.1.11)

B.2 Deriving $a(\vec{k})$ and $b(\vec{k})$

Recall (2.2.6) , (2.2.7) , (2.2.8) , (2.2.9) :

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) && \text{(B.2.1)}\\
\phi^*(x) &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(a^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} + b(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t}) && \text{(B.2.2)}\\
\pi(x) &= \dot{\phi}^*(x)\\
&= \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}(i)(a^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} - b(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t}) && \text{(B.2.3)}\\
\pi^*(x) &= \dot{\phi}(x)\\
&= \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}(-i)(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} - b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) && \text{(B.2.4)}
\end{aligned}$$

Multiplying $\phi(x)$ by $\frac{\sqrt{\omega_k}}{2}$ and $\pi^*(x)$ by $\frac{i}{\sqrt{\omega_k}}$ to get the same factor of a for both and a relative minus sign between the b^* factors i.e.:

$$\begin{aligned}
\sqrt{\frac{\omega_k}{2}}\phi(x) &= \int \frac{d^3k}{2(2\pi)^3}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) && \text{(B.2.5)}\\
\frac{i}{\sqrt{2\omega_k}}\pi^*(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} - b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) && \text{(B.2.6)}
\end{aligned}$$

summing these gives :

$$\sqrt{\frac{\omega_k}{2}}\phi(x) + \frac{i}{\sqrt{2\omega_k}}\pi^*(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) + \frac{1}{2}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} - b^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) \quad \text{(B.2.7)}$$

This simplifies to :

$$\sqrt{\frac{\omega_k}{2}}\phi(x) + \frac{i}{\sqrt{2\omega_k}}\pi^*(x) = \int \frac{d^3k}{(2\pi)^3}a(\vec{k})^{i\vec{k}.\vec{x}-i\omega_k t} \quad \text{(B.2.8)}$$

Fourier transforming gives :

$$\hat{a}(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\phi(x) + \frac{i}{\sqrt{2\omega_k}}\pi^*(x))e^{-i\vec{k}.\vec{x}+i\omega_k t} \quad \text{(B.2.9)}$$

Multiplying $\phi^*(x)$ by $\frac{\sqrt{\omega_k}}{2}$ and $\pi(x)$ by $\frac{i}{\sqrt{\omega_k}}$ to get the same factor of b for both and a relative minus sign between the a^* factors i.e.:

$$\sqrt{\frac{\omega_k}{2}}\phi^*(x) = \int \frac{d^3k}{2(2\pi)^3}(a^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} + b(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t}) \quad \text{(B.2.10)}$$

$$\frac{i}{\sqrt{2\omega_k}}\pi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}(-a^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} + b(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) \quad \text{(B.2.11)}$$

summing these we get:

$$\sqrt{\frac{\omega_k}{2}}\phi^*(x) + \frac{i}{\sqrt{2\omega_k}}\pi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}(b(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + a^*(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) + \frac{1}{2}(a(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} - b(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) \quad \text{(B.2.12)}$$

This simplifies to:

$$\sqrt{\frac{\omega_k}{2}}\phi^*(x) + \frac{i}{\sqrt{2\omega_k}}\pi(x) = \int \frac{d^3k}{(2\pi)^3}b(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} \quad \text{(B.2.13)}$$

Fourier transforming gives :

$$\hat{b}(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\phi^*(x) + \frac{i}{\sqrt{2\omega_k}}\pi(x))e^{-i\vec{k}.\vec{x}+i\omega_k t} \quad \text{(B.2.14)}$$

B.3 Commutator Relations of $\hat{a}$ and $\hat{b}$

We now recall , and their conjugate transposes:

$$\hat{a}(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{-i\vec{k}.\vec{x}+i\omega_k t} \tag{B.3.1}$$

$$\hat{a}^\dagger(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) - \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))e^{i\vec{k}.\vec{x}-i\omega_k t} \tag{B.3.2}$$

$$\hat{b}(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))e^{-i\vec{k}.\vec{x}+i\omega_k t} \tag{B.3.3}$$

$$\hat{b}^\dagger(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) - \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{i\vec{k}.\vec{x}-i\omega_k t} \tag{B.3.4}$$

We consider the commutator of a and its conjugete transpose:

$$\begin{aligned}
[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] &= (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&\times (\int d^3x'(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x'))e^{i\vec{k}'.\vec{x}'-i\omega_{k'} t}) \\
&- (\int d^3x'(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x'))e^{i\vec{k}'.\vec{x}'-i\omega_{k'} t}) \\
&\times (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&= \int d^3x \int d^3x'((\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x)) \\
&\times (\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x')) \\
&- (\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x')) \\
&\times (\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x)))e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_k')t)} \\
&= \int d^3x \int d^3x(\frac{\sqrt{\omega_k\omega_{k'}}}{2}[\hat{\phi}(x), \hat{\phi}^\dagger(x')] \\
&- \frac{1}{2\sqrt{\omega_k\omega_{k'}}}[\hat{\pi}^\dagger(x), \hat{\pi}(x')] \\
&- \frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[\hat{\phi}(x), \hat{\pi}(x)] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[\hat{\pi}^\dagger(x), \hat{\phi}^\dagger(x)])e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_k')t)}
\end{aligned} \tag{B.3.5}$$

Recall (2.2.12):

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t', \vec{x}')] \mid_{t=t'} = i\hbar\delta^{(3)}(\vec{x} - \vec{x}') \tag{B.3.6}$$

transposing and conjugating gives:

$$[\hat{\phi}^{\dagger}(t,\vec{x}),\hat{\pi}^{\dagger}(t',\vec{x}')]_{t=t'} = i\hbar\delta^{(3)}(\vec{x}-\vec{x}') \tag{B.3.7}$$

the ordering exchange imposed by transposing the elements cancels the -1 from conjugating the i
using the Dirac pescription:

$$[\hat{\phi}(t,\vec{x}),\hat{\phi}^{\dagger}(t,\vec{x})]_{t=t'} = [\hat{\pi}(t,\vec{x}),\hat{\pi}^{\dagger}(t,\vec{x})]_{t=t'} = 0 \tag{B.3.8}$$

This means (B.3.5)

$$\begin{aligned}
[\hat{a}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')] &= \int d^3x \int d^3x(-\frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}(i\hbar\delta^{(3)}(\vec{x}-\vec{x}')) \\
&+ \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}(-i\hbar\delta^{(3)}(\vec{x}-\vec{x}')))e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_k')t)} \\
&= \hbar\int d^3x(\frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}+\frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}})e^{i((\vec{k}'.-\vec{k}).\vec{x}+(\omega_k-\omega_k')t)}
\end{aligned} \tag{B.3.9}$$

Thus using the fourier definition of the three-dimensional delta (1.2.12):

$$[\hat{a}(\vec{k}),\hat{a}^{\dagger}(\vec{k})] = (2\pi)^3\hbar\delta^3(\vec{k}-\vec{k}') \tag{B.3.10}$$

We consider the commutator of a and its conjugate transpose:

$$
\begin{aligned}
[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] &= (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&\times (\int d^3x'(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^\dagger(x'))e^{i\vec{k}'.\vec{x}'-i\omega_{k'} t}) \\
&- (\int d^3x'(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^\dagger(x'))e^{i\vec{k}'.\vec{x}'-i\omega_{k'} t}) \\
&\times (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&= \int d^3x \int d^3x'[(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) \\
&+ \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^\dagger(x')) \\
&- ((\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^\dagger(x')) \\
&\times (\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x)))]e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_{k'})t)} \\
&= \int d^3x \int d^3x'[\frac{\sqrt{\omega_k\omega_{k'}}}{2}[\hat{\phi}^\dagger(x), \hat{\phi}(x')] \\
&- \frac{1}{2\sqrt{\omega_k\omega_{k'}}}[\hat{\pi}(x), \hat{\pi}^\dagger(x')] \\
&- \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[\hat{\phi}^\dagger(x), \hat{\pi}^\dagger(x)] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[\hat{\pi}(x), \hat{\phi}(x)]]e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_{k'})t)} \qquad \text{(B.3.11)}
\end{aligned}
$$

using the Dirac prescription to give:

$$[\hat{\phi}^\dagger(x), \hat{\phi}(x')]_{t=t'} = [\hat{\pi}(x), \hat{\pi}^\dagger(x')]_{t=t'} = 0 \qquad \text{(B.3.12)}$$

$$[\phi^\dagger(x), \pi^\dagger(x)]_{t=t'} = [\phi(x), \pi(x)]_{t=t'} = i\hbar\delta^3(\vec{x} - \vec{x}') \qquad \text{(B.3.13)}$$

So (B.3.11) becomes:

$$
\begin{aligned}
[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k})] &= \int d^3x \int d^3x'[-\frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[i\hbar\delta^3(\vec{x} - \vec{x}')] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[-i\hbar\delta^3(\vec{x} - \vec{x}')]e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_{k'})t)} \\
&= \frac{\hbar}{2}\int d^3x[\sqrt{\frac{\omega_{k'}}{\omega_k}} + \sqrt{\frac{\omega_k}{\omega_{k'}}}]e^{i((\vec{k}'-\vec{k}).\vec{x}+(\omega_k-\omega_{k'})t)} \qquad \text{(B.3.14)}
\end{aligned}
$$

Then using the fourier definition of the three dimensional delta (1.2.12) we get:

$$[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = (2\pi)^3 \hbar \delta^3(\vec{k} - \vec{k}') \tag{B.3.15}$$

clearly for any operator $\hat{p}$:

$$[\hat{p}, \hat{p}] = \hat{p}\hat{p} - \hat{p}\hat{p} = 0 \tag{B.3.16}$$

i.e. :

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{B.3.17}$$

The commutator of a with b is:

$$\begin{aligned}
[\hat{a}(\vec{k}), \hat{b}(\vec{k}')] &= (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&\times (\int d^3x(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') + \frac{i}{\sqrt{2\omega_{\prime}k}}\hat{\pi}(x'))e^{-i\vec{k}'.\vec{x}'+i\omega_{k'}t}) \\
&- (\int d^3x(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&\times (\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x))e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&= \int d^3x \int d^3x'[(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x)) \\
&\times (\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') + \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x')) \\
&- ((\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}^\dagger(x') + \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}(x')) \\
&\times (\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(x)))]e^{i(-\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k+\omega_{k'})t)} \\
&= \int d^3x \int d^3x'[\frac{\sqrt{\omega_k\omega_{k'}}}{2}[\hat{\phi}(\mathbf{x}), \hat{\phi}^\dagger(x')] \\
&- \frac{1}{2\sqrt{\omega_k\omega_{k'}}}[\hat{\pi}^\dagger(x), \hat{\pi}(x')] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[\hat{\phi}(x), \hat{\pi}(x')] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[\hat{\pi}^\dagger(\mathbf{x}), \hat{\phi}^\dagger(x')]]e^{i(-\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k+\omega_{k'})t)}
\end{aligned} \tag{B.3.18}$$

using from the Dirac prescription :

$$[\hat{\phi}^\dagger(x), \hat{\phi}(x')]_{t=t'} = [\hat{\pi}(x), \hat{\pi}^\dagger(x')]_{t=t'} = 0 \tag{B.3.19}$$

$$[\hat{\phi}^\dagger(x), \hat{\pi}^\dagger(x)]_{t=t'} = [\hat{\phi}(x), \hat{\pi}(x)]_{t=t'} = i\hbar\delta^3(\vec{x}-\vec{x}') \tag{B.3.20}$$

$$\begin{aligned} [\hat{a}(\vec{k}), \hat{b}(\vec{k}')] &= \int d^3x \int d^3x' (\frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[i\hbar\delta(\vec{x}-\vec{x}')] \\ &+ \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[-i\hbar\delta(\vec{x}-\vec{x}')]])e^{i(-\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k+\omega_{k'})t)} \\ &= \frac{\hbar}{2}\int d^3x(-\sqrt{\frac{\omega_k}{\omega_{k'}}}+\sqrt{\frac{\omega_{k'}}{\omega_k}})e^{i(-(\vec{k}'.+\vec{k}).\vec{x}+(\omega_k+\omega_{k'})t)} \end{aligned} \tag{B.3.21}$$

Then using the fourier definition of the three dimensional delta and realising the sum in brackets no longer depends on $\vec{k}, \vec{k}', \omega_k, \omega_{k'}$ but just on the ratio between ω_k and $\omega_{k'}$ which depend only on $\vec{k}^2$ which is the same for $\vec{k}$ and $-\vec{k}$
we have:

$$[\hat{a}(\vec{k}), \hat{b}(\vec{k}')] = 0 \tag{B.3.22}$$

generally

$$[\hat{a}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = ([\hat{b}(\vec{k}'), \hat{a}(\vec{k})])^\dagger \tag{B.3.23}$$

Thus:

$$[\hat{a}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{B.3.24}$$

The commutator of a with the conjugate transpose for b is:

$$
\begin{aligned}
[\hat{a}(\vec{k}), \hat{b}^{\dagger}(\vec{k}')] &= \left(\int d^3x\left(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^{\dagger}(x)\right)e^{-i\vec{k}.\vec{x}+i\omega_k t}\right) \\
&\times \left(\int d^3x\left(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{'}k}}\hat{\pi}^{\dagger}(x')\right)e^{i\vec{k}'.\vec{x}'-i\omega_{k'}t}\right) \\
&- \left(\int d^3x\left(\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^{\dagger}(x')\right)e^{i\vec{k}.\vec{x}-i\omega_k t}\right) \\
&\times \left(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^{\dagger}(x)\right)e^{-i\vec{k}.\vec{x}+i\omega_k t}) \\
&= \int d^3x \int d^3x'[(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^{\dagger}(x)) \\
&\times (\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^{\dagger}(x') \\
&- ((\sqrt{\frac{\omega_{k'}}{2}}\hat{\phi}(x') - \frac{i}{\sqrt{2\omega_{k'}}}\hat{\pi}^{\dagger}(x')) \\
&\times (\sqrt{\frac{\omega_k}{2}}\hat{\phi}(x) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^{\dagger}(x)))]e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_{k'})t)} \\
&= \int d^3x \int d^3x'[\frac{\sqrt{\omega_k\omega_{k'}}}{2}[\hat{\phi}(x), \hat{\phi}(x')] \\
&+ \frac{1}{2\sqrt{\omega_k\omega_{k'}}}[\hat{\pi}^{\dagger}(x), \hat{\pi}^{\dagger}(x')] \\
&- \frac{i}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}[\hat{\phi}(x), \hat{\pi}^{\dagger}(x')] \\
&+ \frac{i}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}[\hat{\pi}^{\dagger}(x), \hat{\phi}(x')]]e^{i(\vec{k}'.\vec{x}'-\vec{k}.\vec{x}+(\omega_k-\omega_{k'})t)} \qquad \text{(B.3.25)}
\end{aligned}
$$

using :

$$[\hat{\phi}^{\dagger}(x), \hat{\phi}(x')]_{t=t'} = [\hat{\pi}(x), \hat{\pi}^{\dagger}(x')]_{t=t'} = 0 \qquad \text{(B.3.26)}$$

Using the Dirac procedure the analogue of :

$$[\hat{\phi}^{\dagger}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] \qquad \text{(B.3.27)}$$

is

$$\{p^*, q\} = \frac{\partial p^*}{\partial p}\frac{\partial q}{\partial q} - \frac{\partial q}{\partial p}\frac{\partial p^*}{\partial q} = 0 \qquad \text{(B.3.28)}$$

So:

$$[\hat{\phi}^{\dagger}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = 0 \qquad \text{(B.3.29)}$$

and as :

$$[\pi^{\dagger}(\mathbf{x}), \phi(\mathbf{x})] = ([\phi^{\dagger}(\mathbf{x}), \pi(\mathbf{x})])^{\dagger} \qquad \text{(B.3.30)}$$

$$[\phi^{\dagger}(\mathbf{x}), \pi(\mathbf{x})] = 0 \qquad \text{(B.3.31)}$$

Thus:

$$[\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{B.3.32}$$

and

$$[\hat{b}^\dagger(\vec{k}'), \hat{a}(\vec{k})] = -[\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{B.3.33}$$

$$[\hat{a}^\dagger(\vec{k}), \hat{b}(\vec{k}')] = [\hat{b}^\dagger(\vec{k}'), \hat{a}(\vec{k})]^\dagger = 0 \tag{B.3.34}$$

$$[\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{b}(\vec{k}')] = 0 \tag{B.3.35}$$

B.4 $[\hat{a}, \hat{b}]^\dagger = [\hat{b}^\dagger, \hat{a}^\dagger]$

$$[\hat{a}, \hat{b}]^\dagger = (\hat{a}\hat{b})^\dagger - (\hat{b}\hat{a})^\dagger = \hat{b}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger = [\hat{b}^\dagger, \hat{a}^\dagger] \tag{B.4.1}$$

B.5 The Hamiltonian for the Complex Scalar Field

Recall (2.1.13):

$$\hat{H} = \int d^4x(\hat{\pi}(x)\dot{\hat{\phi}}(x) - \mathcal{L}) \tag{B.5.1}$$

We also have (2.2.1):

$$\mathcal{L} = \frac{1}{2}\dot{\hat{\phi}}\dot{\hat{\phi}}^\dagger - \frac{1}{2} \mid \vec{\nabla}\hat{\phi} \mid^2 - \frac{1}{2}m^2 \mid \hat{\phi} \mid^2 \tag{B.5.2}$$

Using (2.2.3):

$$\hat{\pi} = \dot{\hat{\phi}}^\dagger \qquad \hat{\pi}^\dagger = \dot{\hat{\phi}} \tag{B.5.3}$$

Thus;

$$\hat{H} = \int d^4x(\frac{1}{2} \mid \hat{\pi}(x) \mid^2 + \frac{1}{2} \mid \vec{\nabla}\hat{\phi} \mid^2 + \frac{1}{2}m^2 \mid \hat{\phi} \mid^2) \tag{B.5.4}$$

Recall (2.2.6) ,(2.2.7) , (2.2.8) , (2.2.9)

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(\hat{a}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + \hat{b}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) \tag{B.5.5}$$

$$\hat{\phi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(\hat{a}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} + \hat{b}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t}) \tag{B.5.6}$$

$$\hat{\pi}(x) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}(i)(\hat{a}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} - \hat{b}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t}) \tag{B.5.7}$$

$$\hat{\pi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}(-i)(\hat{a}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} - \hat{b}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t}) \tag{B.5.8}$$

we recall that $\omega_k^2 = m^2 + \vec{k}^2$ We can see from the plane wave form of these solutions that:

$$\mid \vec{\nabla}\hat{\phi} \mid^2 = \vec{k}^2 \mid \hat{\phi} \mid^2 \tag{B.5.9}$$

Hence

$$\hat{H} = \int d^4x(\frac{1}{2} \mid \hat{\pi}(x) \mid^2 + \frac{1}{2}\omega_k^2 \mid \hat{\phi}(x) \mid^2) \tag{B.5.10}$$

$$\begin{aligned}
\mid \hat{\phi}(x) \mid^2 &= \underbrace{\int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(\hat{a}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t} + \hat{b}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t})}_{\phi(x)} \\
&\times \underbrace{\int \frac{d^3k'}{(2\pi)^3\sqrt{2\omega_{k'}}}(\hat{a}^\dagger(\vec{k}')e^{-i\vec{k}'.\vec{x}+i\omega_{k'} t} + \hat{b}(\vec{k}')e^{i\vec{k}'.\vec{x}-i\omega_{k'} t})}_{\phi^\dagger(x)} \\
&= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{d^3k'}{(2\pi)^3\sqrt{2\omega_{k'}}}(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}')e^{i(\vec{k}-\vec{k}').\vec{x}-i(\omega_k-\omega_{k'})t} \\
&+ \hat{a}(\vec{k})\hat{b}(\vec{k}')e^{i(\vec{k}+\vec{k}').\vec{x}-i(\omega_k+\omega_{k'})t} + \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k}')e^{-i(\vec{k}+\vec{k}').\vec{x}+i(\omega_k+\omega_{k'})t} \\
&+ \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')e^{-i(\vec{k}-\vec{k}').\vec{x}+i(\omega_k-\omega_{k'})t}) \qquad (B.5.11)
\end{aligned}$$

$$\begin{aligned}
\mid \hat{\pi}(x) \mid^2 &= \underbrace{\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}(i)(\hat{a}^\dagger(\vec{k})e^{-i\vec{k}.\vec{x}+i\omega_k t} - \hat{b}(\vec{k})e^{i\vec{k}.\vec{x}-i\omega_k t})}_{\hat{\pi}(x)} \\
&\times \underbrace{\int \frac{d^3k'}{(2\pi)^3}\sqrt{\frac{\omega_{k'}}{2}}(-i)(\hat{a}(\vec{k}')e^{i\vec{k}'.\vec{x}-i\omega_{k'} t} - \hat{b}^\dagger(\vec{k}')e^{-i\vec{k}'.\vec{x}+i\omega_{k'} t})}_{\hat{p}i^\dagger(x)} \\
&= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\sqrt{\omega_k\omega_{k'}}}{2}(\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}')e^{-i(\vec{k}-\vec{k}').\vec{x}+i(\omega_k-\omega_{k'})t} \\
&- \hat{a}^\dagger(\vec{k})\hat{b}^\dagger(\vec{k}')e^{-i(\vec{k}+\vec{k}').\vec{x}+i(\omega_k+\omega_{k'})t} \\
&- \hat{b}(\vec{k})\hat{a}(\vec{k}')e^{i(\vec{k}+\vec{k}').\vec{x}-i(\omega_k+\omega_{k'})t} \qquad (B.5.12) \\
&+ \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}')e^{i(\vec{k}-\vec{k}').\vec{x}-i(\omega_k-\omega_{k'})t}) \qquad (B.5.13)
\end{aligned}$$

Thus we get:

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\frac{\sqrt{\omega_k\omega_{k'}}}{2}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}')(\int d^4x e^{-i(\vec{k}-\vec{k}').\vec{x}+i(\omega_k-\omega_{k'})t}) \\
&+ (\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}\hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k}') - \frac{\sqrt{\omega_k\omega_{k'}}}{2}\hat{a}^\dagger(\vec{k})\hat{b}^\dagger(\vec{k}'))(\int d^4x e^{-i(\vec{k}+\vec{k}').\vec{x}+i(\omega_k+\omega_{k'})t}) \\
&+ (\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}\hat{a}(\vec{k})\hat{b}(\vec{k}') - \hat{b}(\vec{k})\hat{a}(\vec{k}')\frac{\sqrt{\omega_k\omega_{k'}}}{2})(\int d^4x e^{i(\vec{k}+\vec{k}').\vec{x}-i(\omega_k+\omega_{k'})t}) \\
&+ \frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}')(\int d^4x e^{i(\vec{k}-\vec{k}').\vec{x}-i(\omega_k-\omega_{k'})t}) \\
&+ \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}')\frac{\sqrt{\omega_k\omega_{k'}}}{2}(\int d^4x e^{-i(\vec{k}-\vec{k}').\vec{x}+i(\omega_k-\omega_{k'})t}) \\
&+ \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}(\int d^4x e^{i(\vec{k}-\vec{k}').\vec{x}-i(\omega_k-\omega_{k'})t}) \qquad \text{(B.5.14)}
\end{aligned}$$

using definition of delta (1.2.12):

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\frac{\sqrt{\omega_k\omega_{k'}}}{2}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}')((2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{i(\omega_k-\omega_{k'})t}) \\
&+ (\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}b^\dagger(\vec{k})a^\dagger(\vec{k}') - \frac{\sqrt{\omega_k\omega_{k'}}}{2}\hat{a}^\dagger(\vec{k})\hat{b}^\dagger(\vec{k}'))((2\pi)^3\delta^3(\vec{k}+\vec{k}')e^{i(\omega_k+\omega_{k'})t}) \\
&+ (\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}\hat{a}(\vec{k})\hat{b}(\vec{k}') - \hat{b}(\vec{k})\hat{a}(\vec{k}')\frac{\sqrt{\omega_k\omega_{k'}}}{2})((2\pi)^3\delta^3(\vec{k}+\vec{k}')e^{-i(\omega_k+\omega_{k'})t}) \\
&+ \frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}')((2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{-i(\omega_k-\omega_{k'})t}) \\
&+ \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}')\frac{\sqrt{\omega_k\omega_{k'}}}{2}((2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{i(\omega_k-\omega_{k'})t}) \\
&+ \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')\frac{\omega_k^2}{2\sqrt{\omega_k\omega_{k'}}}((2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{-i(\omega_k-\omega_{k'})t}) \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) \\
&+ \frac{\omega_k}{2}(\hat{b}^\dagger(\vec{k})\hat{a}^\dagger(-\vec{k}) - \hat{a}^\dagger(\vec{k})\hat{b}^\dagger(-\vec{k}))e^{i2\omega_k t} \\
&+ \frac{\omega_k}{2}(\hat{a}(\vec{k})\hat{b}(-\vec{k}') - \hat{b}(\vec{k})\hat{a}(-\vec{k}'))e^{-i2\omega_k t} \\
&+ \frac{\omega_k}{2}\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}') + \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}')\frac{\omega_k}{2} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')\frac{\omega_k}{2} \qquad \text{(B.5.15)}
\end{aligned}$$

using from Appendix B3:

$$[\hat{a}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}(\vec{k}), \hat{b}(\vec{k}')] = 0 \qquad \text{(B.5.16)}$$

we obtain:

$$\hat{H} = \frac{\omega_k}{4}\int \frac{d^3k}{(2\pi)^3}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}') + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) \qquad \text{(B.5.17)}$$

using from Appendix B3:

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = (2\pi)^3\hbar\delta^3(\vec{k}-\vec{k}') \tag{B.5.18}$$

and thus:

$$\hat{H} = \frac{\omega_k}{2}\int \frac{d^3k}{(2\pi)^3}\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) + 2(2\pi)^3\hbar \tag{B.5.19}$$

B.6 $S^{\mu\nu}$ Satisfy the Lorentz Algebra

The general Lorentz algebra is eq 3.17 p39 [9] :

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \tag{B.6.1}$$

For this section we assume that S is in matrix formation we prove this form in Appendix B12
The forms of S are:

$$S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} \tag{B.6.2}$$

and:

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix} \tag{B.6.3}$$

Since (2.3.5) :

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{B.6.4}$$

If $\mu = \nu$:

$$S^{\mu\mu} = \frac{i}{4}[\gamma^\mu, \gamma^\mu] = 0 \tag{B.6.5}$$

Since :

$$S^{\mu\nu} = -S^{\nu\mu} \tag{B.6.6}$$

as:

$$[\gamma^\mu, \gamma^\nu] = -[\gamma^\nu, \gamma^\mu] \tag{B.6.7}$$

We need thereforeonly consider the commutators and self-commutators of S^{0i} and S^{ij}
since by the definition of commutators (1.2.6) $[S^{0i}, S^{jk}] = -[S^{jk}, S^{0i}]$ we just consider $[S^{0i}, S^{jk}]$
The Lorentz algebra is now :

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}) \tag{B.6.8}$$

Thus we need:

$$\begin{aligned}[S^{0i}, S^{0j}] &= i(\underbrace{g^{i0}}_{=0}S^{0i} - \underbrace{g^{00}}_{=1}S^{ij} - g^{ij}\underbrace{S^{00}}_{=0} + \underbrace{g^{0i}}_{=0}S^{0i}) \\ &= -iS^{ij}\end{aligned} \tag{B.6.9}$$

$$
\begin{aligned}
[S^{0i}, S^{jk}] &= i(g^{ij}S^{0k} - \underbrace{g^{0j}}_{=0}S^{ik} - g^{ik}S^{0j} + \underbrace{g^{0k}}_{=0}S^{ij}) \\
&= i(g^{ij}S^{0k} - g^{ik}S^{0j}) \qquad \text{(B.6.10)}
\end{aligned}
$$

$$
[S^{ij}, S^{kl}] = i(g^{jk}S^{il} - g^{ik}S^{jl} - g^{jl}S^{ik} + g^{il}S^{jk}) \qquad \text{(B.6.11)}
$$

using the matrix form:

$$
\begin{aligned}
[S^{0i}, S^{0j}] &= (-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix})(-\frac{i}{2}\begin{pmatrix}\sigma^j & 0\\ 0 & -\sigma^j\end{pmatrix}) - (-\frac{i}{2}\begin{pmatrix}\sigma^j & 0\\ 0 & -\sigma^j\end{pmatrix})(-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}) \\
&= (-\frac{i}{2})^2[\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\begin{pmatrix}\sigma^j & 0\\ 0 & -\sigma^j\end{pmatrix} - \begin{pmatrix}\sigma^j & 0\\ 0 & -\sigma^j\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}] \\
&= (-\frac{i}{2})^2[\begin{pmatrix}\sigma^i\sigma^j & 0\\ 0 & \sigma^i\sigma^j\end{pmatrix} - \begin{pmatrix}\sigma^j\sigma^i & 0\\ 0 & -\sigma^j\sigma^i\end{pmatrix}] \qquad \text{(B.6.12)}
\end{aligned}
$$

We showed above (1.5.7) that

$$
\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k \qquad \text{(B.6.13)}
$$

Thus by the anti-symmetry of epsilon $\sigma^j\sigma^i = -\sigma^i\sigma^j$
Thus:

$$
\begin{aligned}
[S^{0i}, S^{0j}] &= -\frac{1}{4}[\begin{pmatrix}2\sigma^i\sigma^j & 0\\ 0 & 2\sigma^i\sigma^j\end{pmatrix}] \\
&= -i\frac{1}{2}[\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}] \qquad \text{(B.6.14)}
\end{aligned}
$$

So :

$$
[S^{0i}, S^{0j}] = -iS^{ij} \qquad \text{(B.6.15)}
$$

as required

Using the matrix form:

$$
\begin{aligned}
[S^{0i}, S^{jk}] &= (-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix})(\frac{1}{2}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix}) - (\frac{1}{2}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix})(-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}) \\
&= (-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix})(\frac{1}{2}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix}) - (\frac{1}{2}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix})(-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}) \\
&= -\frac{i}{4}\epsilon^{jkl}[\begin{pmatrix}\sigma^i\sigma^l & 0\\ 0 & -\sigma^i\sigma^l\end{pmatrix} - \begin{pmatrix}\sigma^l\sigma^i & 0\\ 0 & -\sigma^k\sigma^i\end{pmatrix}] \qquad \text{(B.6.16)}
\end{aligned}
$$

We define analogously to (1.5.7):

$$
\sigma^i\sigma^l = i\epsilon^{ilm}\sigma^m \qquad \text{(B.6.17)}
$$

thus by the anti-symmetry of epsilon

$$\sigma^i \sigma^l = -\sigma^l \sigma^i \tag{B.6.18}$$

So we get:

$$\begin{aligned}
[S^{0i}, S^{jk}] &= -\frac{i}{4}\epsilon^{jkl}[\begin{pmatrix} 2\sigma^i\sigma^l & 0 \\ 0 & -2\sigma^i\sigma^l \end{pmatrix}] \\
&= -\frac{i}{2}\epsilon^{jkl} i\epsilon^{ilm}[\begin{pmatrix} \sigma^m & 0 \\ 0 & -\sigma^m \end{pmatrix}] \qquad \text{(B.6.19)}
\end{aligned}$$

by the anti-symmetry of epsilon

$$\epsilon^{jkl}\epsilon^{ilm} = -\epsilon^{jkl}\epsilon^{iml} \tag{B.6.20}$$

we know []

$$\epsilon^{jkl}\epsilon^{iml} = \delta_{ij}\delta_{km} - \delta_{jm}\delta_{ki} \tag{B.6.21}$$

$$\begin{aligned}
[S^{0i}, S^{jk}] &= i(-\frac{i}{2}(\delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km})[\begin{pmatrix} \sigma^m & 0 \\ 0 & -\sigma^m \end{pmatrix}]) \\
&= i\delta_{ki}(-\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}) - i\delta_{ij}(-\frac{i}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \qquad \text{(B.6.22)}
\end{aligned}$$

using the definition of S^{0i} (B.6.2) :

$$[S^{0i}, S^{jk}] = i(\delta_{ki}S^{0j} - i\delta_{ij}S^{0k}) \tag{B.6.23}$$

using :

$$\delta_{ki} = -g^{ik} \tag{B.6.24}$$

and

$$\delta_{ij} = -g^{ij} \tag{B.6.25}$$

So :

$$[S^{0i}, S^{jk}] = i(g^{ij}S^{0k} - g^{ik}S^{0j}) \tag{B.6.26}$$

In matrix form:

$$\begin{aligned}
[S^{ij}, S^{lm}] &= \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\frac{1}{2}\epsilon^{lmn}\begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} \\
&- \frac{1}{2}\epsilon^{lmn}\begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix}\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \\
&= \frac{1}{4}\epsilon^{ijk}\epsilon^{lmn}(\begin{pmatrix} \sigma^k\sigma^n & 0 \\ 0 & \sigma^k\sigma^n \end{pmatrix} - \begin{pmatrix} \sigma^n\sigma^k & 0 \\ 0 & \sigma^n\sigma^k \end{pmatrix}) \qquad \text{(B.6.27)}
\end{aligned}$$

We define analogously to (1.5.7):

$$\sigma^k \sigma^n = i\epsilon^{knp}\sigma^p \tag{B.6.28}$$

Then:

$$\begin{aligned}
[S^{ij}, S^{lm}] &= \frac{1}{4}\epsilon^{ijk}\epsilon^{lmn}i\epsilon^{knp}\begin{pmatrix} 2\sigma^p & 0 \\ 0 & 2\sigma^p \end{pmatrix} \\
&= \frac{1}{2}\epsilon^{ijk}\epsilon^{lmn}i\epsilon^{knp}\begin{pmatrix} \sigma^p & 0 \\ 0 & \sigma^p \end{pmatrix}
\end{aligned} \tag{B.6.29}$$

by the antisymmetry of epsilon:

$$\epsilon^{lmn}\epsilon^{knp} = -\epsilon^{lmn}\epsilon^{kpn} \tag{B.6.30}$$

anagolously to (B.6.21) we then have :

$$\epsilon^{lmn}\epsilon^{kpn} = \delta_{lk}\delta_{mp} - \delta_{lp}\delta_{mk} \tag{B.6.31}$$

$$\begin{aligned}
[S^{ij}, S^{lm}] &= i(\frac{1}{2}\epsilon^{ijk}(\delta_{lp}\delta_{mk} - \delta_{lk}\delta_{mp})\begin{pmatrix} \sigma^p & 0 \\ 0 & \sigma^p \end{pmatrix}) \\
&= i(\frac{1}{2}(\epsilon^{ijm}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}\epsilon^{ijl}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix}) \\
&= i(\frac{1}{2}(\epsilon^{ijm}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} - \epsilon^{ijl}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix}))
\end{aligned} \tag{B.6.32}$$

We know that from (B.6.5) : $S^{11} = S^{22} = S^{33} = 0$
and that from (B.6.6) $S^{12} = -S^{21}$, $S^{13} = -S^{31}$, $S^{23} = -S^{32}$
we thus only have to check the three commutators:
1) $[S^{12}, S^{13}]$
2)$[S^{12}, S^{23}]$
3)$[S^{13}, S^{23}]$
taking the first and using (B.6.32):

$$\begin{aligned}
[S^{12}, S^{13}] &= i(\frac{1}{2}(\epsilon^{123}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} - \epsilon^{121}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix})) \\
&= i(\frac{1}{2}(\epsilon^{231}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix})
\end{aligned} \tag{B.6.33}$$

as $\epsilon^{123} = \epsilon^{231}$ and $\epsilon^{121} = 0$
We obtain:

$$[S^{12}, S^{13}] = iS^{23} \tag{B.6.34}$$

Using (B.6.11) we get:

$$\begin{aligned}
[S^{12}, S^{13}] &= i(g^{21}S^{13} - g^{11}S^{23} - g^{23}S^{11} + g^{13}S^{21}) \\
&= iS^{23}
\end{aligned} \tag{B.6.35}$$

as $g^{11} = -1$ $g^{21} = g^{23} = g^{13} = 0$
so there is agreement
Using (B.6.32)

$$\begin{aligned}
[S^{12}, S^{23}] &= i(\frac{1}{2}(\epsilon^{123}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} - \epsilon^{122}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix})) \\
&= -i(\frac{1}{2}(\epsilon^{132}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix})) \qquad \text{(B.6.36)}
\end{aligned}$$

using $\epsilon^{122} = 0$ and $\epsilon^{123} = -\epsilon^{132}$ this becomes:

$$[S^{12}, S^{23}] = -iS^{13} \qquad \text{(B.6.37)}$$

Using (B.6.11) we find:

$$\begin{aligned}
[S^{12}, S^{23}] &= i(g^{22}S^{13} - g^{12}S^{23} - g^{23}S^{12} + g^{13}S^{22}) \\
&= -iS^{13} \qquad \text{(B.6.38)}
\end{aligned}$$

since $g^{22} = -1$ $g^{12} = g^{23} = g^{13} = 0$
so there is agreement

using (B.6.32)

$$\begin{aligned}
[S^{13}, S^{23}] &= i(\frac{1}{2}(\epsilon^{133}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} - \epsilon^{132}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix})) \\
&= i(\frac{1}{2}(\epsilon^{123}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix})) \qquad \text{(B.6.39)}
\end{aligned}$$

using $\epsilon^{133} = 0$ and $\epsilon^{123} = -\epsilon^{132}$ which becomes:

$$[S^{13}, S^{23}] = iS^{12} \qquad \text{(B.6.40)}$$

using (B.6.11) we get:

$$\begin{aligned}
[S^{13}, S^{23}] &= i(g^{32}S^{13} - g^{12}S^{33} - g^{33}S^{12} + g^{13}S^{32}) \\
&= iS^{12} \qquad \text{(B.6.41)}
\end{aligned}$$

since $g^{33} = -1$ $g^{32} = g^{12} = g^{13} = 0$
so there is agreement

B.7 $\Lambda_{\frac{1}{2}}{}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$

Recall (2.3.7):

$$\Lambda_{\frac{1}{2}} = exp[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}] \tag{B.7.1}$$

Consider (p39 eq 3.18 [9]):

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha \tag{B.7.2}$$

Consider the transform of a four-vector :

$$V^\alpha \to exp[-\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_{\alpha\beta}]V^\beta \tag{B.7.3}$$

which is in infinitesimal form :

$$V^\alpha \to (\delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta)V^\beta \tag{B.7.4}$$

i.e. :

$$\begin{aligned} V^\alpha &\to (\delta\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}i(\delta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\delta^{\nu\alpha}))V^\beta \\ V &\to (\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \omega_{10}-\omega_{01} & \omega_{20}-\omega_{02} & \omega_{30}-\omega_{03} \\ \omega_{01}-\omega_{10} & 0 & \omega_{21}-\omega_{12} & \omega_{31}-\omega_{13} \\ \omega_{02}-\omega_{20} & \omega_{12}-\omega_{21} & 0 & \omega_{32}-\omega_{23} \\ \omega_{03}-\omega_{30} & \omega_{13}-\omega_{31} & \omega_{23}-\omega_{32} & 0 \end{pmatrix})V \end{aligned} \tag{B.7.5}$$

assuming $\omega_{\mu\nu} = -\omega_{\nu\mu}$ we get:

$$V \to \begin{pmatrix} 1 & -\omega_{01} & -\omega_{02} & -\omega_{03} \\ \omega_{01} & 1 & -\omega_{12} & -\omega_{13} \\ \omega_{02} & \omega_{12} & 1 & -\omega_{23} \\ \omega_{03} & \omega_{13} & \omega_{23} & 1 \end{pmatrix} V \tag{B.7.6}$$

which can be seen to represent boosts if one $\omega_{0i} = \beta$ and all other ω 's are zero
and represents rotations if one $\omega_{ij} = \theta$ and all other ω's are zero
Thus :

$$\Lambda^\mu{}_\nu = exp[-\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu] \tag{B.7.7}$$

Thus:

$$\Lambda_{\frac{1}{2}}{}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu \tag{B.7.8}$$

is :

$$exp[\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}]\gamma^\mu exp[-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}] = exp[-\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu]\gamma^\nu \tag{B.7.9}$$

or infinitesimally:

$$(1+\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^{\mu}(1-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})=(1-\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu})\gamma^{\nu} \tag{B.7.10}$$

which canceling the ones on each side and ignoring ω^2 terms is:

$$\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\gamma^{\mu}-\gamma^{\mu}\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}=-\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} \tag{B.7.11}$$

which canceling $-\frac{i}{2}\omega_{\rho\sigma}$ on both sides is:

$$\gamma^{\mu}S^{\rho\sigma}-S^{\rho\sigma}\gamma^{\mu}=(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} \tag{B.7.12}$$

which is:

$$[\gamma^{\mu},S^{\rho\sigma}]=(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} \tag{B.7.13}$$

We prove this in matrix form we need to then compute the four commutators:
1)$[\gamma^0,S^{0i}]$
2)$[\gamma^0,S^{ij}]$
3)$[\gamma^i,S^{0j}]$
4)$[\gamma^i,S^{mn}]$
We also need to show that , since by (B.6.5) $[\gamma^{\mu},S^{\rho\rho}]=0$:

$$(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}=0 \tag{B.7.14}$$

if $\rho=\sigma$
Substituting (B.7.2) we get:

$$(\mathcal{J}^{\rho\sigma}))^{\mu}{}_{\nu}=i(\delta^{\rho\mu}\delta^{\sigma}_{\nu}-\delta^{\rho}_{\nu}\delta^{\sigma\mu}) \tag{B.7.15}$$

If $\rho=\sigma$ we have:

$$(\mathcal{J}^{\rho\rho}))^{\mu}{}_{\nu}=i(\delta^{\rho\mu}\delta^{\rho}_{\nu}-\delta^{\rho}_{\nu}\delta^{\rho\mu})=0 \tag{B.7.16}$$

recalling (B.6.2) and (B.6.3) :

$$S^{0i}=-\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} \tag{B.7.17}$$

and

$$S^{ij}=\frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \tag{B.7.18}$$

Also recall (**??**) , (2.3.2)

$$\gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{B.7.19}$$

$$\gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} \tag{B.7.20}$$

Consider $\mu = \rho = 0$ $\sigma = i$ in matrix form:

$$\begin{aligned}
[\gamma^0, S^{0i}] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}) - (-\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= -\frac{i}{2} (\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}) \\
&= i((\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}) = i\gamma^i
\end{aligned} \tag{B.7.21}$$

from (B.7.2)

$$\begin{aligned}
(\mathcal{J}^{0i}))^0{}_\nu &= i(\delta^{00}\delta^i_\nu - \delta^0_\nu \delta^{0i}) \\
&= i\delta^i_\nu
\end{aligned} \tag{B.7.22}$$

as $\delta^{00} = 1$ $\delta^{0i} = 0$

$$(\mathcal{J}^{0i}))^0{}_\nu \gamma^\nu = i\gamma^i \tag{B.7.23}$$

which implies agreement
Consider $\mu = 0$ $\rho = i$ $\sigma = j$ in matrix form

$$\begin{aligned}
[\gamma^0, S^{ij}] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}) - (\frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{1}{2}\epsilon^{ijk} (\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}) \\
&= 0
\end{aligned} \tag{B.7.24}$$

from (B.7.2)

$$\begin{aligned}
(\mathcal{J}^{ij}))^0{}_\nu &= i(\delta^{i0}\delta^j_\nu - \delta^i_\nu \delta^{j0}) \\
&= 0
\end{aligned} \tag{B.7.25}$$

as $\delta^{i0} = \delta^{j0} = 0$

Consider $\mu = i$ $\rho = 0$ $\sigma = j$ in matrix form

$$\begin{aligned}
[\gamma^i, S^{0j}] &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} (-\frac{i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}) - (-\frac{i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}) \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\
&= -\frac{i}{2} \begin{pmatrix} 0 & -\sigma^i\sigma^j \\ -\sigma^i\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j\sigma^i \\ \sigma^j\sigma^i & 0 \end{pmatrix})
\end{aligned} \tag{B.7.26}$$

Recall (1.5.8) which implies:

$$\sigma^i\sigma^j = -\sigma^j\sigma^i \ i \neq j \tag{B.7.27}$$

So:

$$[\gamma^i, S^{0j}] = 0 \qquad i \neq j \tag{B.7.28}$$

Recall (1.5.10) we get:

$$\sigma^{i2} = 1 \tag{B.7.29}$$

So using (B.7.26):

$$\begin{aligned}
[\gamma^i, S^{0i}] &= -\frac{i}{2}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \\
&= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= i\gamma^0
\end{aligned} \tag{B.7.30}$$

Recall (B.7.2):

$$\begin{aligned}
(\mathcal{J}^{0j}))^i{}_\nu &= i(\delta^{0i}\delta^j_\nu - \delta^0_\nu\delta^{ji}) \\
&= -i\delta^0_\nu\delta^{ji}
\end{aligned} \tag{B.7.31}$$

as $\delta^{0i} = 0$
This means:

$$(\mathcal{J}^{0j}))^i{}_\nu\gamma^\nu = 0 \qquad i \neq j \tag{B.7.32}$$

$$(\mathcal{J}^{0j}))^i{}_\nu\gamma^\nu = i\gamma^0 \tag{B.7.33}$$

The two expressions are then equivalent. The minus sign comes from up to down notation, i.e.$\delta^0_\nu = -\delta^{0\nu}$
Consider $\mu = i$ $\rho = m$ $\sigma = n$ in matrix form:

$$\begin{aligned}
[\gamma^i, S^{mn}] &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}(\frac{1}{2}\epsilon^{mnl}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}) - (\frac{1}{2}\epsilon^{mnl}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix})\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\
&= \frac{1}{2}\epsilon^{mnl}(\begin{pmatrix} 0 & \sigma^i\sigma^l \\ -\sigma^i\sigma^l & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^l\sigma^i \\ -\sigma^l\sigma^i & 0 \end{pmatrix})
\end{aligned} \tag{B.7.34}$$

following (1.5.7) we define:

$$\sigma^i\sigma^l = i\epsilon^{ilp}\sigma^p \tag{B.7.35}$$

So:

$$\begin{aligned}
[\gamma^i, S^{mn}] &= i\epsilon^{mnl}\epsilon^{ilp}\begin{pmatrix} 0 & \sigma^p \\ -\sigma^p & 0 \end{pmatrix} \\
&= -i\epsilon^{mnl}\epsilon^{ipl}\gamma^p \\
&= -i(\delta_{mi}\delta_{np} - \delta_{mp}\delta_{ni})\gamma^p
\end{aligned} \tag{B.7.36}$$

Using (B.7.2)

$$(\mathcal{J}^{mn}))^i{}_\nu = i(\delta^{mi}\delta^n_\nu - \delta^m_\nu\delta^{ni} \tag{B.7.37}$$

The two expressions are then equivalent. The minus sign comes from up to down notation i.e. $\delta^n_\nu = -\delta^{n\nu}$ $\quad$ $\delta^m_\nu = -\delta^{m\nu}$

B.8 The Dirac Equation is Lorentz invariant

The lorentz transform of $\psi(x)$ is (2.3.9):

$$\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \tag{B.8.1}$$

The Lorentz transform of ∂_μ is the inverse of the transform on x^μ Thus:

$$\partial_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu \tag{B.8.2}$$

Our Dirac equation is (2.3.8)

$$[i\gamma^\mu\partial_\mu - m]\psi(x) = 0 \tag{B.8.3}$$

This transforms to:

$$[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \tag{B.8.4}$$

Using that:

$$\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1} = I \tag{B.8.5}$$

We get :

$$\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \tag{B.8.6}$$

Re-ordering our lambda's gives:

$$\Lambda_{\frac{1}{2}}[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x) \tag{B.8.7}$$

Using that(2.3.6) $\Lambda_{\frac{1}{2}}{}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. We get:

$$\Lambda_{\frac{1}{2}}[i\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x) \tag{B.8.8}$$

As we are summing over ρ we can change it to ν and re-order the Lambda's to give:

$$\Lambda_{\frac{1}{2}}[i\gamma^\nu\partial_\nu - m]\psi(\Lambda^{-1}x) \tag{B.8.9}$$

This is just a constant $\Lambda_{\frac{1}{2}}$ multiplying the Dirac equation, which is true, everywhere evaluated at $(\Lambda^{-1}x)$ instead of x
so it is a constant multiplied by 0 i.e.:

$$\Lambda_{\frac{1}{2}}[i\gamma^\nu\partial_\nu - m]\psi(\Lambda^{-1}x) = 0 \tag{B.8.10}$$

Thus the Dirac equation is Lorentz invariant

B.9 The Dirac Equation Implies the Klein-Gordon Equation

We know:

$$[i\gamma^\nu \partial_\nu - m]\psi(x) = 0 \tag{B.9.1}$$

the Dirac equation (2.3.8) multiplying by $[-i\gamma^\mu \partial_\mu - m]$ gives:

$$[-i\gamma^\mu \partial_\mu - m][i\gamma^\nu \partial_\nu - m]\psi(x) = 0 \tag{B.9.2}$$

Multiplying out gives:

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi(x) = 0 \tag{B.9.3}$$

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \tag{B.9.4}$$

by just changing the name of the variables
changing the order of the differentiation gives:

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \gamma^\nu \gamma^\mu \partial_\mu \partial_\nu \tag{B.9.5}$$

Thus :

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu \tag{B.9.6}$$

which is:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu \tag{B.9.7}$$

and thus we get for (B.9.3) :

$$(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2)\psi(x) = 0 \tag{B.9.8}$$

by our definition of the gamma's we know: (2.3.4)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{B.9.9}$$

so our equation (B.9.8)becomes:

$$(g^{\mu\nu}\partial_\mu \partial_\nu + m^2)\psi(x) = 0 \tag{B.9.10}$$

which using the definition of g (1.6.3) is :

$$(\partial^2 + m^2)\psi(x) = 0 \tag{B.9.11}$$

i.e. the klein-Gordon equation(2.1.11)

B.10 Lorentz Transformation of $\overline{\psi}$

we define (2.3.10)

$$\overline{\psi} = \psi^\dagger \gamma^0 \tag{B.10.1}$$

and we recall $\psi^\dagger$ transforms to $\psi^\dagger \Lambda^{\frac{1}{2}\dagger}$ by taking the conjugate transpose of (2.3.9) our infinitesimal is then using our definition of $\Lambda^{\frac{1}{2}}$ (2.3.7) is :

$$\psi^\dagger \gamma^0 \to \psi^\dagger (1 - \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma})^\dagger \gamma^0 \tag{B.10.2}$$

assuming $\omega_{\rho\sigma}$ is a real constant :

$$\psi^\dagger \gamma^0 \to \psi^\dagger (1 + \frac{i}{2}\omega_{\rho\sigma} (S^{\rho\sigma})^\dagger) \gamma^0 \tag{B.10.3}$$

$S^{\mu\mu} = 0$ (B.6.5)
$S^{\mu\nu} = -S^{\nu\mu}$ (B.6.6)
in matrix form. Recall (B.6.2) snd (B.6.3)

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{B.10.4}$$

and

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{B.10.5}$$

from this we see that:

$$S^{0i\dagger} = -S^{0i} \tag{B.10.6}$$

Consider (B.6.2) then:

$$S^{0i\dagger} = \frac{i}{2}\begin{pmatrix} \sigma^{i\dagger} & 0 \\ 0 & -\sigma^{i\dagger} \end{pmatrix} \tag{B.10.7}$$

We want:

$$\sigma^{i\dagger} = \sigma^i \tag{B.10.8}$$

$$\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.10.9}$$

$$\sigma^{1T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.10.10}$$

$$\sigma^{1\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.10.11}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.10.12}$$

$$\sigma^{2T} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{B.10.13}$$

$$\sigma^{2\dagger} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.10.14}$$

$$\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.10.15}$$

$$\sigma^{3T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.10.16}$$

$$\sigma^{3\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.10.17}$$

We write

$$\{S^{0i}, \gamma^0\} = S^{0i}\gamma^0 + \gamma^0 S^{0i} \tag{B.10.18}$$

In matrix form

$$\begin{aligned} \{S^{0i}, \gamma^0\} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}) + (-\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\frac{i}{2}(\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}) \\ &= 0 \end{aligned} \tag{B.10.19}$$

So :

$$\gamma^0 S^{0i} = -S^{0i}\gamma^0 \tag{B.10.20}$$

Thus using (B.10.6)

$$\gamma^0 S^{0i} = S^{0i\dagger}\gamma^0 \tag{B.10.21}$$

We also want:

$$S^{ij\dagger} = S^{ij} \tag{B.10.22}$$

using (B.6.3):

$$S^{ij\dagger} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^{k\dagger} & 0 \\ 0 & \sigma^{k\dagger} \end{pmatrix} \tag{B.10.23}$$

Thus using (B.10.8)

$$S^{ij\dagger} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{B.10.24}$$

We showed in appendix B7 (B.7.24):

$$[\gamma^0, S^{ij}] = 0 \tag{B.10.25}$$

and thus:

$$\gamma^0 S^{ij} = S^{ij}\gamma^0 \tag{B.10.26}$$

So our infinitesimal transform goes from:

$$\psi^\dagger\gamma^0 \to \psi^\dagger(1 + \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})^\dagger)\gamma^0 \tag{B.10.27}$$

to

$$\psi^\dagger\gamma^0 \to \psi^\dagger\gamma^0(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) \tag{B.10.28}$$

which becomes the transform:

$$\psi^\dagger\gamma^0 \to \psi^\dagger\gamma^0\Lambda_{\frac{1}{2}}^{-1} \tag{B.10.29}$$

which is the transform :

$$\overline{\psi} \to \overline{\psi}\Lambda_{\frac{1}{2}}^{-1} \tag{B.10.30}$$

B.11 Dirac Equation follows from Dirac Lagrangian

The Dirac-equation is (2.3.8)

$$[i\gamma^\nu\partial_\nu - m]\psi(x) = 0 \tag{B.11.1}$$

The Dirac Lagrangian is (2.3.12) :

$$\mathcal{L}_{Dirac} = \overline{\psi}[i\gamma^\nu\partial_\nu - m]\psi(x) \tag{B.11.2}$$

The Euler-Lagrange equation for $\overline{\psi}$ considering ψ and $\overline{\psi}$ to be different fields is just:

$$\frac{\delta\mathcal{L}}{\delta\overline{\psi}} = 0 \tag{B.11.3}$$

which is obviously just the Dirac equation (2.3.8)

B.12 $S^{\mu\nu}$ are in Block Diagonal Form

We know (2.3.5)

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{B.12.1}$$

We know (2.3.1) , (2.3.2) :

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.12.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{B.12.3}$$

so :

$$
\begin{aligned}
S^{0i} &= \frac{i}{4}[\gamma^0, \gamma^i] \\
&= \frac{i}{4}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\
&= \frac{i}{4}(\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}) \qquad \text{(B.12.4)}
\end{aligned}
$$

So :

$$S^{0i} = \frac{i}{4}(\begin{pmatrix} -2\sigma^i & 0 \\ 0 & 2\sigma^i \end{pmatrix}) \qquad \text{(B.12.5)}$$

and :

$$S^{0i} = -\frac{i}{2}(\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}) \qquad \text{(B.12.6)}$$

$$
\begin{aligned}
S^{ij} &= \frac{i}{4}[\gamma^i, \gamma^j] \\
&= \frac{i}{4}(\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}) \\
&= \frac{i}{4}(\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} \sigma^j\sigma^j & 0 \\ 0 & \sigma^j\sigma^j \end{pmatrix}) \qquad \text{(B.12.7)}
\end{aligned}
$$

from (1.5.7)

$$\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k \qquad \text{(B.12.8)}$$

which means

$$
\begin{aligned}
S^{ij} &= \frac{i}{2}(-i\epsilon^{ijk})(\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}) \\
&= \frac{1}{2}\epsilon^{ijk}(\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}) \qquad \text{(B.12.9)}
\end{aligned}
$$

B.13 Normalising u and v

Recall (2.3.16)

$$u^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix} \qquad \text{(B.13.1)}$$

transposing gives:

$$u^s(p)^\dagger = ((\sqrt{p.\sigma}\xi^s)^\dagger, (\sqrt{p.\overline{\sigma}}\xi^s)^\dagger) = (\xi^{s\dagger}\sqrt{p.\sigma}, \xi^{s\dagger}\sqrt{p.\overline{\sigma}}) \qquad \text{(B.13.2)}$$

we get:

$$\begin{aligned}
u^{r\dagger}(p)u^s(p) &= (\xi^{r\dagger}\sqrt{p.\sigma}, \xi^{r\dagger}\sqrt{p.\overline{\sigma}}) \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix} \\
&= (p.\sigma + p.\overline{\sigma})\xi^{r\dagger}\xi \qquad \text{(B.13.3)}
\end{aligned}$$

Recall (2.3.15) to give:

$$\begin{aligned}
u^{r\dagger}(p)u^s(p) &= (p^0\sigma^0 + p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3 + p^0\sigma^0 - p^1\sigma^1 - p^2\sigma^2 - p^3\sigma^3)\xi^{r\dagger}\xi^s \\
&= 2p^0\xi^{r\dagger}\xi^s \\
&= 2E_{\vec{p}}\xi^{r\dagger}\xi^s \\
&= 2E_{\vec{p}}\delta^{rs} \qquad \text{(B.13.4)}
\end{aligned}$$

by normalising ξ

Recall (2.3.17)

$$v^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\overline{\sigma}}\eta^s \end{pmatrix} \qquad \text{(B.13.5)}$$

its conjugate transpose is:

$$v^s(p)^\dagger = ((\sqrt{p.\sigma}\eta^s)^\dagger, -(\sqrt{p.\overline{\sigma}}\eta^s)^\dagger) = (\eta^{s\dagger}\sqrt{p.\sigma}, -\eta^{s\dagger}\sqrt{p.\overline{\sigma}}) \qquad \text{(B.13.6)}$$

We then get:

$$\begin{aligned}
v^{r\dagger}(p)v^s(p) &= (\eta^{r\dagger}\sqrt{p.\sigma}, -\eta^{r\dagger}\sqrt{p.\overline{\sigma}}) \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\overline{\sigma}}\eta^s \end{pmatrix} \\
&= (p.\sigma + p.\overline{\sigma})\eta^{r\dagger}\eta
\end{aligned}$$

Recall (2.3.15) to give:

$$\begin{aligned}
v^{r\dagger}(p)v^s(p) &= (p^0\sigma^0 + p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3 + p^0\sigma^0 - p^1\sigma^1 - p^2\sigma^2 - p^3\sigma^3)\eta^{r\dagger}\eta^s \\
&= 2p^0\eta^{r\dagger}\eta^s \\
&= 2E_{\vec{p}}\eta^{r\dagger}\eta^s \\
&= 2E_{\vec{p}}\delta^{rs} \qquad \text{(B.13.7)}
\end{aligned}$$

by normalising η

Using (2.3.16) , recall (2.3.10) and (B.13.2)

$$\overline{u}^s(p) = u^s(p)^\dagger\gamma^0 \qquad \text{(B.13.8)}$$

$$\begin{aligned}
\overline{u}^s(p) &= u^s(p)^\dagger\gamma^0 = (\xi^{s\dagger}\sqrt{p.\sigma}, \xi^{s\dagger}\sqrt{p.\overline{\sigma}}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= (\xi^{s\dagger}\sqrt{p.\overline{\sigma}}, \xi^{s\dagger}\sqrt{p.\sigma}) \qquad \text{(B.13.9)}
\end{aligned}$$

Then we have:

$$\begin{aligned}\overline{u}^s(p)u^r(p) &= (\xi^{s\dagger}\sqrt{p.\overline{\sigma}}, \xi^{s\dagger}\sqrt{p.\sigma})\begin{pmatrix}\sqrt{p.\sigma}\xi^r\\ \sqrt{p.\overline{\sigma}}\xi^r\end{pmatrix}\\ &= 2\xi^{s\dagger}\xi\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \qquad \text{(B.13.10)}\end{aligned}$$

Consider using (2.3.15):

$$\begin{aligned}p.\overline{\sigma}p.\sigma &= (p^0\sigma^0 + p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3)(p^0\sigma^0 - [p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3])\\ &= (p^0\sigma^0)^2 - [p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3]^2\\ &= (p^0\sigma^0)^2 - [(p^1\sigma^1)^2 + (p^2\sigma^2)^2 + (p^3\sigma^3)^2]\\ &- p_1p_2\{\sigma^1, \sigma^3\} - p_2p_3\{\sigma^2\sigma^3\} - p_1p_3\{\sigma^1, \sigma^3\} \qquad \text{(B.13.11)}\end{aligned}$$

recall (1.5.7):

$$\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k \qquad \text{(B.13.12)}$$

recall (1.5.8) and (1.5.10)

$$\{\sigma^i, \sigma^j\} = 0 \qquad \text{(B.13.13)}$$

$$\sigma^{i^2} = 1 \qquad \text{(B.13.14)}$$

then use (B.13.11) to give:

$$\begin{aligned}p.\overline{\sigma}p.\sigma &= (p^0\ )^2 - [(p^1)^2 + (p^2)^2 + (p^3)^2]\\ &= p^2 = m^2 \qquad \text{(B.13.15)}\end{aligned}$$

So:

$$\sqrt{p.\overline{\sigma}p.\sigma} = m \qquad \text{(B.13.16)}$$

Thus (B.13.10) becomes:

$$\begin{aligned}\overline{u}^s(p)u^r(p) &= 2\xi^{s\dagger}\xi^r m\\ &= 2\delta^{rs}m \qquad \text{(B.13.17)}\end{aligned}$$

by normalising ξ

Recall (2.3.17) , (B.13.6) and (2.3.10)

$$\overline{v}^s(p) = v^s(p)^\dagger\gamma^0 \qquad \text{(B.13.18)}$$

$$\begin{aligned}\overline{v}^s(p) &= v^s(p)^\dagger\gamma^0 = (\eta^{s\dagger}\sqrt{p.\sigma}, -\eta^{s\dagger}\sqrt{p.\overline{\sigma}})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= (-\eta^{s\dagger}\sqrt{p.\overline{\sigma}}, \eta^{s\dagger}\sqrt{p.\sigma}) \qquad \text{(B.13.19)}\end{aligned}$$

Then:

$$\begin{aligned}\overline{v}^s(p)v^r(p) &= (-\eta^{s\dagger}\sqrt{p.\overline{\sigma}}, \eta^{s\dagger}\sqrt{p.\sigma})\begin{pmatrix}\sqrt{p.\sigma}\eta^r\\ -\sqrt{p.\overline{\sigma}}\eta^r\end{pmatrix}\\ &= -2\eta^{s\dagger}\eta\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \qquad \text{(B.13.20)}\end{aligned}$$

Recall (B.13.16)

$$\sqrt{p.\overline{\sigma}p.\sigma} = m \tag{B.13.21}$$

to get :

$$\begin{aligned}\overline{v}^s(p)b^r(p) &= -2\eta^{s\dagger}\eta^r m \\ &= -2\delta^{rs}m\end{aligned} \tag{B.13.22}$$

by normalising η

Recall (B.13.9) (2.3.17)

$$\begin{aligned}\overline{u}^s(p)v^r(p) &= (\xi^{s\dagger}\sqrt{p.\overline{\sigma}}, \xi^{s\dagger}\sqrt{p.\sigma})\begin{pmatrix}\sqrt{p.\sigma}\eta^r \\ -\sqrt{p.\overline{\sigma}}\eta^r\end{pmatrix} \\ &= \xi^{s\dagger}\eta^r(\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} - \sqrt{p.\sigma}\sqrt{p.\overline{\sigma}}\ ) \\ &= 0\end{aligned} \tag{B.13.23}$$

Recall (B.13.19) ,(2.3.16)

$$\begin{aligned}\overline{v}^r(p)u^s(p) &= (-\eta^{r\dagger}\sqrt{p.\overline{\sigma}}, \eta^{r\dagger}\sqrt{p.\sigma})\begin{pmatrix}\sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s\end{pmatrix} \\ &= \eta^{r\dagger}\xi^s(-\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} + \sqrt{p.\sigma}\sqrt{p.\overline{\sigma}}) \\ &= 0\end{aligned} \tag{B.13.24}$$

Recall (B.13.2), (2.3.17):

$$\begin{aligned}u^{r\dagger}(p)v^s(-\vec{p}) &= (\xi^{s\dagger}\sqrt{p.\sigma}, \xi^{s\dagger}\sqrt{p.\overline{\sigma}})\begin{pmatrix}\sqrt{p.\overline{\sigma}}\eta^s \\ -\sqrt{p.\sigma}\eta^s\end{pmatrix} \\ &= \xi^{s\dagger}\eta^s(\sqrt{p.\sigma}\sqrt{p.\overline{\sigma}} - \sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \\ &= 0\end{aligned} \tag{B.13.25}$$

Recall (B.13.6), (2.3.16)

$$\begin{aligned}v^{r\dagger}(-\vec{p})u^s(p) &= (\eta^{s\dagger}\sqrt{p.\overline{\sigma}}, -\eta^{s\dagger}\sqrt{p.\sigma})\begin{pmatrix}\sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s\end{pmatrix} \\ &= \xi^{s\dagger}\eta^s(\sqrt{p.\sigma}\sqrt{p.\overline{\sigma}} - \sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \\ &= 0\end{aligned} \tag{B.13.26}$$

B.14 The Spin Sums

Using (2.3.16):

$$u^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix} \tag{B.14.1}$$

and (B.13.9) :

$$\overline{u}^s(p) = (\xi^{s\dagger}\sqrt{p.\overline{\sigma}}, \xi^{s\dagger}\sqrt{p.\sigma}) \tag{B.14.2}$$

Thus :

$$\begin{aligned} \sum_{s=1,2} u^s(p)\overline{u}^s(p) &= \sum_{s=1,2} \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix} (\xi^{s\dagger}\sqrt{p.\overline{\sigma}}, \xi^{s\dagger}\sqrt{p.\sigma}) \\ &= \sum_{s=1,2} \begin{pmatrix} \sqrt{p.\sigma}\sqrt{p.\overline{\sigma}}\xi^s\xi^{s\dagger} & \sqrt{p.\sigma}\sqrt{p.\sigma}\xi^s\xi^{s\dagger} \\ \sqrt{p.\overline{\sigma}}\sqrt{p.\overline{\sigma}}\xi^s\xi^{s\dagger} & \sqrt{p.\overline{\sigma}}\sqrt{p.\sigma}\xi^s\xi^{s\dagger} \end{pmatrix} \end{aligned} \tag{B.14.3}$$

using the normalisation of ξ:

$$\sum_{s=1,2} \xi^s\xi^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.14.4}$$

We get:

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\sqrt{p.\overline{\sigma}} & \sqrt{p.\sigma}\sqrt{p.\sigma} \\ \sqrt{p.\overline{\sigma}}\sqrt{p.\overline{\sigma}} & \sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \end{pmatrix} \tag{B.14.5}$$

Recall (B.13.16):

$$\sqrt{p.\overline{\sigma}p.\sigma} = m \tag{B.14.6}$$

Recall (2.3.1) , (2.3.2)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.14.7}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{B.14.8}$$

in our new notation (2.3.15) we have:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{B.14.9}$$

(B.14.5) becomes:

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \begin{pmatrix} m & p.\sigma \\ p.\overline{\sigma} & m \end{pmatrix} \tag{B.14.10}$$

$$= \begin{pmatrix} 0 & p.\sigma \\ p.\overline{\sigma} & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \tag{B.14.11}$$

Thus :

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \gamma.p + m \tag{B.14.12}$$

Using (2.3.17) , (B.13.19)

$$v^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\overline{\sigma}}\eta^s \end{pmatrix} \tag{B.14.13}$$

$$\overline{v}^s(p) = (-\eta^{s\dagger}\sqrt{p.\overline{\sigma}}, \eta^{s\dagger}\sqrt{p.\sigma}) \tag{B.14.14}$$

Thus :

$$\begin{aligned} \sum_{s=1,2} v^s(p)\overline{v}^s(p) &= \sum_{s=1,2} \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\overline{\sigma}}\eta^s \end{pmatrix} (-\eta^{s\dagger}\sqrt{p.\overline{\sigma}}, \eta^{s\dagger}\sqrt{p.\sigma}) \\ &= \sum_{s=1,2} \begin{pmatrix} -\sqrt{p.\sigma}\sqrt{p.\overline{\sigma}}\eta^s\eta^{s\dagger} & \sqrt{p.\sigma}\sqrt{p.\sigma}\eta^s\eta^{s\dagger} \\ \sqrt{p.\overline{\sigma}}\sqrt{p.\overline{\sigma}}\eta^s\eta^{s\dagger} & -\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma}\eta^s\eta^{s\dagger} \end{pmatrix} \end{aligned} \tag{B.14.15}$$

using the normalisation of η

$$\sum_{s=1,2} \eta^s\eta^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.14.16}$$

We get:

$$\sum_{s=1,2} v^s(p)\overline{v}^s(p) = \begin{pmatrix} -\sqrt{p.\sigma}\sqrt{p.\overline{\sigma}} & \sqrt{p.\sigma}\sqrt{p.\sigma} \\ \sqrt{p.\overline{\sigma}}\sqrt{p.\overline{\sigma}} & -\sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \end{pmatrix} \tag{B.14.17}$$

Recall (B.13.16)

$$\sqrt{p.\overline{\sigma}p.\sigma} = m \tag{B.14.18}$$

$$\begin{aligned} \sum_{s=1,2} v^s(p)\overline{v}^s(p) &= \begin{pmatrix} -m & p.\sigma \\ p.\overline{\sigma} & -m \end{pmatrix} \\ &= \begin{pmatrix} 0 & p.\sigma \\ p.\overline{\sigma} & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \end{aligned} \tag{B.14.19}$$

Thus :

$$\sum_{s=1,2} v^s(p)\overline{v}^s(p) = \gamma.p - m \tag{B.14.20}$$

B.15 The Dirac Conjugate Momentum and Hamiltonian

the conjugate momentum is given by :

$$\frac{\delta \mathcal{L}}{\delta \partial_0 \psi} \tag{B.15.1}$$

Recall (2.3.12)

$$\mathcal{L}_{Dirac} = \overline{\psi}[i\gamma^\nu \partial_\nu - m]\psi(x) \tag{B.15.2}$$

Thus our conjugate momentum is given by (3.1.1):

$$\frac{\delta \mathcal{L}}{\delta \partial_0 \psi} = \overline{\psi} i \gamma^0 \tag{B.15.3}$$

Recall (2.3.10)

$$\overline{\psi} = \psi^\dagger \gamma^0 \tag{B.15.4}$$

So:

$$\frac{\delta \mathcal{L}}{\delta \partial_0 \psi} = i\psi^\dagger \gamma^0 \gamma^0 \tag{B.15.5}$$

$$\begin{aligned} \gamma^0\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{B.15.6}$$

So :

$$\frac{\delta \mathcal{L}}{\delta \partial_0 \psi} = \psi^\dagger i \tag{B.15.7}$$

So the conjugate momentum is :

$$\psi^\dagger i \tag{B.15.8}$$

The hamiltonian is given by (2.1.13)

$$H = \int d^4 x \mathcal{H} = \int d^4 x \dot{p} q - \mathcal{L} \tag{B.15.9}$$

q is the conjugate momentum
So using (B.15.8):

$$\begin{aligned} H &= \int d^4 i\psi^\dagger \partial_0 \psi - (\overline{\psi}[i\gamma^\nu \partial_\nu - m]\psi(x)) \\ &= \int d^4 i\psi^\dagger \partial_0 \psi - i\psi^\dagger \gamma^0 \gamma^0 \partial_0 \psi - i\overline{\psi}\vec{\gamma}.\vec{\nabla}\psi + \overline{\psi} m \psi \\ &= \int d^4 \overline{\psi}(-i\vec{\gamma}.\vec{\nabla} + m)\psi \end{aligned} \tag{B.15.10}$$

B.16 $\psi_c = \gamma^2\psi^*$

As we already have (2.3.29):

$$\psi_c = C\gamma^0\psi^* \tag{B.16.1}$$

We want to prove:

$$C\gamma^0 = \gamma^2 \tag{B.16.2}$$

We define C as the operator such that (l13 p101 [8])

$$-\gamma^{\mu *} = (C\gamma^0)^{-1}\gamma^\mu(C\gamma^0) \tag{B.16.3}$$

then :

$$C\gamma^0\gamma^{\mu *} = -\gamma^\mu C\gamma^0 \tag{B.16.4}$$

multiplying both sides from the left by $C\gamma^0$
as γ^2 is the only complex matrix for $\mu \neq 2$:

$$C\gamma^0\gamma^\mu = -\gamma^\mu C\gamma^0 \tag{B.16.5}$$

i.e.

$$\{C\gamma^0, \gamma^\mu\} = 0 \;\; \mu \neq 2 \tag{B.16.6}$$

γ^2 recall (2.3.2) is completely imaginary as it is made of blocks of σ^2 which are completely imaginary recall (1.3.14) so:

$$\gamma^{2*} = -\gamma^2 \tag{B.16.7}$$

So (B.16.5) becomes:

$$C\gamma^0\gamma^2 = \gamma^2 C\gamma^0 \tag{B.16.8}$$

which is :

$$[C\gamma^0, \gamma^2] = 0 \tag{B.16.9}$$

Obviously $C \propto \gamma^2$satisfies this
by the property stated above recall (2.3.4)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times 1_{n\times n} \tag{B.16.10}$$

i.e.

$$\{\gamma^2, \gamma^\nu\} = 0 \;\; \nu \neq 2 \tag{B.16.11}$$

which means :

$$C\gamma^0 \propto \gamma^2 \tag{B.16.12}$$

satisfies the equation (B.16.5) because C is known to be unitary we gain:

$$C\gamma^0 = \gamma^2 \tag{B.16.13}$$

B.17 Dirac Equation Implies the Majorana Equation

take the majorana equation (2.3.32)

$$i \not{\partial}\psi = m\psi_c \tag{B.17.1}$$

using (2.3.24) and (2.3.29) we get :

$$i\gamma^\mu \partial_\mu \psi = m\gamma^2\psi^* \tag{B.17.2}$$

conjugating we get:

$$-i\gamma^{\mu*}\partial_\mu \psi^* = m(-\gamma^2\psi) \tag{B.17.3}$$

as γ^2 is imaginary see (2.3.2) and thus :

$$\gamma^{2^*} = -\gamma^2 \tag{B.17.4}$$

multiplying (B.17.3) by γ^2 gives:

$$\gamma^2(-i\gamma^{\mu*}\partial_\mu\psi^*) = \gamma^2 m(-\gamma^2\psi) \tag{B.17.5}$$

as all other matrices are real:

$$\gamma^{*\mu} = \gamma^\mu \;\; \mu \neq 2 \tag{B.17.6}$$

and (B.16.11) implies:

$$\gamma^\mu\gamma^2 = -\gamma^2\gamma^\mu \;\; \mu \neq 2 \tag{B.17.7}$$

(B.17.4) implies:

$$\gamma^2\gamma^{2*} = -\gamma^2\gamma^2 \tag{B.17.8}$$

So combing the two previous equations and remembering all the matrices except γ^2 are real we get:

$$\gamma^2\gamma^{\mu*} = -\gamma^\mu\gamma^2 \tag{B.17.9}$$

recall (2.3.4)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times 1_{n\times n} \tag{B.17.10}$$

This implies:

$$\{\gamma^2, \gamma^2\} = -2I \tag{B.17.11}$$

since from (**??**) $g^{22} = -1$ this means :

$$-\gamma^2\gamma^2 = I \tag{B.17.12}$$

after dividing both sides by -2
So (**??**) becomes:

$$i\gamma^\mu\gamma^2\partial_\mu\psi^* = m\psi \tag{B.17.13}$$

which commuting the derivative and γ^2 becomes:

$$i\gamma^\mu \partial_\mu \gamma^2 \psi^* = m\psi \tag{B.17.14}$$

which using (2.3.24) and (2.3.29) becomes:

$$i\partial\!\!\!/ \psi_c = m\psi \tag{B.17.15}$$

so the Klein-gordan equation (2.1.11) :

$$-\partial^2 \psi = m^2 \psi \tag{B.17.16}$$

becomes:

$$i\partial\!\!\!/ (i\partial\!\!\!/ \psi) = i \ \partial\!\!\!/ (m\psi_c) = m^2\psi \tag{B.17.17}$$

using (B.17.15) twice

Appendix C

Appendix C : Chapter 3

C.1 The Commutator Relation for the Majorana Field

The fact that a majorana field obeys the dirac equation means we can write (2.3.29):

$$\psi_c = C\gamma^0\psi \tag{C.1.1}$$

as (p237 eq 37.18 [5]):

$$\overline{\psi} = \psi^T C \tag{C.1.2}$$

We know from (B.15.3) that the conjugate momentum is:

$$\overline{\psi}\gamma^0 \tag{C.1.3}$$

which is then using (C.1.1)

$$\psi^T C\gamma^0 \tag{C.1.4}$$

if C is defined as (p237 eq 37.19 [5]) :

$$C = \begin{pmatrix} -\epsilon^{ac} & 0 \\ 0 & -\epsilon_{\dot{a}\dot{c}} \end{pmatrix} \tag{C.1.5}$$

This commutes through γ^0 (2.3.1)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{C.1.6}$$

so:

$$\psi^T C\gamma^0 = \psi^T\gamma^0 C \tag{C.1.7}$$

if following eq37.16 p237 [5]

$$\psi^T = (\psi_1^T, \psi_1^*) \tag{C.1.8}$$

$$\psi^T \gamma^0 = \begin{pmatrix} \psi_1^* \\ \psi_1^T \end{pmatrix} \tag{C.1.9}$$

assuming these terms are real then this is ψ and our equation

$$\{\psi_\alpha(\vec{x}, t), \psi_\beta(\vec{x}, t)\} = (C\gamma^0)_{\alpha\beta}\delta^3(\vec{x} - \vec{y}) \tag{C.1.10}$$

follows

Appendix D

Appendix D : Chapter 4

D.1 Properties of U

we define by comparison with (4.2.10):

$$U(t,t') = e^{iH_0(t-t')}e^{-iH(t-t')} \tag{D.1.1}$$

Thus:

$$\begin{aligned} U(t_1,t_2)U(t_2,t_3) &= e^{iH_0(t_1-t_2)}e^{-iH(t_1-t_2)}e^{iH_0(t_2-t_3)}e^{-iH(t_2-t_3)} \\ &= e^{iH_0(t_1-t_3)}e^{-iH(t_1-t_3)} \end{aligned} \tag{D.1.2}$$

which is $U(t_1,t_3)$
using (4.2.10) we get:

$$U(t,t')^\dagger = e^{-iH_0(t-t')}e^{iH(t-t')} \tag{D.1.3}$$

we get:

$$\begin{aligned} U(t_1,t_3)U(t_2,t_3)^\dagger &= e^{iH_0(t_1-t_3)}e^{-iH(t_1-t_3)}e^{-iH_0(t_2-t_3)}e^{iH(t_2-t_3)} \\ &= e^{iH_0(t_1-t_2)}e^{-iH(t_1-t_2)} \end{aligned} \tag{D.1.4}$$

which is $U(t_1,t_2)$

D.2 Wick's Theorem

We know that wick's theorem is true for $\phi_0(x)\phi(y)$ to prove it is true prove the case n+1 given the case n, i.e. prove by induction

$$T\{\phi_0(x_2)....\phi_0(x_{n+1})\} = N\{\phi_0(x_2)....\phi_0(x_{n+1}) \quad + \quad (\text{ all contractions of } \phi_0(x_2)....\phi_0(x_{n+1})\}$$

is n case we take to be proven.
assuming x_1^0 is a time greater then all the other times
We write:

$$\phi_0(x_1) = \phi_0^{cr}(x_1) + \phi_0^{ani}(x_1) \tag{D.2.1}$$

splitting the field into creation and annihilation parts.
obviously theϕ_0^{cr} can be brought inside the normal-ordering as it is a creation operator and so should come from the left.
Therefore we only consider the term:

$$\phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1}) + (\text{ all contractions of } \phi_0(x_2)....\phi_0(x_{n+1})\}$$

We consider only:

$$\phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1})\} \tag{D.2.2}$$

the contraction terms follow in exactly the same way except obviously we can't contract a term that is already contracted . Thus we get:

$$\begin{aligned}\phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1})\} &= N\{\phi_0(x_2)....\phi_0(x_{n+1})\}\phi_0^{ani} \\ &- N\{\phi_0(x_2)....\phi_0(x_{n+1})\}\phi_0^{ani} \\ &+ \phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1})\} \\ &= N\{\phi_0(x_2)....\phi_0(x_{n+1})\}\phi_0^{ani}(x_1) \\ &+ [\phi_0^{ani}(x_1), N\{\phi_0(x_2)....\phi_0(x_{n+1})\}]\end{aligned} \tag{D.2.3}$$

assuming the anihilation parts of each field commute we get .
we can commute off the field with each of the members of the normal ordered part to get:

$$\begin{aligned}\phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1})\} &= N\{\phi_0^{ani}(x_1)\phi_0(x_2)....\phi_0(x_{n+1})\} \\ &+ N\{[\phi_0^{ani}(x_1), \phi_0^{cr}(x_2)]....\phi_0(x_{n+1})\} \\ &+ N\{\phi_0(x_2)[\phi_0^{ani}(x_1), \phi_0^{cr}(x_3)]....\phi_0(x_{n+1})\} \\ &+ ......\end{aligned} \tag{D.2.4}$$

where the first term is now normal ordered
and the commutators are brought inside the normal ordering as they are contractions and thus are δ s and are unaffected by ordering
Thus we write in terms of contractions as

$$\begin{aligned}\phi_0^{ani}(x_1)N\{\phi_0(x_2)....\phi_0(x_{n+1})\} &= N\{\phi_0^{ani}(x_1)\phi_0(x_2)....\phi_0(x_{n+1})\} \\ &+ N\{\phi_0(x_1)\phi_0(x_2)...\phi_0(x_{n+1})\} \\ &+ N\{\phi_0(x_2)\phi_0(x_1)\phi_0(x_3)....\phi_0(x_{n+1})\} \\ &+ ......\end{aligned} \tag{D.2.5}$$

This recombines with our term coming from the ϕ_0^{cr}
To give :

$$N\{\phi_0(x_1)\phi_0(x_2)....\phi_0(x_{n+1}) + \text{the contractions of } \phi_0(x_1) \text{ with } \phi_0(x_2)....\phi_0(x_{n+1}) \tag{D.2.6}$$

The other terms follow by applying $\phi_0(x_1)$ to them in the same way , except if they are already contracted they can't be contracted again

D.3 a in Terms of Differentials

We know that (B.3):

$$a(\vec{k}) = \int d^3x(\sqrt{\frac{\omega_k}{2}}\phi(x) + \frac{i}{\sqrt{2\omega_k}}\pi^\dagger(x))e^{ikx} \tag{D.3.1}$$

$k^0 = \omega_k$
We know (3.1.1) and (2.2.1):

$$\pi^\dagger = \frac{\delta\mathcal{L}}{\delta\partial_0\phi^\dagger} \tag{D.3.2}$$

$$\mathcal{L} = \frac{1}{2} \mid \partial^\mu\phi \mid^2 - \frac{1}{2}m^2 \mid \phi \mid^2 \tag{D.3.3}$$

$$\pi^\dagger = \dot{\phi} \tag{D.3.4}$$

using (1.2.9)

$$e^{ikx}\overleftrightarrow{\partial_0}\phi = e^{ikx}\pi^\dagger - i\omega_k\phi e^{ikx} \tag{D.3.5}$$

So :

$$a(\vec{k}) = \int d^3x\frac{i}{\sqrt{2\omega_k}}e^{ikx}\overleftrightarrow{\partial_0}\phi \tag{D.3.6}$$

conjugate transposing means:

$$a^\dagger(\vec{k}) = \int d^3x\frac{-i}{\sqrt{2\omega_k}}e^{-ikx}\overleftrightarrow{\partial_0}\phi \tag{D.3.7}$$

as $i^2 = -1$

D.4 Forms of $a_1(-\infty)$ and $a_1^\dagger(\infty)$

Recall (4.4.4):

$$a_1^\dagger = \int d^3k g_1(\vec{k})a^\dagger(\vec{k}) \tag{D.4.1}$$

$$g_1(\vec{k}) \propto exp[\frac{-(\vec{k}-\vec{k}_1)^2}{4\sigma^2}] \tag{D.4.2}$$

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\partial_0 a_1^\dagger(t) \tag{D.4.3}$$

Recall (D.3.6)

$$a(\vec{k}) = \int d^3x\frac{i}{\sqrt{2\omega_k}}e^{ikx}\overleftrightarrow{\partial_0}\phi \tag{D.4.4}$$

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x\partial_0(e^{-ikx}\overleftrightarrow{\partial_0}\phi(x)) \tag{D.4.5}$$

recall (3.1.1) , (1.2.9)

$$e^{-ikx}\overleftrightarrow{\partial_0}\phi(x) = e^{-ikx}\partial_0\phi(x) + i\omega_k e^{-ikx}\phi(x) \tag{D.4.6}$$

We get:

$$\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x\partial_0(e^{-ikx}\partial_0\phi(x) + i\omega_k e^{-ikx}\phi(x)) \\
&= \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial_0^2 + \omega_k^2)\phi(x) \\
&= \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial_0^2 + \vec{k}^2 + m^2)\phi(x)
\end{aligned} \tag{D.4.7}$$

as $\omega^2 = \vec{k}^2 + m^2$

$$\nabla e^{-ikx} = ike^{-ikx} \tag{D.4.8}$$

So:

$$-\nabla^2 e^{-ikx} = k^2 e^{-ikx} \tag{D.4.9}$$

which means:

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2)\phi(x) \tag{D.4.10}$$

integrating the nabla term by parts twice and iassuming the associated boundary terms vanish we get:

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi(x) \tag{D.4.11}$$

using the definition of ∂^2 (from ∂_μ and (1.6.4)):

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \frac{-i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial^2 + m^2)\phi(x) \tag{D.4.12}$$

We then have :

$$a_1^\dagger(-\infty) = a_1^\dagger(\infty) + \frac{i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{-ikx}(\partial^2 + m^2)\phi(x) \tag{D.4.13}$$

conjugate transposing gives :

$$a_1(-\infty) = a_1(\infty) - \frac{i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{ikx}(\partial^2 + m^2)\phi(x) \tag{D.4.14}$$

$$a_1(\infty) = a_1(-\infty) + \frac{i}{\sqrt{2\omega_k}}\int d^3k g_1(\vec{k})\int d^4x e^{ikx}(\partial^2 + m^2)\phi(x) \tag{D.4.15}$$

D.5 Creation, Annihilation Operators for the Dirac Spinor Field

Recall (2.3.13)

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} (a^s_{\vec{p}} u^s(p) e^{-ipx} + b^{s\dagger}_{\vec{p}} v^s(p) e^{ipx} \tag{D.5.1}$$

Fourier transform:

$$\int d^3x e^{ipx} \psi = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} (a^s_{\vec{p}} u^s(p) + b^{s\dagger}_{\vec{p}} v^s(p) e^{2i\omega t} \tag{D.5.2}$$

We multiply on the left by (recall (2.3.10)) $\overline{u}^s(p)\gamma^0$ i.e. $u^{s\dagger}(p)$
we use the spin sums, recall (2.3.18) , (2.3.21)

$$u^{r\dagger}(p) u^s(p) = 2E_{\vec{p}} \delta^{rs} \tag{D.5.3}$$

$$u^{r\dagger}(p) v^s(-\vec{p}) = 0 \tag{D.5.4}$$

and:

$$\int d^3x e^{ipx} \psi = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} (a^s_{\vec{p}} u^s(p) + b^{s\dagger}_{-\vec{p}} v^s(-\vec{p}) e^{2i\omega t} \tag{D.5.5}$$

which comes from changing the variable on b and v Thus:

$$a^s_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}} \int d^3x e^{ipx} \overline{u}^s(\vec{p}) \gamma^0 \psi \tag{D.5.6}$$

Fourier transforming (2.3.13) and by changing the variable on a and u

$$\int d^3x e^{-ipx} \psi = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} (a^s_{-\vec{p}} u^s(-\vec{p}) e^{2i\omega t} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) \tag{D.5.7}$$

multiplying on the left by $\overline{v}^s(p)\gamma^0$ i.e. $v^{s\dagger}(p)$
and using the spin sums recall (2.3.21) ,(2.3.19):

$$v^{r\dagger}(p) v^s(p) = 2E_{\vec{p}} \delta^{rs} \tag{D.5.8}$$

$$v^{r\dagger}(p) u^s(-\vec{p}) = 0 \tag{D.5.9}$$

We get:

$$b^{s\dagger}_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}} \int d^3x e^{-ipx} \overline{v}^s(\vec{p}) \gamma^0 \psi \tag{D.5.10}$$

$$\begin{aligned} (\overline{v}^s(\vec{p})\gamma^0\psi)^\dagger &= \psi^\dagger \gamma^{0\dagger} \overline{v}^s(\vec{p})^\dagger \\ &= \overline{\psi} \gamma^0 v^s(\vec{p}) \end{aligned} \tag{D.5.11}$$

So:

$$b^s_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}} \int d^3x e^{ipx} \overline{\psi} \gamma^0 v^s(\vec{p}) \tag{D.5.12}$$

D.6 $a_1^\dagger(\infty)$ for the Dirac Field

Recall (4.4.14) and its transpose:

$$\begin{aligned} a^s_{\vec{p}} &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3x e^{ipx}\overline{u}^s(\vec{p})\gamma^0\psi \\ a^{s\dagger}_{\vec{p}} &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3x e^{-ipx}\overline{\psi}\gamma^0 u^s(\vec{p}) \end{aligned} \tag{D.6.1}$$

as using (2.3.10):

$$(\overline{u}^s(\vec{p})\gamma^0\psi)^\dagger = \psi^\dagger\gamma^{0\dagger}(\overline{u}^s(\vec{p}))^\dagger = \psi^\dagger\gamma^{0\dagger}\gamma^{0\dagger}u^{s\dagger}(\vec{p}) \tag{D.6.2}$$

Recall (2.3.1) we see :

$$\gamma^{0\dagger} = \gamma^0 \tag{D.6.3}$$

So:

$$(\overline{u}^s(\vec{p})\gamma^0\psi)^\dagger = \overline{\psi}\gamma^0 u^s(\vec{p}) \tag{D.6.4}$$

We then have:

$$\begin{aligned} a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\int_{-\infty}^{\infty} dt\partial_0 a_1^\dagger(t) \\ a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3p g_1(\vec{p})\int d^4x\partial_0(e^{-ipx}\overline{\psi}\gamma^0 u^s(\vec{p})) \\ a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3p g_1(\vec{p})\int d^4x\partial_0(e^{-ipx})\overline{\psi}\gamma^0 u^s(\vec{p})) + e^{-ipx}\partial_0\overline{\psi}\gamma^0 u^s(\vec{p}) \\ a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3p g_1(\vec{p})\int d^4x - ip^0(e^{-ipx})\overline{\psi}\gamma^0 u^s(\vec{p})) + e^{-ipx}\partial_0\overline{\psi}\gamma^0 u^s(\vec{p}) \\ a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3p g_1(\vec{p})\int d^4x\overline{\psi}(\gamma^0\overleftarrow{\partial_0} - i\gamma^0 p^0)u^s(\vec{p})e^{-ipx} \end{aligned} \tag{D.6.5}$$

take:

$$(\not{p} - m)u_s(\vec{p}) = 0 \tag{D.6.6}$$

this is follows from using ue^{-ipx} in the dirac equation (2.3.8) and implies using (2.3.24) that:

$$\gamma^0 p^0 u_s(\vec{p}) = (\gamma^i p_i + m)u_s(\vec{p}) \tag{D.6.7}$$

This means (D.6.5) becomes:

$$a_1^\dagger(-\infty) - a_1^\dagger = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3p g_1(\vec{p})\int d^4x\overline{\psi}(\gamma^0\overleftarrow{\partial_0} - i\gamma^i p_i - im)u^s(\vec{p})e^{-ipx}$$

ip^i can be written as ∂_i acting on e^{ipx} so:

$$a_1^\dagger(-\infty)-a_1^\dagger(\infty) = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{\psi}(\gamma^0\overleftarrow{\partial_0}-\gamma^i\vec{\partial_i}-im)u^s(\vec{p})e^{-ipx}$$

integrating the $\gamma^i\vec{\partial_i}$ piece by parts and taking the constant term to vanish at the boundary .

$$a_1^\dagger(-\infty)-a_1^\dagger(\infty) = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{\psi}(\gamma^0\overleftarrow{\partial_0}+\gamma^i\overleftarrow{\partial}_i-im)u^s(\vec{p})e^{-ipx}$$

using the definition of $\not{\partial}$ (2.3.24):

$$\begin{aligned}
a_1^\dagger(-\infty) - a_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{\psi}(\overleftarrow{\not{\partial}} - im)u^s(\vec{p})e^{-ipx} \\
&= i\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{\psi}(i\overleftarrow{\not{\partial}} + m)u^s(\vec{p})e^{-ipx} \\
&= i\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p}) \\
&\times \int d^4x\overline{\psi}(i\overleftarrow{\not{\partial}} - m)v^s(\vec{p})e^{-ipx} \qquad \text{(D.6.8)}
\end{aligned}$$

D.7 $b_1^\dagger(\infty)$ for the Dirac Field

recall (4.4.16)

$$b_{1\vec{p}}^{s\dagger} = \frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3xe^{-ipx}\overline{v}^s(\vec{p})\gamma^0\psi \qquad \text{(D.7.1)}$$

We then have:

$$\begin{aligned}
b_1^\dagger(-\infty) - b_1^\dagger(\infty) &= -\int_{-\infty}^{\infty}\partial_0 b_1^\dagger(t) \\
&= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\partial_0(e^{-ipx}\overline{v}^s(\vec{p})\gamma^0\psi) \\
&= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\partial_0(e^{-ipx})\overline{v}^s(\vec{p})\gamma^0\psi + e^{-ipx}\overline{v}^s(\vec{p})\gamma^0\partial_0\psi \\
&= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x - ip^0(e^{-ipx})\overline{v}^s(\vec{p})\gamma^0\psi) + e^{-ipx}\overline{v}^s(\vec{p})\gamma^0\partial_0\psi \\
&= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{v}^s(\vec{p})e^{-ipx}(\gamma^0\vec{\partial_0} - i\gamma^0p^0)\psi \qquad \text{(D.7.2)}
\end{aligned}$$

$(\not{p} + m)v_s(\vec{p}) = 0$ this follows from considering the dirac equation (2.3.8) acting on $ve^{ipx}$and implies:

$$\gamma^0p^0\overline{v}^s = (\gamma^ip_i - m)v^s \qquad \text{(D.7.3)}$$

$$b_1^\dagger(-\infty) - b_1^\dagger = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4xv^s(\vec{p})e^{-ipx}(\gamma^0\vec{\partial_0} - i\gamma^i p_i + im)\overline{\psi} \tag{D.7.4}$$

ip^i can be written as ∂_i acting on e^{ipx}

$$b_1^\dagger(-\infty)-b_1^\dagger(\infty) = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{v}^s(\vec{p})e^{-ipx}(\gamma^0\vec{\partial_0}-\gamma^i\overleftarrow{\partial}_i+im)\overline{\psi} \tag{D.7.5}$$

integrating the $\gamma^i\overleftarrow{\partial}_i$ piece by parts and taking the constant term to vanish at the boundary :

$$b_1^\dagger(-\infty) - b_1^\dagger(\infty) = -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{v}^s(\vec{p})e^{-ipx}(\gamma^0\vec{\partial_0}+\gamma^i\vec{\partial_i}+im)\psi \tag{D.7.6}$$

using the definition of $\vec{\not\partial}$ (2.3.24)

$$\begin{aligned}
b_1^\dagger(-\infty) - b_1^\dagger(\infty) &= -\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{v}^s(\vec{p})e^{-ipx}(\vec{\not\partial}+im)\psi \\
&= -i\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p})\int d^4x\overline{v}^s(\vec{p})e^{-ipx}(-i\vec{\not\partial}+m)\psi \\
&= -i\frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3pg_1(\vec{p}) \\
&\times \int d^4x\overline{v}^s(\vec{p})e^{-ipx}(-i\vec{\not\partial}+m)\psi
\end{aligned} \tag{D.7.7}$$

D.8 $G_F(x,x')$ in Terms of Four-Integrals

Recall the first line of (**??**):

$$G_F(x,x') = \langle 0 \mid T(\phi_0(x')\phi_0^\dagger(x)) \mid 0\rangle \tag{D.8.1}$$

where T is the time-ordering symbol: We know (2.2.6) and its conjugate transpose:

$$\phi = \int \frac{d^3k}{(2\pi)^3}[\hat{a}(\vec{k})e^{-ikx} + \hat{b}^\dagger(\vec{k})e^{ikx}] \tag{D.8.2}$$

$$\phi^\dagger = \int \frac{d^3k}{(2\pi)^3}[\hat{a}^\dagger(\vec{k})e^{ikx} + \hat{b}(\vec{k})e^{-ikx}] \tag{D.8.3}$$

We also recall the set of creation operator commutators from Appendix B3:

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')\hbar \tag{D.8.4}$$

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = [\hat{b}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{b}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{D.8.5}$$

$$[\hat{a}(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{b}(\vec{k}')] = [\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0 \tag{D.8.6}$$

where we have restored the $\hbar$ for convenience
multiplying (2.2.6) and its transpose we get:

$$\begin{aligned}
\phi\phi^\dagger &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}[\hat{a}(\vec{k})e^{-ikx}+\hat{b}^\dagger(\vec{k})e^{ikx}][\hat{a}^\dagger(\vec{k})e^{ik'x'}+\hat{b}(\vec{k})e^{-ik'x'}] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}[\hat{a}(\vec{k})\hat{a}^\dagger e^{-ikx+ik'x'}+\hat{a}(\vec{k})\hat{b}(\vec{k})e^{-ikx-ikk'x'} \\
&+ \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k})e^{ikx+ik'x'}\hat{a}^\dagger(\vec{k})\hat{b}(\vec{k})e^{ikx-ik'x'}] \qquad \text{(D.8.7)}
\end{aligned}$$

So taking the vacuum expectation we get:

$$\langle 0 \mid \phi\phi^\dagger \mid 0\rangle = \langle 0 \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})]e^{-ikx+ik'x'} \mid 0\rangle \tag{D.8.8}$$

Using the definition of the creation operators:

$$\hat{a} \mid 0\rangle = \hat{b} \mid 0\rangle = 0 \tag{D.8.9}$$

$$\langle 0 \mid a^\dagger = \langle 0 \mid b^\dagger = 0 \tag{D.8.10}$$

We can add to (D.8.8)

$$\langle 0 \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}[\hat{a}\dagger(\vec{k})\hat{a}(\vec{k})]e^{-ikx+ik'x'} \mid 0\rangle = 0 \tag{D.8.11}$$

to give:

$$\begin{aligned}
\langle 0 \mid \phi\phi^\dagger \mid 0\rangle &= \langle 0 \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k})]e^{-ikx+ik'x'} \mid 0\rangle \\
&= \langle 0 \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k'}}(2\pi)^3\hbar\delta^3(\vec{k}-\vec{k}')e^{-ikx+ik'x'} \mid 0\rangle \\
&= \langle 0 \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\hbar e^{-ik(x-x')} \mid 0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\hbar e^{-ik(x-x')} \qquad \text{(D.8.12)}
\end{aligned}$$

by taking out the constant and realising that $\langle 0 \mid 0\rangle = 1$. also introducing the heaviside function to account for the time ordering symbol we get

$$\langle 0 \mid T(\mid \phi\phi^\dagger \mid) \mid 0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\theta(t'-t)\hbar e^{-ik(x-x')}+\theta(t-t')\hbar e^{-ik(x'-x)}$$

Using the integral form of the heaviside function (1.2.2):

$$\theta(t) = \lim_{\epsilon\to 0}\int \frac{d\omega}{2\pi i}\frac{e^{i\omega t}}{\omega - i\epsilon} \tag{D.8.13}$$

We get:

$$G(x',x) = \lim_{\epsilon\to 0}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi i}\frac{1}{2\omega_k}\hbar\frac{1}{\omega - i\epsilon}[e^{i(\omega-\omega_k)(t'-t)}e^{i\vec{k}.(\vec{x}'-\vec{x})}+e^{-i(\omega-\omega_k)(t'-t)}e^{-i\vec{k}.(\vec{x}'-\vec{x})}]$$

We split into two terms:

$$\begin{aligned} G(x',x) &= \lim_{\epsilon\to 0}\int \frac{d^3k'}{(2\pi)^3}\int \frac{d\omega}{2\pi i}\frac{1}{2\omega_k}\hbar\frac{1}{\omega - i\epsilon}e^{i(\omega-\omega_k)(t'-t)}e^{i\vec{k}.(\vec{x}'-\vec{x})} \\ &+ \lim_{\epsilon\to 0}\int \frac{d^3k'}{(2\pi)^3}\int \frac{d\omega}{2\pi i}\frac{1}{2\omega_k}\hbar\frac{1}{\omega - i\epsilon}e^{-i(\omega-\omega_k)(t'-t)}e^{-i\vec{k}.(\vec{x}'-\vec{x})} \end{aligned} \tag{D.8.14}$$

in the first term we take:

$$k_0 = \omega_k - \omega \tag{D.8.15}$$

In the second we take:

$$k_0 = \omega - \omega_k \tag{D.8.16}$$

So:

$$\begin{aligned} G(x',x) &= \lim_{\epsilon\to 0}(-i\hbar)\int \frac{d^4k}{(2\pi)^4}\frac{1}{2\omega_k}\frac{1}{\omega_k - k_0 - i\epsilon}e^{-ik_0(t'-t)}e^{i\vec{k}.(\vec{x}'-\vec{x})} \\ &+ \lim_{\epsilon\to 0}(-i\hbar)\int \frac{d^4k}{(2\pi)^4}\frac{1}{2\omega_k}\hbar\frac{1}{\omega_k + k_0 - i\epsilon}e^{-ik_0(t'-t)}e^{-i\vec{k}.(\vec{x}'-\vec{x})} \end{aligned} \tag{D.8.17}$$

swapping $\vec{k} \to -\vec{k}$ in the second term we get:

$$G(x',x) = \lim_{\epsilon\to 0}(-i\hbar)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x'-x)}}{2\omega_k}(\frac{1}{\omega_k - k_0 - i\epsilon} + \frac{1}{\omega_k + k_0 - i\epsilon}) \tag{D.8.18}$$

Consider

$$\begin{aligned} (\frac{1}{\omega_k - k_0 - i\epsilon} + \frac{1}{\omega_k + k_0 - i\epsilon}) &= \frac{\omega_k + k_0 - i\epsilon + \omega_k - k_0 - i\epsilon}{(\omega_k - k_0 - i\epsilon)(\omega_k + k_0 - i\epsilon)} \\ &= \frac{2\omega_k}{\omega_k^2 - k_0^2 - i\epsilon 2\omega_k - \epsilon^2} \end{aligned} \tag{D.8.19}$$

where $\omega_k^2 = k^2 + m^2$
ϵ^2 term is ignored
We rewrite $\epsilon 2\omega_k$ as ϵ
and get :

$$\begin{aligned} (\frac{1}{\omega_k - k_0 - i\epsilon} + \frac{1}{\omega_k + k_0 - i\epsilon}) &= \frac{2\omega_k}{k^2 - k_0^2 + m^2 - i\epsilon} \\ &= \frac{2\omega_k}{-k_\nu k^\nu + m^2 - i\epsilon} \end{aligned} \tag{D.8.20}$$

using the definition of the momentum four vector (1.6.15)
Thus (D.8.18) becomes:

$$G(x',x) = \lim_{\epsilon\to 0}(-i\hbar)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x'-x)}}{-k_\nu k^\nu + m^2 - i\epsilon} \tag{D.8.21}$$

D.9 $S_F(x,y)$ in Terms of Four-Integrals

following [5] p268 eq 42.6:

$$S_F = i\langle 0 \mid T\psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle \tag{D.9.1}$$

We know (2.3.13)

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \tag{D.9.2}$$

knowing (2.3.10)

$$\overline{\psi} = \psi^\dagger\gamma^0 \tag{D.9.3}$$

We know :

$$\overline{\psi} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(e^{ipx}\overline{u}^s(p)a^{s\dagger}_{\vec{p}} + e^{-ipx}\overline{v}^s(p)b^{s\dagger}_{\vec{p}} \tag{D.9.4}$$

Thus :

$$\begin{aligned}\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle &= \langle 0 \mid \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \\ &\times \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{r=1,2}(e^{ip'y}\overline{u}^r(p')a^{r\dagger}_{\vec{p}} + e^{-ip'y}\overline{v}^r(p')b^{r\dagger}_{\vec{p}}\end{aligned} \tag{D.9.5}$$

since any terms in normal order will vanish , we only get the term :

$$\begin{aligned}\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle &= \langle 0 \mid \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}} \\ &\times \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s=1,2}\sum_{r=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx}e^{ip'y}\overline{u}^r(p')a^{r\dagger} \mid 0\rangle\end{aligned} \tag{D.9.6}$$

We can add a term of the form :

$$\begin{aligned}\langle 0 \mid &\int \tfrac{d^3p}{(2\pi)^3} \times \\ \frac{1}{\sqrt{2E_{\vec{p}}}} &\int \tfrac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s=1,2}\sum_{r=1,2}(u^s(p)e^{-ipx}e^{ip'y}\overline{u}^r(p')a^{s\dagger}a^r_{\vec{p}} \mid 0\rangle \\ &= 0\end{aligned} \tag{D.9.7}$$

To give:

$$\begin{aligned}\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s=1,2}\sum_{r=1,2}u^s(p)e^{-ipx}e^{ip'y}\overline{u}^r(p') \\ &\times \langle 0 \mid \{a^s_{\vec{p}}, a^{r\dagger}\} \mid 0\rangle\end{aligned} \tag{D.9.8}$$

Our anti-commutator for the creation operator a is by analogue with the scalar case Appendix B3 to be :

$$\{a^s_{\vec{p}}, a^{s\dagger}\} = \hbar(2\pi)^3\delta^3(\vec{p} - \vec{p}')2\sqrt{2E_{\vec{p}}}\delta_{sr} \tag{D.9.9}$$

Thus we get :

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2} u^s(p)e^{-ip(x-y)}\overline{u}^s(p)\langle 0 \mid\mid 0\rangle$$

which as the vacuum states are orthogonal is:

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2} u^s(p)e^{-ip(x-y)}\overline{u}^s(p) \tag{D.9.10}$$

using the spin sum (2.3.22)

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = (\not{p} + m) \tag{D.9.11}$$

Thus:

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(\not{p} + m)(p)e^{-ip(x-y)}) \tag{D.9.12}$$

recalling again (2.3.13) and (D.9.4)

$$\begin{aligned}\langle 0 \mid \overline{\psi}_\alpha(x)\psi_\beta(y) \mid 0\rangle &= \langle 0 \mid \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{r=1,2}(e^{ip'y}\overline{u}^r(p')a^{r\dagger}_{\vec{p}} + e^{-ip'y}\overline{v}^r(p)b^{r\dagger}_{\vec{p}'} \\ &\times \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \mid 0\rangle\end{aligned}$$

since any terms in normal order will vanish , we only get the term :

$$\langle 0 \mid \overline{\psi}_\alpha(x)\psi_\beta(y) \mid 0\rangle = \langle 0 \mid \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s=1,2}\sum_{r=1,2}\overline{v}^r(p')a^{r\dagger}e^{-ip'y}e^{ipx}a^s_{\vec{p}}v^s(p) \mid 0\rangle$$

We can add a term of the form :

$$\langle 0 \mid \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s=1,2}\sum_{r=1,2} v^s(p)e^{-ip'y}e^{ipx}\overline{v}^r(p')b^r_{\vec{p}}b^{s\dagger}_{\vec{p}} \mid 0\rangle = 0$$

To make our term:

$$\begin{aligned}\langle 0 \mid \overline{\psi}_\alpha(x)\psi_\beta(y) \mid 0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}\sum_{r=1,2} e^{-ip'y}e^{ipx}\overline{v}^r v^s(p)(p') \\ &\times \langle 0 \mid \{a^s_{\vec{p}}, a^{r\dagger}\} \mid 0\rangle \end{aligned} \tag{D.9.13}$$

recall (D.9.9) to get :

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}\overline{v}^s(p)e^{ip(x-y)}v^s(p)\langle 0 \mid\mid 0\rangle$$

which as the vacuum states are orthogonal:

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}\overline{v}^s(p)e^{ip(x-y)} \tag{D.9.14}$$

using the spin sum (2.3.23)

$$\sum_{s=1,2}\overline{v}^s(p)v^s(p) = (\not{p} - m) \tag{D.9.15}$$

we get:

$$\langle 0 \mid \psi_\alpha(x)\overline{\psi}_\beta(y) \mid 0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(\not{p} - m)e^{ip(x-y)} \tag{D.9.16}$$

We can multiply our propagator expression for the scalar field derived in Appendix D8 by f(p) to give the identity

$$\begin{aligned}\int \quad \frac{d^4p}{(2\pi)^4}\frac{f(p)e^{-ip(x-y)}}{-p^2+m^2-i\epsilon} &= i\theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}e^{-ip(x-y)}f(p)\\ + \quad i\theta(y^0-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}e^{ip(x-y)}f(-p)\end{aligned} \tag{D.9.17}$$

where the minus sign on the second term comes from changing the dummy variable to -p in the d^3p and because the p^0 is already negative due to the time ordering

Thus recalling (4.4.40) , (D.9.12) , (D.9.16) to give:

$$\begin{aligned}\langle 0 \mid T \quad (\overline{\psi}_\beta(y)\psi_\alpha(x)) \quad \mid 0\rangle &= \theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(\not{p}+m)e^{-ip(x-y)}\\ - \qquad & \theta(y^0-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(\not{p}-m)e^{ip(x-y)}\end{aligned} \tag{D.9.18}$$

So we get considering $f(p) = \not{p} + m$:

$$S_F(x-y) = \frac{1}{i}\langle 0 \mid \overline{\psi}_\beta(y)\psi_\alpha(x) \mid 0\rangle = \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}(\not{p}+m)}{-p^2+m^2-i\epsilon} \tag{D.9.19}$$

D.10 Deriving the Vector Propagator

This follows from Section 9.4 of [9]
We consider the action :

$$S = \int d^4x[-\frac{1}{4}(F_{\mu\nu})^2] \tag{D.10.1}$$

We know $F_{\mu\nu}$ is a function of A_μ and both $F_{\mu\nu}$ and $A_{\mu\nu}$ are invariant under the transform (see section 1.7)

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{D.10.2}$$

Recall (1.7.13)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu \tag{D.10.3}$$

is obviously invariant as under the gauge transform this becomes :

$$\partial_\mu A_\nu(x) + \partial_\mu\frac{1}{e}\partial_\nu\alpha(x) - \partial_\nu A_\mu(x) + \partial_\nu\frac{1}{e}\partial_\mu\alpha(x) \tag{D.10.4}$$

which is just :

$$\partial_\mu A_\nu - \partial_\mu A_\mu \tag{D.10.5}$$

as the partial derivatives of α commute
We consider the Fadeev-poppov method which we will come back to in Section 8.4
We introduce the definition :

$$1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \tag{D.10.6}$$

This is just the continuum generalisation of the identity of the n-dimensional dirac delta function with a switch from $\vec{g}$ co-ordinates to $\vec{a}$ co-ordinates (p295 [9] l23):

$$1 = (\prod_i da_i)\delta^n(\vec{g}(\vec{a}))det(\frac{\partial g_i}{\partial a_j}) \tag{D.10.7}$$

A^α_μ are the transformed fields i.e:

$$A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{D.10.8}$$

G is a function that when set equal to zero corresponds to some gauge -fixing condition
We assume that G has no terms of the form $A\alpha$ or higher order terms and is linear in $alpha$ and its derivatives , i.e. $\frac{\delta G(A^\alpha)}{\delta\alpha}$ is independent of both α and A
Thus we can write :

$$1 = det(\frac{\delta G(A^\alpha)}{\delta\alpha})\int \mathcal{D}\alpha(x)\delta(G(A^\alpha)) \tag{D.10.9}$$

Now consider :

$$Z = \int \mathcal{D}Ae^{iS[A]} \tag{D.10.10}$$

by inserting one (D.10.9) we can write this as:

$$Z = det(\frac{\delta G(A^{\alpha})}{\delta\alpha}) \int \mathcal{D}\alpha(x)\mathcal{D}Ae^{iS[A]}\delta(G(A^{\alpha})) \tag{D.10.11}$$

Since A^{α} differs from A by only a term linear in the derivative of α (D.10.8)

$$\mathcal{D}A^{\alpha} = \mathcal{D}A \tag{D.10.12}$$

since S (D.10.1) depends only on $F_{\mu\nu}$ it is gauge invariant ,i.e.:

$$S[A] = S[A^{\alpha}] \tag{D.10.13}$$

So :

$$Z = det(\frac{\delta G(A^{\alpha})}{\delta\alpha}) \int \mathcal{D}\alpha(x)\mathcal{D}A^{\alpha}e^{iS[A^{\alpha}]}\delta(G(A^{\alpha})) \tag{D.10.14}$$

Changing the dummy integration variable from A^{α} to A we can take out the α dependence and get

$$Z = N \int \mathcal{D}Ae^{iS[A]}\delta(G(A)) \tag{D.10.15}$$

where N is a normalisation constant
We suppose (eq 9.55 p296 [9]):

$$G(A) = \partial A_{\mu}(x) - w(x) \tag{D.10.16}$$

providing w is scalar this does not affect our results. i.e.: $\frac{\delta G(A^{\alpha})}{\delta\alpha}$ so

$$Z = N \int \mathcal{D}Ae^{iS[A]}\delta(\partial A_{\mu}(x) - w(x)) \tag{D.10.17}$$

We multiply Z by 1 in the form of a normalised gaussian function generalised to the continuum centered around w withe width ξ
to get :

$$Z = NN(\xi) \int \mathcal{D}A\mathcal{D}we^{-i\int d^4x\frac{w^2}{2\xi}}e^{iS[A]}\delta(\partial A_{\mu}(x) - w(x)) \tag{D.10.18}$$

where $N(\xi)$ is the normalisation constant associated with the Gaussian function
using the delta to eliminate the w integral we get:

$$Z = NN(\xi) \int \mathcal{D}Ae^{iS[A]}e^{-i\int d^4x\frac{(\partial^{\mu}A_{\mu})^2}{2\xi}} \tag{D.10.19}$$

We need now only consider :

$$S[A] - \int d^4x \frac{(\partial^\mu A_\mu)^2}{2\xi} \tag{D.10.20}$$

i.e.

$$\int d^4x - \frac{1}{4}(F_{\mu\nu})^2 - \frac{(\partial^\mu A_\mu)^2}{2\xi} \tag{D.10.21}$$

which describes the action of the A's with gauge fixed if we can write this as:

$$\int d^4x A_\mu(x) f A_\nu(x) \tag{D.10.22}$$

for some function f
Then our propagator will be the delta function of this f i.e.:

$$f D_{F\nu\rho}(x-y) = i\delta^\mu_\rho \delta^4(x-y) \tag{D.10.23}$$

as the propagator describes the motion of a particle from x to y
You can see this is true of the Dirac-propagator and the scalar propagator by seeing that in position space the exponential is the solution of both the dirac and the Klein -gordan equation and that the other parts give the klein-gordan or dirac equation in momentum space multiplied by $i\hbar$ i.e applying the relevant equation to the exponential will give the form of the equation in momentum space which cancels with rest of the equation to give $i\hbar$ multiplying the exponential which is just the dirac delta multiplied by $i\hbar$. The rearrangement to get f will be simply integrating by parts on the first A which as we ignore boundary terms is the same as differentiation on the second A. Differentiation on the second A is the same as varying it with repect to the derivative of the first A which would be the Euler-lagrange equations. Hence f will be the equivalent equation: Consider using (1.7.13) and (1.6.4) in the exponent of (D.10.21):

$$-\frac{1}{4}(F_{\mu\nu})^2 - \frac{(\partial^\mu A_\mu)^2}{2\xi} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{(\partial^\mu A_\mu)^2}{2\xi}$$

$$-\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\mu \partial^\nu A^\mu + \partial_\nu A_\mu \partial^\nu A^\mu) - \frac{(\partial^\mu A_\mu)^2}{2\xi}$$

The terms $\partial_\mu A_\nu \partial^\mu A^\nu$ and $\partial_\nu A_\mu \partial^\nu A^\mu$ can be considered to be the same by renaming the summation variables
We integrate each term by parts and ignore the boundary terms as A vanishes at the boundary this is equivalent to integrating and multiplying by - 1

$$\partial_\mu A_\nu \partial^\mu A^\nu \tag{D.10.24}$$

integrates to :

$$-A_\nu \partial_\mu \partial^\mu A^\nu \tag{D.10.25}$$

which is :

$$A_\nu \partial^2 A^\nu \tag{D.10.26}$$

by the definition of ∂^2 ((1.6.4) with ∂_μ): Consider

$$A_\nu \partial^2 A_\mu = A_\nu \partial^2 A^\nu g_{\mu\nu} \tag{D.10.27}$$

using the g to change A^ν to A_μ since $g_{\mu\nu} g^{\mu\nu} = 1$ this is basically just (1.6.4)

$$A_\mu \partial^2 A_\nu g^{\mu\nu} = A_\mu \partial^2 A^\nu \tag{D.10.28}$$

where we have renamed variables for convenience.
Thus as

$$\partial_\mu A_\nu \partial^\mu A^\nu \tag{D.10.29}$$

and

$$\partial_\nu A_\mu \partial^\nu A^\mu \tag{D.10.30}$$

can be considered to be the same :

$$-\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) = \frac{1}{2} A_\mu \partial^2 A_\nu g^{\mu\nu} \tag{D.10.31}$$

where the extra minus sign comes from integration by parts
The :

$$(\partial^\mu A_\mu)^2 \tag{D.10.32}$$

term
can be written

$$\partial^\mu A_\mu \partial^\mu A_\mu \tag{D.10.33}$$

which is:

$$\partial^\mu A_\mu \partial^\nu A_\nu \tag{D.10.34}$$

changing the second summation variable
integrating by parts and ignoring the boundary term this becomes:

$$-A_\mu \partial^\mu \partial^\nu A_\nu \tag{D.10.35}$$

So :

$$-\frac{(\partial^\mu A_\mu)^2}{2\xi} = \frac{1}{2\xi} A_\mu \partial^\mu \partial^\nu A_\nu \tag{D.10.36}$$

The terms

$$\partial_\nu A_\mu \partial^\mu A^\nu \tag{D.10.37}$$

and

$$\partial_\mu A_\nu \partial^\mu \partial^\nu A^\mu \tag{D.10.38}$$

can be considered to be the same by for example exchanging μ and ν in the (D.10.38)
taking :

$$\partial_\nu A_\mu \partial^\mu A^\nu \tag{D.10.39}$$

and integrating by parts and ignoring the boundary term we get

$$-\ A_\mu \partial_\nu \partial^\mu A^\nu \tag{D.10.40}$$

as partial derivatives commute this becomes;

$$-\ A_\mu \partial^\mu \partial_\nu A^\nu \tag{D.10.41}$$

a property of four-vectors is that (1.6.4):

$$a_\mu b^\mu = a^\mu b_\mu \tag{D.10.42}$$

we get:

$$\partial_\nu A^\nu = \partial^\nu A_\nu \tag{D.10.43}$$

Thus our two terms (D.10.37) and (D.10.38) are:

$$-\ A_\mu \partial^\mu \partial^\nu A_\nu \tag{D.10.44}$$

and so:

$$-\frac{1}{4}(-\partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\mu \partial^\nu A^\mu) = -\frac{1}{2} A_\mu \partial^\mu \partial^\nu A_\nu \tag{D.10.45}$$

substituting (D.10.31) , (D.10.36) , (D.10.45) into the exponent of (D.10.21) we get:

$$\begin{aligned} -\frac{1}{4}(F_{\mu\nu})^2 - \frac{(\partial^\mu A_\mu)^2}{2\xi} &= \frac{1}{2}(A_\mu \partial^2 A_\nu g^{\mu\nu} - A_\mu \partial^\mu \partial^\nu A_\nu + \frac{1}{\xi} A_\mu \partial^\mu \partial^\nu A_\nu \\ &= A_\mu (\frac{1}{2}[\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu (1 - \frac{1}{\xi}]) A_\nu \end{aligned} \tag{D.10.46}$$

hence our f is compare (D.10.23)

$$(\frac{1}{2}[\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu (1 - \frac{1}{\xi}]) \tag{D.10.47}$$

changing to momentum space which considering the exponential e^{ikx} is equivalent to replacing the differentials by ik with the appropriate index gives our f in momentum space as :

$$(-k^2 g^{\mu\nu} + (1 - \frac{1}{\xi}) k^\mu k^\nu) \tilde{D}_{F\nu\rho}(k) = i\delta^\mu_\rho \tag{D.10.48}$$

Thus :

$$\tilde{D}_{F\nu\rho}(k) = \frac{-i}{k^2 + i\epsilon}(g_{\nu\rho} - (1 - \xi)\frac{k_\nu k_\rho}{k^2}) \tag{D.10.49}$$

We show this by substiting into (D.10.48) i.e:

$$(-k^2 g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu)\frac{-i}{k^2+i\epsilon}(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2}) \tag{D.10.50}$$

multiplying out gives:

$$\tfrac{-i}{k^2+i\epsilon}(-k^2 g^{\mu\nu} g_{\nu\rho} + (1-\tfrac{1}{\xi})k^\mu k^\nu g_{\nu\rho} + k^2 g^{\mu\nu}(1-\xi)\tfrac{k_\nu k_\rho}{k^2} - (1-\tfrac{1}{\xi})k^\mu k^\nu(1-\xi)\tfrac{k_\nu k_\rho}{k^2})$$

rewriting the product of g and cancelling the k^2 using (1.6.4) on the last term:

$$\frac{-i}{k^2+i\epsilon}(-k^2 {\delta^\mu}_\rho + (1-\frac{1}{\xi})k^\mu k_\rho + (1-\xi)k^\mu k_\rho - (1-\frac{1}{\xi})k^\mu k_\rho \nu(1-\xi)) \tag{D.10.51}$$

taking out ${\delta^\mu}_\rho$ as a factor:

$$\frac{-i}{k^2+i\epsilon}{\delta^\mu}_\rho(-k^2 + (1-\frac{1}{\xi})k^\mu k_\mu + k^2(1-\xi)\frac{k^\mu k_\mu}{k^2} - (1-\frac{1}{\xi})k^\mu k_\mu(1-\xi)) \tag{D.10.52}$$

using (1.6.4):

$$\frac{-i}{k^2+i\epsilon}{\delta^\mu}_\rho(-k^2 + (1-\frac{1}{\xi})k^2 + k^2(1-\xi) - (1-\frac{1}{\xi})k^2(1-\xi)) \tag{D.10.53}$$

isolating the coefficients of k^2 :

$$\frac{-i}{k^2+i\epsilon}{\delta^\mu}_\rho(-k^2 + 2k^2 - k^2(\xi+\frac{1}{\xi}) - 2k^2 + (\xi+\frac{1}{\xi})k^2) \tag{D.10.54}$$

simplifying:

$$\frac{ik^2}{k^2+i\epsilon}{\delta^\mu}_\rho \tag{D.10.55}$$

ignoring the epsilon

$$i{\delta^\mu}_\rho \tag{D.10.56}$$

Therefore in the feynman gauge $\xi = 1$ (l18 p297 [9])
from (D.10.49):

$$\tilde{D}_{F\nu\rho}(k) = \frac{-i}{k^2+i\epsilon}(g_{\nu\rho}) \tag{D.10.57}$$

We now just need to show (4.4.47):

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{1}{\vec{k}^2}\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\epsilon}\sum_{\lambda=\pm}\epsilon_\lambda^{\mu *}(\vec{k})\epsilon_\lambda^\nu(\vec{k}) \tag{D.10.58}$$

is equivalent to our D ignoring an i
following eq 55.15 p337 [5]:

$$\sum_{\lambda=\pm}\epsilon_{i\lambda}^*(\vec{k})\epsilon_{j\lambda}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \tag{D.10.59}$$

we consider :

$$\hat{t}^\mu = (1,0) \tag{D.10.60}$$

then :

$$(0.\vec{k}) = k^\mu - (\hat{t}.k)\hat{t}^\mu \tag{D.10.61}$$

rearranging:

$$(0.\vec{k}) + (\hat{t}.k)\hat{t}^\mu = k^\mu \tag{D.10.62}$$

using (1.6.4) to square:

$$-\vec{k}^2 + (\hat{t}.k)^2 = k^2 \tag{D.10.63}$$

rearranging:

$$\vec{k}^2 = -k^2 + (\hat{t}.k)^2 \tag{D.10.64}$$

we want a normalised four-vector consider

$$\hat{z}^\mu = \frac{i(k^\mu - (\hat{t}.k)\hat{t}^\mu)}{[-k^2 + (\hat{t}.k)^2]^{\frac{1}{2}}} \tag{D.10.65}$$

$\hat{z}^\mu \hat{z}^\nu$ becomes $-\frac{k_i k_j}{\vec{k}^2}$ in the i,j limit from (D.10.64) and (D.10.60) which gives:

$$\hat{t}^i = 0 \tag{D.10.66}$$

consider (1.6.1): $-g^{\mu\nu} = \delta_{ij}$ on i,j
Thus we postulate :

$$\sum_{\lambda=\pm} \epsilon_\lambda^{\mu *}(\vec{k})\epsilon_\lambda^\nu(\vec{k}) = -g^{\mu\nu} + \hat{t}^\mu t^\nu + z^\mu z^\nu \tag{D.10.67}$$

as both sides vanish when μ or ν =0 we get complete agreement
Thus :

$$\frac{1}{\vec{k}^2}\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\epsilon}\sum_{\lambda=\pm} \epsilon_\lambda^{\mu *}(\vec{k})\epsilon_\lambda^\nu(\vec{k}) \tag{D.10.68}$$

becomes from (D.10.64) and (D.10.60) :

$$\frac{t^\mu t^\nu}{-k^2 + (t.k)^2} + \frac{-g^{\mu\nu} + \hat{t}^\mu t^\nu + z^\mu z^\nu}{k^2 - i\epsilon} \tag{D.10.69}$$

in position representation our k^μ become ∂^μ acting on the exponential integration by parts means we can ignore them see l7-14 p342 [5]
Thus we can write (D.10.65):

$$\hat{z}^\mu = \frac{-i(\hat{t}.k)\hat{t}^\mu}{[-k^2 + (\hat{t}.k)^2]^{\frac{1}{2}}} \tag{D.10.70}$$

Thus our expression (D.10.69) is :

$$\frac{t^\mu t^\nu}{-k^2 + (t.k)^2} + \frac{-g^{\mu\nu} + \hat{t}^\mu t^\nu - \frac{(t.k)t^\mu t^\nu}{[-k^2 + (\hat{t}.k)^2]}}{k^2 - i\epsilon} \tag{D.10.71}$$

This becomes ignoring the difference between k^2 and $k^2 - i\epsilon$:

$$\frac{1}{k^2 - i\epsilon}[-g^{\mu\nu} + (\frac{k^2}{-k^2 + (\hat{t}.k)^2} + 1 - \frac{(t.k)^2}{-k^2 + (\hat{t}.k)^2})t^\mu t^\nu] \tag{D.10.72}$$

which is just :

$$\frac{1}{k^2 - i\epsilon} - g^{\mu\nu} \tag{D.10.73}$$

as predicted (D.10.57) divided by i

D.11 Contracting Fields with their Momentum States

switching the final state creation operators to the expressions given we gain a field and a derivative function (either the klein-gordan (2.1.11) or Dirac equation (2.3.8)) . Moving the fields inside the time ordering symbol gives you a sum of terms made of of n over 2 contractions for n of a given field type these contractions correspond to the green's functions of the relevant equation thus we pick up a delta for each field paie and the remaining term is the one we associate with the incoming particles

Appendix E

Appendix E :Chapter 5

E.1 Conservation of Current in the Dirac Field

We need to show that :

$$\partial_\mu j^\mu = 0 \tag{E.1.1}$$

given (5.2.24)

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{E.1.2}$$

Thus:

$$\partial_\mu j^\mu = \partial_\mu(\overline{\psi}\gamma^\mu\psi) \tag{E.1.3}$$

separating into derivatives on ψ and $\overline{\psi}$ we get:

$$\partial_\mu j^\mu = (\partial_\mu\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi \tag{E.1.4}$$

The Dirac equation is (2.3.8):

$$(i\not{\partial} - m)\psi = 0 \tag{E.1.5}$$

i.e. :

$$i\not{\partial}\psi = m\psi \tag{E.1.6}$$

multiplying both sides by -i

$$\not{\partial}\psi = -im\psi \tag{E.1.7}$$

remembering (2.3.24):

$$\gamma^\mu\partial_\mu\psi = -im\psi \tag{E.1.8}$$

we then take the transpose :

$$\partial_\mu\psi^\dagger\gamma^{\mu\dagger} = im\psi^\dagger \tag{E.1.9}$$

multiplying on the left by γ^0 and recalling (2.3.10):

$$\partial_\mu\psi^\dagger\gamma^{\mu\dagger}\gamma^0 = im\overline{\psi} \tag{E.1.10}$$

recall (2.3.1) , (2.3.2):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{E.1.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ 0 & -\sigma^i \end{pmatrix} \tag{E.1.12}$$

clearly

$$\gamma^0 = \gamma^{0\dagger} \tag{E.1.13}$$

and

$$\gamma^{i\dagger} = \begin{pmatrix} 0 & -\sigma^{i\dagger} \\ 0 & \sigma^{i\dagger} \end{pmatrix} \tag{E.1.14}$$

Recall (B.10.8)

$$\sigma^{i\dagger} = \sigma^i \tag{E.1.15}$$

So

$$\gamma^{i\dagger} = -\gamma^{\dagger i} \tag{E.1.16}$$

Recall from (2.3.4):

$$\{\gamma^0, \gamma^i\} = 0 \tag{E.1.17}$$

So from (E.1.16):

$$\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \tag{E.1.18}$$

Then (E.1.10) recalling (2.3.10) becomes:

$$\partial_\mu \overline{\psi}\gamma^\mu = im\overline{\psi} \tag{E.1.19}$$

using the Feynman slash (2.3.24) and the property of four-vectors (1.6.4) $a^\mu b_\mu = a_\mu b^\mu$ we get

$$\not{\partial}\overline{\psi} = im\overline{\psi} \tag{E.1.20}$$

using the property of four -vectors (1.6.4) we write this as Thus (E.1.4) becomes using also (E.1.7) :

$$\partial_\mu j^\mu = (im\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu(-im\psi) \tag{E.1.21}$$

which is :

$$\partial_\mu j^\mu = 0 \tag{E.1.22}$$

and thus the current is conserved

E.2 $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$

Recall (2.3.25)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{E.2.1}$$

To write this in matrix from we remember (2.3.1) , (2.3.2)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{E.2.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{E.2.3}$$

So :

$$\begin{aligned} \gamma^5 &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= i\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix} \\ &= i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^2\sigma^3 \end{pmatrix} \end{aligned} \tag{E.2.4}$$

Recall (1.5.7)

$$\sigma^i\sigma^j = i\epsilon^{ijk}\epsilon^k \tag{E.2.5}$$

i.e.

$$\sigma^2\sigma^3 = i\epsilon^1 \tag{E.2.6}$$

$\epsilon^{231} = 1$
So:

$$\begin{aligned} \gamma^5 &= i\begin{pmatrix} \sigma^1 i\sigma^1 & 0 \\ 0 & -\sigma^1 i\sigma^1 \end{pmatrix} \\ &= \begin{pmatrix} -(\sigma^1)^2 & 0 \\ 0 & (\sigma^1)^2 \end{pmatrix} \end{aligned} \tag{E.2.7}$$

recall (1.5.10)

$$(\sigma^1)^2 = I \tag{E.2.8}$$

Thus :

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.2.9}$$

Recall (2.3.2)

$$\begin{aligned} \gamma^5\gamma^i &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} \end{aligned} \tag{E.2.10}$$

$$\begin{aligned}\gamma^i\gamma^5 &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \end{aligned} \tag{E.2.11}$$

so :

$$\gamma^i\gamma^5 = -\gamma^i\gamma^5 \tag{E.2.12}$$

Recall (2.3.1) and (2.3.26):

$$\begin{aligned}\gamma^5\gamma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{E.2.13}$$

$$\begin{aligned}\gamma^0\gamma^5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{E.2.14}$$

so :

$$\gamma^0\gamma^5 = -\gamma^0\gamma^5 \tag{E.2.15}$$

and thus recalling (E.2.12)

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \tag{E.2.16}$$

E.3 $\gamma^5 = \gamma^{5\dagger}$

from above (2.3.26)

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.3.1}$$

So:

$$\gamma^{5\dagger} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.3.2}$$

Thus :

$$\gamma^5 = \gamma^{5\dagger} \tag{E.3.3}$$

E.4 $(\gamma^5)^2 = 1$

from above (2.3.26)

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.4.1}$$

Thus:

$$(\gamma^5)^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.4.2}$$

and thus :

$$(\gamma^5)^2 = 1 \tag{E.4.3}$$

E.5 $\partial_\mu j^{\mu 5}(x) = 2im\overline{\psi}\gamma^5\psi$

writing :(5.2.29):

$$j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi \tag{E.5.1}$$

the derivative is:

$$\partial_\mu j^{\mu 5} = \partial_\mu(\overline{\psi}\gamma^\mu\gamma^5\psi) \tag{E.5.2}$$

separating into derivatives on ψ and $\overline{\psi}$ we get:

$$\partial_\mu j^\mu = (\partial_\mu\overline{\psi})\gamma^\mu\gamma^5\psi + \overline{\psi}\gamma^\mu\gamma^5\partial_\mu\psi \tag{E.5.3}$$

We found above in appendix E2 (E.2.12) that :

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \tag{E.5.4}$$

So:

$$\partial_\mu j^\mu = (\partial_\mu\overline{\psi})\gamma^\mu\gamma^5\psi - \overline{\psi}\gamma^5\gamma^\mu\partial_\mu\psi \tag{E.5.5}$$

using the property of four -vectors (1.6.4) on the first term we find

$$\partial_\mu j^\mu = \gamma^\mu(\partial_\mu\overline{\psi})\gamma^5\psi - \overline{\psi}\gamma^5\gamma^\mu\partial_\mu\psi \tag{E.5.6}$$

which becomes using (2.3.24)

$$\partial_\mu j^\mu = \not{\partial}\overline{\psi}\gamma^5\psi - \overline{\psi}\gamma^5\not{\partial}\psi \tag{E.5.7}$$

We found in section E1 that (E.1.20) , (E.1.7)

$$\not{\partial}\overline{\psi} = im\overline{\psi} \tag{E.5.8}$$

and

$$\not{\partial}\psi = -im\psi \tag{E.5.9}$$

Thus we have :

$$\partial_\mu j^\mu = im\overline{\psi}\gamma^5\psi - \overline{\psi}\gamma^5(-im\psi) \tag{E.5.10}$$

which is :

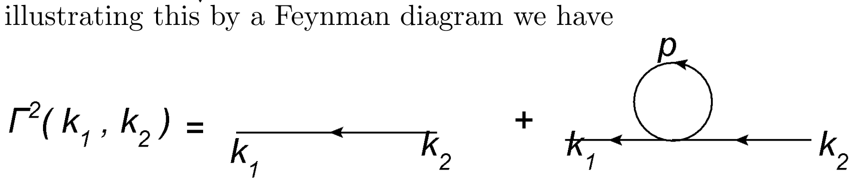
$$\partial_\mu j^\mu = 2im\overline{\psi}\gamma^5\psi \tag{E.5.11}$$

Appendix F

Appendix F: Chapter 6

F.1 The Divergent Terms of ϕ^4 Scalar Theory

We here merely prove that statement that was given in section 6.3 that the N=2 and N=4 terms between them result in three divergent terms
This follows section 7.1 and 7.2 of [7]
We first consider the case N=2 which we call Γ^2
We look only at the one loop correction case as this is all we have considered in section 6.3 and ignore the overall momentum conserving delta functions and factors of $\frac{1}{\sqrt{\omega}}$ as these are common to all our graphs
illustrating this by a Feynman diagram we have



Our relevant Feynman rules are
1) associate a four-momentum to each external propagator
2) Associate a factor of $(\frac{-i\hbar}{\lambda})$ to each vertex
3) associate with each internal propagator a factor of $\frac{-i\hbar}{-p^2+m^2-i\epsilon}$
4) integrate over every undetermined loop momentum
5) Divide by the symmetry factor
using this we get :

$$-k_1^2 + m^2 + \frac{1}{2}(-\frac{i}{\hbar}\lambda) \int \frac{d^4p}{(2\pi)^4} \frac{-i\hbar}{-p^2 + m^2 - i\epsilon} \qquad \text{(F.1.1)}$$

since the first term is clearly non -divergent and the coefficient of the integral is non-divergent we consider only the term :

$$\int \frac{d^4p}{(2\pi)^4} \frac{-i\hbar}{-p^2 + m^2 - i\epsilon} \qquad \text{(F.1.2)}$$

since this depends only on p^2 we can take out the angular dependence and thus take out the volume integral which is finite and write :

$$Vol(S^3)\int \frac{dp}{(2\pi)^4}p^3\frac{-i\hbar}{-p^2+m^2-i\epsilon} \tag{F.1.3}$$

We consider ther term depending on p:

$$\frac{p^3}{-p^2+m^2-i\epsilon} \tag{F.1.4}$$

as we can take out the $i\hbar$ as this is non-divergent to get:

$$\frac{p^3}{-p^2+m^2-i\epsilon} \tag{F.1.5}$$

can be written:

$$\frac{-p(-p^2+m^2-i\epsilon)}{-p^2+m^2-i\epsilon}+\frac{p(m^2-i\epsilon)}{-p^2+m^2-i\epsilon} \tag{F.1.6}$$

which is just:

$$-p+\frac{p(m^2-i\epsilon)}{-p^2+m^2-i\epsilon} \tag{F.1.7}$$

This integrates to:

$$\frac{-p^2}{2}-\frac{m^2}{2}log(-p^2+m^2) \tag{F.1.8}$$

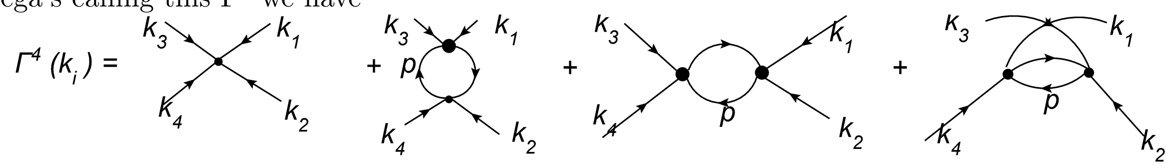
where we ignore the ϵ 's : evaluating at the two end-points 0 and the cut-off momentum Λ :

$$\frac{-\Lambda^2}{2}-\frac{m^2}{2}log(\frac{-\Lambda^2+m^2}{m^2}) \tag{F.1.9}$$

this diverges in two ways both quadratically and logarithmically

We consider the case N=4

We again only consider one loop corrections and ignoring the delta's and omega's calling this Γ^4 we have



Our relevant Feynman rules are

1) Associate a factor of $(\frac{-i\hbar}{\lambda})$ to each vertex

2) associate with each internal propagator a factor of $\frac{-i\hbar}{-p^2+m^2-i\epsilon}$

3) integrate over every undetermined loop momentum

4) Divide by the symmetry factor

if we consider where k_1 joins the loop we see that the loop terms all have the form :

$$V(q) = \frac{1}{2}(-\frac{i}{\hbar}\lambda)^2 \int \frac{d^4p}{(2\pi)^4} \frac{-i\hbar}{-p^2+m^2-i\epsilon} \frac{-i\hbar}{-(q+p)^2+m^2-i\epsilon} \tag{F.1.10}$$

where q is the four-momentum sum of k_1 and the other four-momentum that joins the loop at the same vertex
Thus our overall diagram gives:

$$\Gamma^4(k_i) = -(\frac{i\lambda}{\hbar}) + V(k_1+k_3) + V(k_1+k_2) + V(k_1+k_4) \tag{F.1.11}$$

If we assume the four-momentum is much larger then the mass and the mass is much lower then out cut-off momentum we can take $m \approx 0$ in the integrals . We can ignore the $-(\frac{i\lambda}{\hbar})$ term and the $\frac{1}{2}(-\frac{i}{\hbar}\lambda)^2(-i\hbar)^2$ prefactor of the integrals as they clearly do not diverge.Thus we consider only the terms

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2-i\epsilon} \frac{1}{-(q+p)^2-i\epsilon} \tag{F.1.12}$$

We wish to show :

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2+m^2 i\epsilon} \frac{1}{-(q+p)^2+m^2-i\epsilon} \tag{F.1.13}$$

is equivalent to:

$$\int_0^1 dx_1 dx_2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{[x_1(-p^2+m^2-i\epsilon)+x_2(-(p+q)^2+m^2-i\epsilon)]^2} \delta(1-x_1-x_2)$$

getting rid of the x_2 using the delta function and reordering the derivatives we obtain:

$$\int \frac{d^4p}{(2\pi)^4} \int_0^1 dx_1 \frac{1}{[x_1(-p^2+m^2-i\epsilon)+(1-x_1)(-(p+q)^2+m^2-i\epsilon)]^2} \tag{F.1.14}$$

we consider the denominator :

$$[x_1(-p^2+m^2-i\epsilon)+(1-x_1)(-(p+q)^2+m^2-i\epsilon)]^2 \tag{F.1.15}$$

multiplying out this becomes :

$$x_1^2(-p^2+m^2-i\epsilon)^2+2x_1(1-x_1)(-(p+q)^2+m^2-i\epsilon)(-p^2+m^2-i\epsilon)+(1-x_1)^2(-(p+q)^2+m^2-i\epsilon)^2$$

We re-arrange in terms of coefficients of the different orders of x_1 to get:

$$\begin{aligned} & x_1^2 && [(-p^2+m^2-i\epsilon)^2+(-(p+q)^2+m^2-i\epsilon)^2-2(-(p+q)^2+m^2-i\epsilon)(-p^2+m^2-i\epsilon)]+ \\ 2\ & x_1 && (-(p+q)^2+m^2-i\epsilon)(-p^2+m^2-i\epsilon-(-(p+q)^2+m^2-i\epsilon)) \\ & + && (-(p+q)^2+m^2-i\epsilon)^2 \end{aligned} \tag{F.1.16}$$

consider the x_1 coefficient which is :

$$2x_1(-(p+q)^2+m^2-i\epsilon)(-p^2+m^2-i\epsilon-(-(p+q)^2+m^2-i\epsilon)) \quad \text{(F.1.17)}$$

This becomes:

$$2x_1(-(p+q)^2+m^2-i\epsilon)(-p^2+(p+q)^2)) \quad \text{(F.1.18)}$$

multiplying out the square this becomes:

$$2x_1(-(p+q)^2+m^2-i\epsilon)(-p^2+p^2+2pq+q^2)) \quad \text{(F.1.19)}$$

which is:

$$2x_1(-(p+q)^2+m^2-i\epsilon)(2pq+q^2)) \quad \text{(F.1.20)}$$

or:

$$2x_1(-(p+q)^2+m^2-i\epsilon)q(2p+q)) \quad \text{(F.1.21)}$$

The coefficient of x_1^2 in (F.1.16) is

$$[(-p^2+m^2-i\epsilon)^2+(-(p+q)^2+m^2-i\epsilon)^2-2(-(p+q)^2+m^2-i\epsilon)(-p^2+m^2-i\epsilon)]$$

doing the squares as $(a+m^2-i\epsilon)^2 = a^2+2a(m^2-i\epsilon)$ this becomes :

$$\begin{aligned}[\quad & p^4 \quad -2p^2(m^2-i\epsilon)+(m^2-i\epsilon)^2+(p+q)^4-2(p+q)(m^2-i\epsilon)+(m^2-i\epsilon)^2 \\ - \quad & 2p^2(p+q)^2+2p^2(m^2-i\epsilon)+2(p+q)^2(m^2-i\epsilon)-2(m^2-i\epsilon)] \end{aligned} \quad \text{(F.1.22)}$$

canceling terms we get:

$$[p^4+(p+q)^4-2p^2(p+q)^2] \quad \text{(F.1.23)}$$

multiplying out the squares we get:

$$[p^4+p^4+4p^3q+6p^2q^2+4pq^3+q^4-2p^4-4p^3q-2p^2q^2] \quad \text{(F.1.24)}$$

which is :

$$[4p^2q^2+4pq^3+q^4] \quad \text{(F.1.25)}$$

which is :

$$q^2[4p^2+4pq+q^2] \quad \text{(F.1.26)}$$

which is:

$$q^2[2p+q]^2 \quad \text{(F.1.27)}$$

and thus our overall expression for the denominator (F.1.16) is:

$$q^2[2p+q]^2x_1^2+2x_1(-(p+q)^2+m^2-i\epsilon)q(2p+q))+(-(p+q)^2+m^2-i\epsilon)^2$$

which can be written as a square as:

$$[q[2p+q]x_1 + (-(p+q)^2 + m^2 - i\epsilon)]^2 \tag{F.1.28}$$

Our integral (F.1.14) is then :

$$\int \frac{d^4p}{(2\pi)^4} \int_0^1 dx_1 \frac{1}{[q[2p+q]x_1 + (-(p+q)^2 + m^2 - i\epsilon)]^2} \tag{F.1.29}$$

consider only the integral over x

$$\int_0^1 dx_1 \frac{1}{[q[2p+q]x_1 + (-(p+q)^2 + m^2 - i\epsilon)]^2} \tag{F.1.30}$$

using the co-ordinate transform :

$$[q[2p+q]x_1 + (-(p+q)^2 + m^2 - i\epsilon)] = x \tag{F.1.31}$$

which has:

$$dx = q[2p+q]dx_1 \tag{F.1.32}$$

our integral (F.1.30) becomes :

$$\frac{1}{q[2p+q]} \int_{(-(p+q)^2+m^2-i\epsilon)}^{[q[2p+q]+(-(p+q)^2+m^2-i\epsilon)]} dx \frac{1}{x^2} \tag{F.1.33}$$

We can see that:

$$[q[2p+q]+(-(p+q)^2+m^2-i\epsilon)] = 2pq+q^2-p^2-2pq+q^2+m^2-i\epsilon \tag{F.1.34}$$

our integral (F.1.30) is thus:

$$\frac{1}{q[2p+q]} \int_{(-(p+q)^2+m^2-i\epsilon)}^{-p^2+m^2-i\epsilon)]} dx \frac{1}{x^2} \tag{F.1.35}$$

This integrates to :

$$-\frac{1}{q[2p+q]} [\frac{1}{x}]_{(-(p+q)^2+m^2-i\epsilon)}^{-p^2+m^2-i\epsilon)]} \tag{F.1.36}$$

which is :

$$\frac{1}{q[2p+q]} [\frac{1}{(-(p+q)^2+m^2-i\epsilon)} - \frac{1}{-p^2+m^2-i\epsilon}] \tag{F.1.37}$$

simplifying to:

$$\frac{1}{q[2p+q]} [\frac{-p^2+m^2-i\epsilon-(-(p+q)^2+m^2-i\epsilon)}{(-p^2+m^2-i\epsilon)(-(p+q)^2+m^2-i\epsilon)}] \tag{F.1.38}$$

calculate:

$$-p^2+m^2-i\epsilon-(-(p+q)^2+m^2-i\epsilon) = -p^2+(p+q)^2 = 2pq+q^2 = q(2p+q) \tag{F.1.39}$$

So our integral (F.1.14) is :

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{q[2p+q]}[\frac{q(2p+q)}{(-p^2+m^2-i\epsilon)(-(p+q)^2+m^2-i\epsilon)}] \tag{F.1.40}$$

which is :

$$\int \frac{d^4p}{(2\pi)^4}[\frac{1}{(-p^2+m^2-i\epsilon)(-(p+q)^2+m^2-i\epsilon)}] \tag{F.1.41}$$

as required.
returning to the $m \approx 0$ case we have (F.1.12) which is by our identity:

$$\int_0^1 dx_1 dx_2 \int \frac{d^4p}{(2\pi)^4}\frac{1}{[x_1(-p^2-i\epsilon)+x_2(-(p+q)^2-i\epsilon)]^2}\delta(1-x_1-x_2) \tag{F.1.42}$$

using the delta and x_1 integral to take x_1 to $1-x_2$ and calling x_2 x . We get :

$$\int_0^1 dx \int \frac{d^4p}{(2\pi)^4}\frac{1}{[(1-x)(-p^2-i\epsilon)+x(-(p+q)^2-i\epsilon)]^2} \tag{F.1.43}$$

the denominator is:

$$[(1-x)(-p^2-i\epsilon)+x(-(p+q)^2-i\epsilon)]^2 \tag{F.1.44}$$

the root of this expands to:

$$(-p^2-i\epsilon)+xp^2+xi\epsilon-xp^2-2xpq-xq^2-xi\epsilon \tag{F.1.45}$$

which is :

$$-p^2-2xpq-xq^2-i\epsilon \tag{F.1.46}$$

Thus our integral (F.1.43) is then:

$$\int_0^1 dx \int \frac{d^4p}{(2\pi)^4}\frac{1}{[p^2+2xpq+xq^2+i\epsilon]^2} \tag{F.1.47}$$

we introduce

$$l = p+xq \tag{F.1.48}$$

which means

$$dl = dp \tag{F.1.49}$$

Consider:

$$\begin{aligned} l^2 + x(1-x)q^2 &= (p+xq)^2 + xq^2 - x^2q^2 \\ &= p^2 + 2xpq + x^2q^2 + xq^2 - x^2q^2 \\ &= p^2 + 2xpq + xq^2 \end{aligned} \tag{F.1.50}$$

which is the root of the denominator of (F.1.47) and our integral becomes :

$$\int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{[l^2 + x(1-x)q^2 + i\epsilon)^2} \tag{F.1.51}$$

our integral now only depends on l^2 so we can take out the angular integrals as the volume of the three-sphere which we ignore and our integral becomes:

$$\int_0^1 dx \int_{xq}^{\Lambda+xq} \frac{dl}{(2\pi)^4} l^3 \frac{1}{[l^2 + x(1-x)q^2 + i\epsilon]^2} \tag{F.1.52}$$

let:

$$m^2 = l^2 + x(1-x)q^2 + i\epsilon \tag{F.1.53}$$

for the arbitary variable m which has nothing to do with mass, so:

$$2mdm = 2ldl \tag{F.1.54}$$

and:

$$l^2 = m^2 - x(1-x)q^2 - i\epsilon \tag{F.1.55}$$

So our integral becomes (F.1.51) becomes:

$$\int_0^1 dx \int_{\sqrt{xq^2}}^{\sqrt{\Lambda^2+2xq\Lambda+xq^2}} \frac{dm}{(2\pi)^4} m(m^2 - x(1-x)q^2 - i\epsilon)\frac{1}{m^4} \tag{F.1.56}$$

This is :

$$\int_0^1 dx \int_{\sqrt{xq^2}}^{\sqrt{\Lambda^2+2xq\Lambda+xq^2}} \frac{dm}{(2\pi)^4} [\frac{1}{m} - \frac{(x(1-x)q^2 - i\epsilon)}{m^3}] \tag{F.1.57}$$

which integrates to:

$$\int_0^1 dx \frac{1}{(2\pi)^4} [logm + \frac{1}{2}\frac{(x(1-x)q^2 - i\epsilon)}{m^2}]_{\sqrt{xq^2}}^{\sqrt{\Lambda^2+2xq\Lambda+xq^2}} \tag{F.1.58}$$

and then evaluating becomes:

$$\int_0^1 dx \frac{1}{(2\pi)^4} [\frac{1}{2} log[\Lambda^2 + 2xq\Lambda] + \frac{1}{2}\frac{(x(1-x)q^2 - i\epsilon)}{xq^2} - \frac{1}{2}\frac{(x(1-x)q^2 - i\epsilon)}{\Lambda^2 + 2xq\Lambda + xq^2}]$$

which is

$$\int_0^1 dx \frac{1}{(2\pi)^4}[\frac{1}{2}log[\Lambda^2 + 2xq\Lambda] + \frac{1}{2}(1-x) - \frac{1}{2}\frac{(x(1-x)q^2)}{\Lambda^2 + 2xq\Lambda + xq^2} \quad (F.1.59)$$

suppressing the epsilons as they are no longer relevant
Thus then only possible divergence is in the first term thus there is only one divergent terms ,since our x integral is finite for finite terms as we have no terms of the form x^{-n} n integers
we then have as stated only three possible divergences

F.2 $Cj^\mu(x)C^\dagger = -j^\mu(x)$

recall (5.2.24)

$$j^\mu(x) = \overline{\psi}\gamma^\mu\psi \quad (F.2.1)$$

In section 2.3 (2.3.30)we stated that C gave:

$$Ca^s_{\vec{p}}C = b^s_{\vec{p}} \qquad Cb^s_{\vec{p}}C = a^s_{\vec{p}} \quad (F.2.2)$$

We also know that (**??**)

$$\psi = \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \quad (F.2.3)$$

We earlier stated that the b creates the anti-particles of those particles created by a
We can interpret anti-particles as particles traveling backwards in time p23 l3-5 [7] . This is equivalent to applying the time reversal operator which reverses the momentum and the spin l32-33 p67 [9]
if we associate spin we a given axis with polar co-ordinates θ and ϕ Then we have :

$$\xi^s = (\xi(\uparrow), \xi(\downarrow)) \quad (F.2.4)$$

where the different spins up and down correspond to the different component s of the spinor
we have following p68 l3 [9]

$$\xi(\uparrow) = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} \qquad \xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \quad (F.2.5)$$

The flipped spin version of (F.2.4) is :

$$\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow)) \quad (F.2.6)$$

Thus as we know b creates a positron whose spinor v contains the inversed spin spinor ϵ^{-s}
Recall (2.3.17):

$$v(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta \\ -\sqrt{p.\overline{\sigma}}\eta \end{pmatrix} \tag{F.2.7}$$

This becomes :

$$v(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^{-s} \\ -\sqrt{p.\overline{\sigma}}\xi^{-s} \end{pmatrix} \tag{F.2.8}$$

We can show that :

$$\epsilon^{-s} = -i\sigma^2(\xi^s)^* \tag{F.2.9}$$

Conjugate (F.2.5)

$$\xi(\uparrow)^* = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{-i\phi}sin\frac{\theta}{2} \end{pmatrix} \qquad \xi(\uparrow)^* = \begin{pmatrix} -e^{i\phi}sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \tag{F.2.10}$$

as the trigonometric functions are real for real angles Recall (1.3.14) so that:

$$-i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{F.2.11}$$

Hence applying to the two parts of (F.2.10)

$$\begin{aligned} -i\sigma^2\xi(\uparrow)^* &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2} \\ e^{-i\phi}sin\frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} -e^{-i\phi}sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \\ &= \xi(\downarrow) \end{aligned} \tag{F.2.12}$$

$$\begin{aligned} -i\sigma^2\xi(\downarrow)^* &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -e^{i\phi}sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} -cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} \\ &= -\xi(\uparrow) \end{aligned} \tag{F.2.13}$$

We can now write using (2.3.17) and (**??**)

$$v(p) = \begin{pmatrix} \sqrt{p.\sigma}(-i\sigma^2\xi^{s*}) \\ -\sqrt{p.\overline{\sigma}}(-i\sigma^2\xi^{s*}) \end{pmatrix} \tag{F.2.14}$$

We now wish to show :

$$\sqrt{p.\sigma}\sigma^2 = \sigma^2\sqrt{p.\overline{\sigma}^*} \tag{F.2.15}$$

and :

$$\sqrt{p.\overline{\sigma}}\sigma^2 = \sigma^2\sqrt{p.\sigma^*} \tag{F.2.16}$$

We can show that for a general matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{F.2.17}$$

its positive square roots are given by i.e. $C^2 = A$ the negative square roots are -C:

$$C^{\pm} = \frac{1}{\sqrt{a+d\pm 2\Delta}}\begin{pmatrix} a\pm\Delta & b \\ c & d\pm\Delta \end{pmatrix} \tag{F.2.18}$$

where $\Delta = \sqrt{ad-bc}$
prove by substitution:

$$\begin{aligned}
C^{\pm 2} &= \frac{1}{\sqrt{a+d\pm 2\Delta}}\begin{pmatrix} a\pm\Delta & b \\ c & d\pm\Delta \end{pmatrix}\frac{1}{\sqrt{a+d\pm 2\Delta}}\begin{pmatrix} a\pm\Delta & b \\ c & d\pm\Delta \end{pmatrix} \\
&= \frac{1}{a+d\pm 2\Delta}\begin{pmatrix} (a\pm\Delta)^2+bc & ab+bd\pm 2\Delta b \\ ca+cd\pm 2\Delta c & (d\pm\Delta)^2+bc \end{pmatrix} \\
C^{\pm 2} &= \frac{1}{a+d\pm 2\Delta}\begin{pmatrix} a^2+ad-bc\pm 2a\Delta+bc & b(a+d\pm 2\Delta) \\ c(a+d\pm 2\Delta) & d^2+ad-bc\pm 2d\Delta+bc \end{pmatrix} \\
&= \frac{1}{a+d\pm 2\Delta}\begin{pmatrix} a(a+d\pm 2a\Delta) & b(a+d\pm 2\Delta) \\ c(a+d\pm 2\Delta) & d(d+a\pm 2d\Delta) \end{pmatrix}
\end{aligned} \tag{F.2.19}$$

So $C^{\pm 2} = A$ as required

Consider using (2.3.15)

$$\begin{aligned}
p.\sigma &= p^0 - p^1\sigma^1 - p^2\sigma^2 - p^3\sigma^3 \\
&= p^0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - p^1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - p^2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - p^3\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{F.2.20}$$

using the pauli spin matrices (1.3.14)
Thus recombining

$$p.\sigma = \begin{pmatrix} p^0-p^3 & -p^1+ip^2 \\ -p^1-ip^2 & p^0+p^3 \end{pmatrix} \tag{F.2.21}$$

and

$$det(p.\sigma) = (p^0)^2-(p^3)^2-(p^1)^2-(p^2)^2 = p^2 \tag{F.2.22}$$

So considering (F.2.18)

$$\sqrt{p.\sigma} = \frac{1}{2p^0\pm p}\begin{pmatrix} p^0-p^3\pm p & -p^1+ip^2 \\ -p^1-ip^2 & p^0+p^3\pm p \end{pmatrix} \tag{F.2.23}$$

Consider using (2.3.15)

$$p.\overline{\sigma}^* = p^0 + p^1\sigma^{1*} + p^2\sigma^{2*} + p^3\sigma^{3*} \tag{F.2.24}$$

$\sigma^{1*} = \sigma^1$ $\sigma^{2*} = \sigma^2$ as σ^1, σ^2 are real
$\sigma^{2*} = -\sigma^2$ as σ^2 is complex

So:

$$\begin{aligned} p.\overline{\sigma}^* &= p^0 + p^1\sigma^1 - p^2\sigma^2 + p^3\sigma^3 \qquad\qquad (F.2.25) \\ &= p^0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + p^1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - p^2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p^3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{aligned}$$

using the pauli spin matrices (1.3.14) Recombining:

$$p.\overline{\sigma}^* = \begin{pmatrix}p^0 + p^3 & p^1 + ip^2\\ p^1 - ip^2 & p^0 - p^3\end{pmatrix} \tag{F.2.26}$$

with

$$det(p.\overline{\sigma}^*) = (p^0)^2 - (p^3)^2 - (p^1)^2 - (p^2)^2 = p^2 \tag{F.2.27}$$

So considering (F.2.18)

$$\sqrt{p.\overline{\sigma}^*} = \frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 + p^3 \pm p & p^1 + ip^2\\ p^1 - ip^2 & p^0 - p^3 \pm p\end{pmatrix} \tag{F.2.28}$$

Using (F.2.23)

$$\begin{aligned} \sqrt{p.\sigma}\sigma^2 &= \frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 - p^3 \pm p & -p^1 + ip^2\\ -p^1 - ip^2 & p^0 + p^3 \pm p\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\ &= \frac{1}{2p^0 \pm p}\begin{pmatrix}-ip^1 - p^2 & -ip^0 + ip^3 \mp ip\\ ip^0 + ip^3 \pm ip & ip^1 - p^2\end{pmatrix} \qquad (F.2.29) \end{aligned}$$

Using (F.2.28)

$$\begin{aligned} \sigma^2\sqrt{p.\overline{\sigma}^*} &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 + p^3 \pm p & p^1 + ip^2\\ p^1 - ip^2 & p^0 - p^3 \pm p\end{pmatrix} \\ &= \frac{1}{2p^0 \pm p}\begin{pmatrix}-ip^1 - p^2 & -ip^0 + ip^3 \mp ip\\ ip^0 + ip^3 \pm ip & ip^1 - p^2\end{pmatrix} \qquad (F.2.30) \end{aligned}$$

So comparing (F.2.29) , and (F.2.30)

$$\sigma^2\sqrt{p.\overline{\sigma}^*} = \sigma^2\sqrt{p.\overline{\sigma}^*} \tag{F.2.31}$$

Recall (2.3.15)

$$\begin{aligned} p.\overline{\sigma} &= p^0 + p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3 && \text{(F.2.32)} \\ &= p^0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + p^1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p^2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p^3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{aligned}$$

using the pauli spin matrices (1.3.14)
Recombining:

$$p.\overline{\sigma} = \begin{pmatrix}p^0 + p^3 & p^1 - ip^2\\ p^1 + ip^2 & p^0 - p^3\end{pmatrix} \tag{F.2.33}$$

with

$$det(p.\overline{\sigma}) = (p^0)^2 - (p^3)^2 - (p^1)^2 - (p^2)^2 = p^2 \tag{F.2.34}$$

So recalling (F.2.18)

$$\sqrt{p.\overline{\sigma}} = \frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 + p^3 \pm p & p^1 - ip^2\\ p^1 + ip^2 & p^0 - p^3 \pm p\end{pmatrix} \tag{F.2.35}$$

Recall (2.3.15)

$$p.\sigma^* = p^0 - p^1\sigma^{1*} - p^2\sigma^{2*} - p^3\sigma^{3*} \tag{F.2.36}$$

$\sigma^{1*} = \sigma^1$ $\sigma^{2*} = \sigma^2$ as σ^1, σ^2 are real
$\sigma^{2*} = -\sigma^2$ as σ^2 is complex
So:

$$\begin{aligned} p.\sigma^* &= p^0 - p^1\sigma^1 + p^2\sigma^2 - p^3\sigma^3 && \text{(F.2.37)} \\ &= p^0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - p^1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p^2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - p^3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{aligned}$$

using the pauli spin matrices (1.3.14)
recombining:

$$p.\sigma^* = \begin{pmatrix}p^0 - p^3 & -p^1 - ip^2\\ -p^1 + ip^2 & p^0 + p^3\end{pmatrix} \tag{F.2.38}$$

with

$$det(p.\sigma^*) = (p^0)^2 - (p^3)^2 - (p^1)^2 - (p^2)^2 = p^2 \tag{F.2.39}$$

Recall (F.2.18)

$$\sqrt{p.\sigma^*} = \frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 - p^3 \pm p & -p^1 - ip^2\\ -p^1 + ip^2 & p^0 + p^3 \pm p\end{pmatrix} \tag{F.2.40}$$

Recall (F.2.35)

$$\begin{aligned} \sqrt{p.\overline{\sigma}}\sigma^2 &= \frac{1}{2p^0 \pm p}\begin{pmatrix}p^0 + p^3 \pm p & p^1 - ip^2\\ p^1 + ip^2 & p^0 - p^3 \pm p\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\ &= \frac{1}{2p^0 \pm p}\begin{pmatrix}ip^1 + p^2 & -ip^0 - ip^3 \mp ip\\ ip^0 - ip^3 \pm ip & -ip^1 + p^2\end{pmatrix} && \text{(F.2.41)} \end{aligned}$$

Recall (F.2.40)

$$\begin{aligned}
\sigma^2\sqrt{p.\sigma^*} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{2p^0 \pm p} \begin{pmatrix} p^0 - p^3 \pm p & -p^1 - ip^2 \\ -p^1 + ip^2 & p^0 + p^3 \pm p \end{pmatrix} \\
&= \frac{1}{2p^0 \pm p} \begin{pmatrix} ip^1 + p^2 & -ip^0 - ip^3 \mp ip \\ ip^0 - ip^3 \pm ip & -ip^1 + p^2 \end{pmatrix}
\end{aligned} \tag{F.2.42}$$

So comparing (F.2.42) and (F.2.41) :

$$\sigma^2\sqrt{p.\sigma^*} = \sqrt{p.\overline{\sigma}}\sigma^2 \tag{F.2.43}$$

Now recall (F.2.14)

$$v(p) = \begin{pmatrix} \sqrt{p.\sigma}(-i\sigma^2\xi^{s*}) \\ -\sqrt{p.\overline{\sigma}}(-i\sigma^2\xi^{s*}) \end{pmatrix} \tag{F.2.44}$$

which becomes recalling (F.2.15) , (F.2.16):

$$v(p) = \begin{pmatrix} -i\sigma^2\sqrt{p.\overline{\sigma}^*}\xi^{s*} \\ i\sigma^2\sqrt{p.\sigma^*}\xi^{s*} \end{pmatrix} \tag{F.2.45}$$

Thus :

$$[v(p)]^* = [\begin{pmatrix} -i\sigma^2\sqrt{p.\overline{\sigma}^*}\xi^{s*} \\ i\sigma^2\sqrt{p.\sigma^*}\xi^{s*} \end{pmatrix}]^* = \begin{pmatrix} i\sigma^{2*}\sqrt{p.\overline{\sigma}}\xi^{s} \\ -i\sigma^{2*}\sqrt{p.\sigma}\xi^{s} \end{pmatrix} \tag{F.2.46}$$

$\sigma^{2*} = -\sigma^2$ as σ^2 is imaginary giving:

$$[v(p)]^* = \begin{pmatrix} -i\sigma^2\sqrt{p.\overline{\sigma}}\xi^{s} \\ i\sigma^2\sqrt{p.\sigma}\xi^{s} \end{pmatrix} \tag{F.2.47}$$

We can write this as:

$$[v(p)]^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p.\sigma}\xi^{s} \\ \sqrt{p.\overline{\sigma}}\xi^{s} \end{pmatrix} \tag{F.2.48}$$

remembering from (**??**)

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \tag{F.2.49}$$

So recalling the definition of u (2.3.16)

$$[v(p)]^* = -i\gamma^2 u^s(p) \tag{F.2.50}$$

taking the conjugate we get:

$$v(p) = i\gamma^{2*}[u^s(p)]^* \tag{F.2.51}$$

$$\gamma^{2*} = \begin{pmatrix} 0 & \sigma^{2*} \\ -\sigma^{2*} & 0 \end{pmatrix} \tag{F.2.52}$$

$\sigma^{2*} = -\sigma^2$ as σ^2 is imaginary
Thus :

$$\gamma^{2*} = -\gamma^2 \tag{F.2.53}$$

Thus (F.2.51):

$$v(p) = -i\gamma^2[u^s(p)]^* \tag{F.2.54}$$

taking (F.2.51) multiplying each side by i we get:

$$i[v(p)]^* = \gamma^2 u^s(p) \tag{F.2.55}$$

using (2.3.4)

$$\gamma^2\gamma^2 = -I_4 \tag{F.2.56}$$

So:

$$i\gamma^2[v(p)]^* = -u^s(p) \tag{F.2.57}$$

i.e.

$$-i\gamma^2[v(p)]^* = u^s(p) \tag{F.2.58}$$

recalling (**??**) using (F.2.51) , (F.2.58)

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \tag{F.2.59}$$

So using (F.2.51) , (F.2.58)

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}(-i\gamma^2[v(p)]^*)e^{-ipx} + b^{s\dagger}_{\vec{p}}(-i\gamma^2)[u^s(p)]^*e^{ipx}) \tag{F.2.60}$$

So using the C 's on the a and b we get i.e. (2.3.30)

$$C\psi C = \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(b^s_{\vec{p}}(-i\gamma^2[v(p)]^*)e^{-ipx} + a^{s\dagger}_{\vec{p}}(-i\gamma^2)[u^s(p)]^*e^{ipx}) \tag{F.2.61}$$

which is :

$$-i\gamma^2\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(b^s_{\vec{p}}([v(p)]^*)e^{-ipx} + a^{s\dagger}_{\vec{p}}[u^s(p)]^*e^{ipx}) \tag{F.2.62}$$

which is considering (**??**)

$$-i\gamma^2\psi^* \tag{F.2.63}$$

$$-i\gamma^2\psi^* = -i\gamma^2(\psi^\dagger)^T \tag{F.2.64}$$

We know following Appendix B15 for the Dirac conjugate momentum that:

$$\overline{\psi}\gamma^0 = \psi^\dagger \tag{F.2.65}$$

Consider:

$$\begin{aligned}(\overline{\psi}\gamma^0\gamma^2)^T &= (\psi^\dagger\gamma^2)^T \\ &= \gamma^{2T}(\psi^\dagger)^T\end{aligned} \tag{F.2.66}$$

Recall (2.3.2) as γ^2 is purely imaginary :

$$\gamma^{2T} = -\gamma^{2\dagger} \tag{F.2.67}$$

Recall from (**??**)

$$\gamma^2 = -\gamma^{2\dagger} \tag{F.2.68}$$

So

$$\gamma^2 = \gamma^{2T} \tag{F.2.69}$$

recall (F.2.66)

$$-i\gamma^2(\psi^\dagger)^T = -i(\overline{\psi}\gamma^0\gamma^2)^T \tag{F.2.70}$$

and so recalling (F.2.62)

$$C\psi C = -i(\overline{\psi}\gamma^0\gamma^2)^T \tag{F.2.71}$$

Consider

$$C\overline{\psi}C = C\psi^\dagger C\gamma^0 \tag{F.2.72}$$

We recall (**??**)

$$\psi = \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^s_{\vec{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ipx}) \tag{F.2.73}$$

So:

$$\psi^\dagger = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^{s\dagger}_{\vec{p}}u^{s\dagger}(p)e^{ipx} + b^s_{\vec{p}}v^{s\dagger}(p)e^{-ipx}) \tag{F.2.74}$$

Recalling (F.2.50)

$$[v(p)]^* = -i\gamma^2u^s(p) \tag{F.2.75}$$

transposing we get :

$$\begin{aligned}[v(p)]^\dagger &= -i(\gamma^2u^s(p))^T \\ &= -iu^{sT}(p)\gamma^{2T}\end{aligned} \tag{F.2.76}$$

using (**??**)

$$[v(p)]^\dagger = -iu^{sT}(p)\gamma^2 \tag{F.2.77}$$

using from above (F.2.58)

$$-i\gamma^2[v(p)]^* = u^s(p) \tag{F.2.78}$$

transposing we get:

$$\begin{aligned} u^{s\dagger}(p) &= (-i\gamma^2[v(p)]^*)^\dagger \\ &= i[v(p)]^T\gamma^{2\dagger} \end{aligned} \tag{F.2.79}$$

Recall (**??**)

$$u^{s\dagger}(p) = -i[v(p)]^T\gamma^2 \tag{F.2.80}$$

Thus using (F.2.80) , (F.2.77) in (F.2.74) we get:

$$\psi^\dagger = -i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(a^{s\dagger}_{\vec{p}}v^{sT}(p)e^{ipx} + b^s_{\vec{p}}u^{sT}(p)e^{-ipx})\gamma^2 \tag{F.2.81}$$

Using the C on the creation operators as in (2.3.30) we get:

$$C\psi^\dagger C = -i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}(b^{s\dagger}_{\vec{p}}v^{sT}(p)e^{ipx} + a^s_{\vec{p}}v^{sT}(p)e^{-ipx})\gamma^2 \tag{F.2.82}$$

which is recalling (**??**)

$$-i(\psi^\dagger)^*\gamma^2 \tag{F.2.83}$$

which is

$$-i\psi^T\gamma^2 \tag{F.2.84}$$

This can be written:

$$-i(\gamma^{2T}\psi^T)^T \tag{F.2.85}$$

and as $\gamma^{2T} = \gamma^2$ (F.2.69) is :

$$(-i\gamma^2\psi)^T \tag{F.2.86}$$

so :

$$\begin{aligned} C\overline{\psi}C &= C\psi^\dagger C\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 \\ &= (-i\gamma^{0T}\gamma^2\psi)^T \end{aligned} \tag{F.2.87}$$

Recall (2.3.1)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{F.2.88}$$

So:

$$\gamma^{0T} = \gamma^0 \tag{F.2.89}$$

Thus (F.2.87) becomes:

$$C\overline{\psi}C = (-i\gamma^0\gamma^2\psi)^T \tag{F.2.90}$$

Recall from (5.2.24) that:

$$Cj^\mu(x)C = C\overline{\psi}\gamma^\mu\psi C \tag{F.2.91}$$

$$CC = I \tag{F.2.92}$$

as charge conjugation twice on the a operators takes a to b and then back to a and similarly takes b to a to b
Thus:

$$Cj^\mu(x)C = C\overline{\psi}C\gamma^\mu C\psi C \tag{F.2.93}$$

Recalling (F.2.90) and (F.2.71) this becomes:

$$\begin{aligned} Cj^\mu(x)C &= (-i\gamma^0\gamma^2\psi)^T\gamma^\mu[-i(\overline{\psi}\gamma^0\gamma^2)^T] \\ &= -(\gamma^0\gamma^2\psi)^T\gamma^\mu(\overline{\psi}\gamma^0\gamma^2)^T \end{aligned} \tag{F.2.94}$$

following equation p70 eq 3.147 [9] we can write with indices as :

$$Cj^\mu(x)C = -\gamma^0_{ab}\gamma^2_{bc}\psi_c\gamma^\mu\overline{\psi}_d\gamma^0_{de}\gamma^2_{ea} \tag{F.2.95}$$

re-ordering and using the anti-commutation of fermions to swap the order of ψ_c and $\overline{\psi}_d$ we get :

$$Cj^\mu(x)C = \overline{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^\mu\gamma^0_{ab}\gamma^2_{bc}\psi_c \tag{F.2.96}$$

as γ^2 and γ^0 are chosen to anti-commute recall (2.3.4):

$$Cj^\mu(x)C = -\overline{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^\mu\gamma^2_{bc}\gamma^0_{ab}\psi_c \tag{F.2.97}$$

(2.3.4) also gives:
γ^2_{bc} anti-commutes with all the γ^μ except $\mu = 2$
with which it commutes
γ^0_{ab} anti-commutes with all the γ^μ except $\mu = 0$
with which it commutes
Recall that as all matrices are real except γ^2 see (2.3.2) and (2.3.1) (**??**)

$$\gamma^\mu = -\gamma^{\mu\dagger} \tag{F.2.98}$$

implies

$$\gamma^{\mu} = -\gamma^{\mu T} \;\; \mu \neq 2 \tag{F.2.99}$$

Thus as our matrices commute between the gamma matrices and their transposes recalling also (F.2.69) we find:
commuting for our symmetric matrices γ^0, γ^2 therefore remains the same
commuting for our anti-symmetric matrices γ^1, γ^3 therefore picks up an extra minus sign
all our terms pick up a minus sign so (F.2.97) becomes:

$$Cj^{\mu}(x)C = \overline{\psi}_d \gamma^0_{de} \gamma^2_{ea} \gamma^2_{bc} \gamma^0_{ab} \gamma^{\mu} \psi_c \tag{F.2.100}$$

which is :

$$Cj^{\mu}(x)C = \overline{\psi} \gamma^0 \gamma^2 \gamma^2 \gamma^0 \gamma^{\mu} \psi \tag{F.2.101}$$

(2.3.4) implies:

$$(\gamma^2)^2 = -I \tag{F.2.102}$$

$$(\gamma^0)^2 = I \tag{F.2.103}$$

meaning:

$$Cj^{\mu}(x)C = -\overline{\psi} \gamma^{\mu} \psi \tag{F.2.104}$$

F.3 $\frac{\partial}{\delta \phi_{cl}(x)} \Gamma[\phi_{cl}] = -J(x)$

Recall: (6.4.5)

$$\Gamma[\phi_{cl}] = -E[J] - \int d^4y J(y) \phi_{cl}(y) \tag{F.3.1}$$

So:

$$\frac{\delta}{\delta \phi_{cl}(x)} \Gamma[\phi_{cl}] = -\frac{\delta}{\delta \phi_{cl}(x)} E[J] - \int d^4y \frac{\delta J(y)}{\delta \phi_{cl}(x)} \phi_{cl}(y) - J(x) \tag{F.3.2}$$

changing the first differential from ϕ to J:

$$\frac{\delta}{\delta \phi_{cl}(x)} \Gamma[\phi_{cl}] = -\int d^4y \frac{\delta J(y)}{\delta \phi_{cl}(x)} \frac{\delta E[J]}{\delta J(y)} - \int d^4y \frac{\delta J(y)}{\delta \phi_{cl}(x)} \phi_{cl}(y) - J(x) \tag{F.3.3}$$

We know (6.4.4)

$$\phi_{cl}(x) = \langle \Omega \mid \phi(x) \mid \Omega \rangle_J \tag{F.3.4}$$

Recall (6.4.3)

$$\begin{aligned} \frac{\delta}{\delta J(x)} E[J] &= -\langle \Omega \mid \phi(x) \mid \Omega \rangle_J \\ &= -\phi_{cl}(x) \end{aligned} \tag{F.3.5}$$

Thus (F.3.3) becomes:

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = -\int d^4y \frac{\delta J(y)}{\delta\phi_{cl}(x)}(-\phi_{cl}(x)) - \int d^4y \frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) - J(x) \tag{F.3.6}$$

So the first two terms cancel and we get (6.4.6):

$$\frac{\partial}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = -J(x) \tag{F.3.7}$$

Appendix G

Appendix G: Chapter 7

G.1 Simplification of the Lagrangian

Recall (7.1.19) , (7.1.21) , (7.1.23) , (7.1.20)

$$\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^\dagger + \mu^2 \Phi \Phi^\dagger - \lambda(\Phi\Phi^\dagger)^2 \tag{G.1.1}$$

$$\partial_\mu \Phi \partial^\mu \Phi^\dagger = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1^\dagger + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2^\dagger \tag{G.1.2}$$

$$(\Phi\Phi^\dagger) = \frac{{\phi_1'}^2 + 2v\phi_1' + v^2 + {\phi_2'}^2}{2} \tag{G.1.3}$$

$$\begin{aligned}(\Phi\Phi^\dagger)^2 &= \frac{{\phi_1'}^4 + 4v{\phi_1'}^3 + 2v^2{\phi_1'}^2 + 2{\phi_1'}^2{\phi_2'}^2 + 4v^2{\phi_1'}^2}{4} \\ &+ \frac{4v^3\phi_1' + 4v\phi_1'{\phi_2'}^2 + v^4 + 2v^2{\phi_2'}^2 + {\phi_2'}^4}{4}\end{aligned} \tag{G.1.4}$$

with $v = \sqrt{\frac{\mu^2}{\lambda}}$
eliminating v

$$\mu^2(\Phi\Phi^\dagger) = \mu^2\frac{{\phi_1'}^2}{2} + \frac{\mu^3\phi_1'}{\lambda^{\frac{1}{2}}} + \frac{\mu^4}{2\lambda} + \frac{{\phi_2'}^2}{2} \tag{G.1.5}$$

using (7.1.23) we get:

$$\begin{aligned}\lambda(\Phi\Phi^\dagger)^2 &= \frac{\lambda{\phi_1'}^4}{4} + \lambda^{\frac{1}{2}}\mu{\phi_1'}^3 + \frac{\mu^2{\phi_1'}^2}{2} + \frac{\lambda{\phi_1'}^2{\phi_2'}^2}{2} \\ &+ \mu^2{\phi_1'}^2 + \frac{\mu^3\phi_1'}{\lambda^{\frac{1}{2}}} + \mu^2\lambda^{\frac{1}{2}}\phi_1'{\phi_2'}^2 + \frac{\mu^4}{4\lambda} + \frac{\mu^2{\phi_2'}^2}{2} + \frac{\lambda{\phi_2'}^4}{4}\end{aligned} \tag{G.1.6}$$

Thus combining these :

$$\begin{aligned}\mu^2(\Phi\Phi^\dagger) - \lambda(\Phi\Phi^\dagger)^2 &= \frac{\mu^4}{4\lambda} - \mu^2{\phi_1'}^2 - \frac{\lambda{\phi_1'}^4}{4} - \mu^2\lambda^{\frac{1}{2}}\phi_1'{\phi_2'}^2 \\ &- \lambda^{\frac{1}{2}}\mu{\phi_1'}^3 - \frac{\lambda{\phi_1'}^2{\phi_2'}^2}{2} - \frac{\lambda{\phi_2'}^4}{4}\end{aligned} \tag{G.1.7}$$

Thus using (7.1.19):

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1^\dagger + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2^\dagger + \frac{\mu^4}{4\lambda} - \mu^2{\phi_1'}^2 - \frac{\lambda{\phi_1'}^4}{4} - \mu^2\lambda^{\frac{1}{2}}\phi_1'{\phi_2'}^2 \\ &- \lambda^{\frac{1}{2}}\mu{\phi_1'}^3 - \frac{\lambda{\phi_1'}^2{\phi_2'}^2}{2} - \frac{\lambda{\phi_2'}^4}{4} \qquad \text{(G.1.8)}\end{aligned}$$

G.2 $\delta(k^2) = \frac{\delta(|\vec{k}|-k_0)+\delta(|\vec{k}|+k_0)}{2|\vec{k}|}$

writing our definition of the delta function(1.2.11)

$$\int \delta(u)f(u)du = f(0) \qquad \text{(G.2.1)}$$

and chose :

$$u = g(x) \qquad \text{(G.2.2)}$$

we get :

$$\int \delta(g(x))f(g(x)) \mid g'(x) \mid dx = f(0) \qquad \text{(G.2.3)}$$

Thus if the zero's of g are at x_i and $g'(x_i) \neq 0$
we can define :

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{\mid g'(x_i)} \qquad \text{(G.2.4)}$$

which gives us the appropriate result (G.2.3) Consider (7.2.24):

$$\delta(k^2) = \delta((k^0)^2 - \mid \vec{k} \mid^2) \qquad \text{(G.2.5)}$$

This has two solutions :

$$k_0 = \mid \vec{k} \mid \qquad \text{(G.2.6)}$$

and

$$k_0 = - \mid \vec{k} \mid \qquad \text{(G.2.7)}$$

as we integrating over k_0

$$(k^2)' = 2k_0 \qquad \text{(G.2.8)}$$

so we have using (G.2.4) :

$$\delta(k^2) = \frac{\delta(k_0 + \mid \vec{k} \mid) + \delta(k_0 - \mid \vec{k} \mid)}{\mid 2k_0 \mid} \qquad \text{(G.2.9)}$$

evaluating at $k_0 = \pm \mid \vec{k} \mid$ we get :

$$\delta(k^2) = \frac{\delta(k_0 + \mid \vec{k} \mid) + \delta(k_0 - \mid \vec{k} \mid)}{2 \mid \vec{k} \mid} \qquad \text{(G.2.10)}$$

G.3 $c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$ violates time symmetry

We consider only time symmetry
in matrix form recall (**??**):

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{-E_x}{c} & 0 & -B_z & B_y \\ \frac{-E_y}{c} & B_z & 0 & B_x \\ \frac{-E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{G.3.1}$$

Thus:

$$\begin{aligned} F_{\alpha\beta}F_{\nu\mu} &= \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{-E_x}{c} & 0 & -B_z & B_y \\ \frac{-E_y}{c} & B_z & 0 & B_x \\ \frac{-E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{-E_x}{c} & 0 & -B_z & B_y \\ \frac{-E_y}{c} & B_z & 0 & B_x \\ \frac{-E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{(E_x^2+E_y^2+E_z^2)}{c^2} & \frac{E_yB_z}{c} - \frac{E_zB_y}{c} & -\frac{E_xB_z}{c} + \frac{E_zB_x}{c} & \frac{E_xB_y}{c} - \frac{E_yB_x}{c} \\ \frac{B_zE_y}{c} - \frac{B_yE_z}{c} & -\frac{E_x^2}{c^2} - B_z^2 - B_y^2 & -\frac{E_xE_y}{c^2} + B_yB_x & -\frac{E_xE_z}{c^2} + B_zB_x \\ \frac{B_yE_x}{c} - \frac{B_xE_y}{c} & -\frac{E_zE_x}{c^2} + B_xB_z & -\frac{E_zE_y}{c^2} + B_yB_z & \frac{E_z^2}{c^2} - B_y^2 - B_x^2 \end{pmatrix} \end{aligned} \tag{G.3.2}$$

assuming the B and E fields are time symmetric then our alterations just occur in the top row and left column as these where the only ones we used the time co-ordinates in see Appendix A11 Thus in our time transfered co-ordinates we have (the time transfer means time components pick up a minus sign)

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{G.3.3}$$

and then

$$F_{\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{G.3.4}$$

which means :

$$F_{\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} -\frac{(E_x^2+E_y^2+E_z^2)}{c^2} & -\frac{E_yB_z}{c} + \frac{E_zB_y}{c} & \frac{E_xB_z}{c} - \frac{E_zB_x}{c} & -\frac{E_xB_y}{c} + \frac{E_yB_x}{c} \\ -\frac{B_zE_y}{c} + \frac{B_yE_z}{c} & -\frac{E_x^2}{c^2} - B_z^2 - B_y^2 & -\frac{E_xE_y}{c^2} + B_yB_x & -\frac{E_xE_z}{c^2} + B_zB_x \\ -\frac{B_yE_x}{c} + \frac{B_xE_y}{c} & -\frac{E_zE_x}{c^2} + B_xB_z & -\frac{E_zE_y}{c^2} + B_yB_z & \frac{E_z^2}{c^2} - B_y^2 - B_x^2 \end{pmatrix}$$

clearly this is not the same as the expression before the time change (G.3.2)

G.4 Dirac equation in Weyl matrix form

Recall the Dirac equation (**??**)

$$(i\partial\!\!\!/ - m)\psi \tag{G.4.1}$$

clearly $-m\psi$ is equivalent to :

$$\begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix} \psi \tag{G.4.2}$$

as -m is a constant Recall the Feynman slash (2.3.24) , (2.3.1) and (2.3.2):

$$\partial\!\!\!/ = \gamma^\mu \partial_\mu \tag{G.4.3}$$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{G.4.4}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{G.4.5}$$

so :

$$\partial\!\!\!/ = \begin{pmatrix} 0 & \partial_0 + \sigma^i \partial_i \\ \partial_0 - \sigma^i \partial_i & 0 \end{pmatrix} \tag{G.4.6}$$

Thus using (G.4.6) and (G.4.2)

$$i\partial\!\!\!/ - m = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}.\vec{\nabla} & -m \end{pmatrix} \tag{G.4.7}$$

as :

$$\vec{\nabla} = (\partial_1, \partial_2, \partial_3) \tag{G.4.8}$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^2) \tag{G.4.9}$$

writing (7.6.1)

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{G.4.10}$$

we gain using (G.4.7) :

$$(i\partial\!\!\!/ - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{G.4.11}$$

which is (7.6.2)

G.5 $\overline{\psi} i \not{\partial} \psi = \overline{\psi}_L i \not{\partial} \psi_L + \overline{\psi}_R i \not{\partial} \psi_R$

Recall (G.4.6) , (2.3.24),(7.6.1) , (2.3.1) , (2.3.10)

$$\not{\partial} = \begin{pmatrix} 0 & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & 0 \end{pmatrix} \tag{G.5.1}$$

$$\overline{\psi} = \psi^\dagger \gamma^0 \tag{G.5.2}$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{G.5.3}$$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{G.5.4}$$

Thus using (7.6.1) , (2.3.1) , (2.3.10) :

$$\overline{\psi} = \psi^\dagger \gamma^0 = (\psi_L^\dagger, \psi_R^\dagger) \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{G.5.5}$$

combining with (G.4.6) we get:

$$\begin{aligned} \overline{\psi} \not{\partial} &= (\psi_L^\dagger, \psi_R^\dagger) \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & 0 \end{pmatrix} \\ &= (\psi_L^\dagger, \psi_R^\dagger) \begin{pmatrix} i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & 0 \\ 0 & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \end{pmatrix} \end{aligned} \tag{G.5.6}$$

multiplying from the left by ψ gives:

$$\begin{aligned} \overline{\psi} i \not{\partial} \psi &= (\psi_L^\dagger, \psi_R^\dagger) \begin{pmatrix} i(\partial_0 - \vec{\sigma}.\vec{\nabla}) & 0 \\ 0 & i(\partial_0 + \vec{\sigma}.\vec{\nabla}) \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= (\psi_L^\dagger, \psi_R^\dagger) \begin{pmatrix} i(\partial_0 - \vec{\sigma}.\vec{\nabla})\psi_L \\ i(\partial_0 + \vec{\sigma}.\vec{\nabla})\psi_R \end{pmatrix} \\ &= \psi_L^\dagger i(\partial_0 - \vec{\sigma}.\vec{\nabla})\psi_L + \psi_R^\dagger i(\partial_0 + \vec{\sigma}.\vec{\nabla})\psi_R \end{aligned} \tag{G.5.7}$$

call

$$\psi_L = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \tag{G.5.8}$$

by exact analogue with (G.5.7) we get :

$$\overline{\psi}_L i \not{\partial} \psi_L = \psi_L^\dagger i(\partial_0 - \vec{\sigma}.\vec{\nabla})\psi_L \tag{G.5.9}$$

call :

$$\psi_R = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \tag{G.5.10}$$

by exact analogue we get :

$$\overline{\psi}_R i\partial\!\!\!/\psi_R = \psi_R^\dagger i(\partial_0 + \vec{\sigma}.\vec{\nabla})\psi_R \tag{G.5.11}$$

Thus comparing (G.5.9) and (G.5.11) with (G.5.7) we get (7.6.3):

$$\overline{\psi} i\partial\!\!\!/\psi = \overline{\psi}_L i\partial\!\!\!/\psi_L + \overline{\psi}_R i\partial\!\!\!/\psi_R \tag{G.5.12}$$

G.6 kinetic part of scalar lagrangian in component form

Recall (7.7.31)

$$\begin{aligned}
\Delta\mathcal{L} &= \frac{1}{2}(0,v)(gYa_\mu + g'T^i A_\mu{}^i)(gYa^\mu + g'T^i A^{\mu i})\begin{pmatrix}0\\v\end{pmatrix}\\
&= \frac{1}{2}(0,v)(gYa_\mu + \frac{g'}{2}(\sigma^1 A_\mu{}^1 + \sigma^2 A_\mu{}^2 + \sigma^3 A_\mu{}^3))\\
&\times\ (gYa^\mu + \frac{g'}{2}(\sigma^1 A^{\mu 1} + \sigma^2 A^{\mu 2} + \sigma^3 A^{\mu 3}))\begin{pmatrix}0\\v\end{pmatrix}\\
&= \frac{1}{2}v^2 g^2 Y^2 (a_\mu)^2 + \frac{1}{4}gYg'(a_\mu A^{\mu 1} + A_\mu{}^1 a^\mu)(0,v)\sigma^1\begin{pmatrix}0\\v\end{pmatrix}\\
&+\ \frac{1}{4}gYg'(a_\mu A^{\mu 2} + A_\mu{}^2 a^\mu)(0,v)\sigma^2\begin{pmatrix}0\\v\end{pmatrix} + \frac{1}{4}gYg'(a_\mu A^{\mu 3} + A_\mu{}^3 a^\mu)(0,v)\sigma^3\begin{pmatrix}0\\v\end{pmatrix}\\
&+\ \frac{1}{2}\frac{g'^2}{4}(A_\mu^1)^2(0,v)(\sigma^1)^2\begin{pmatrix}0\\v\end{pmatrix} + \frac{1}{2}\frac{g'^2}{4}(A_\mu^2)^2(0,v)(\sigma^2)^2\begin{pmatrix}0\\v\end{pmatrix}\\
&+\ \frac{1}{2}\frac{g'^2}{4}(A_\mu^3)^2(0,v)(\sigma^3)^2\begin{pmatrix}0\\v\end{pmatrix}
\end{aligned} \tag{G.6.1}$$

where (1.6.4) is used to define the squares. Recall (2.3.15) :

$$(\sigma^i)^2 = \hat{I} \tag{G.6.2}$$

using (1.3.14):

$$\sigma^1\begin{pmatrix}0\\v\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\v\end{pmatrix} = \begin{pmatrix}v\\0\end{pmatrix} \tag{G.6.3}$$

so:

$$(v,0)\sigma^1\begin{pmatrix}0\\v\end{pmatrix} = 0 \tag{G.6.4}$$

again using (1.3.14):

$$\sigma^2\begin{pmatrix}0\\v\end{pmatrix} = \begin{pmatrix}0&i\\-i&0\end{pmatrix}\begin{pmatrix}0\\v\end{pmatrix} = \begin{pmatrix}iv\\0\end{pmatrix} \tag{G.6.5}$$

so:

$$(v,0)\sigma^2 \begin{pmatrix} 0 \\ v \end{pmatrix} = 0 \tag{G.6.6}$$

using again (1.3.14):

$$\sigma^3 \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = - \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{G.6.7}$$

so:

$$(v,0)\sigma^3 \begin{pmatrix} 0 \\ v \end{pmatrix} = -v^2 \tag{G.6.8}$$

and using (2.3.15) ,(G.6.4), (G.6.5) , (G.6.7) (G.6.1) becomes:

$$\begin{aligned} \Delta\mathcal{L} &= \frac{1}{2}v^2 g^2 Y^2 (a_\mu)^2 - \frac{1}{4}v^2 gYg'(a_\mu A^{\mu 3} + A_\mu{}^3 a^\mu) \\ &+ \frac{1}{2}[\frac{g'^2}{4}v^2 (A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^3)^2] \end{aligned} \tag{G.6.9}$$

using the property of four-vectors (1.6.4)

$$a_\mu A^{\mu 3} = A_\mu{}^3 a^\mu \tag{G.6.10}$$

So (G.6.9) becomes:

$$\begin{aligned} \Delta\mathcal{L} &= \frac{1}{2}v^2 (g^2 Y^2 (a_\mu)^2 + \frac{g'^2}{4}(A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^3)^2) - gYg' a_\mu A^{\mu 3}) \\ &= \frac{1}{2}v^2 (\frac{g'^2}{4}((A_\mu^1)^2 + (A_\mu^2)^2) + (gYa_\mu - \frac{g'}{2}A^{\mu 3})^2) \end{aligned} \tag{G.6.11}$$

which is (7.7.32)

G.7 re-writing D_μ

Recall (7.7.3) , (7.7.36) , (7.7.37) , (7.7.33) , (7.7.48)

$$D_\mu = \partial_\mu - igYa_\mu - ig'T^i B_\mu^i \tag{G.7.1}$$

$$Z_\mu^0 = \frac{1}{\sqrt{g^2 + g'^2}}(g'B_\mu^3 - ga_\mu) \tag{G.7.2}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gB_\mu^3 + g'a_\mu) \tag{G.7.3}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(B_\mu^1 \mp iB_\mu^2) \tag{G.7.4}$$

$$T^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) \tag{G.7.5}$$

We want :

$$D_\mu = \partial_\mu - i\frac{g'}{\sqrt{2}}(W_\mu^+T^+ + W_\mu^-T^-) - i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g'^2T^3 - g^2Y) - i\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(T^3+Y) \quad \text{(G.7.6)}$$

comparing with (7.7.3) canceling the derivative terms as they are common to both and dividing by -i:

$$gYa_\mu + g'T^iB_\mu^i = \frac{g'}{\sqrt{2}}(W_\mu^+T^+ + W_\mu^-T^-) + \frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g'^2T^3 - g^2Y) + \frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(T^3+Y) \quad \text{(G.7.7)}$$

Recall (7.7.33) , (7.7.48):

$$\begin{aligned} W_\mu^\pm T^\pm &= \frac{1}{\sqrt{2}}(B_\mu^1 \mp iB_\mu^2)\frac{1}{2}(\sigma^1 \pm i\sigma^2) \\ &= \frac{1}{\sqrt{2}}[B_\mu^1T^1 + B_\mu^2T^2 \pm i(B_\mu^1T^2 - B_\mu^2T^1)] \end{aligned} \quad \text{(G.7.8)}$$

So :

$$(W_\mu^+T^+ + W_\mu^-T^-) = \sqrt{2}[B_\mu^1T^1 + B_\mu^2T^2] \quad \text{(G.7.9)}$$

Considering (7.7.36) :

$$\begin{aligned} \frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g'^2T^3 - g^2Y) &= \frac{1}{\sqrt{g^2+g'^2}}\frac{1}{\sqrt{g^2+g'^2}}(g'B_\mu^3 - ga_\mu)(g'^2T^3 - g^2Y) \\ &= \frac{1}{g^2+g'^2}g'^3B_\mu^3T^3 + \frac{1}{g^2+g'^2}g^3a_\mu Y \\ &- \frac{1}{g^2+g'^2}(gg'^2a_\mu T^3 + g^2g'B_\mu^3Y) \end{aligned} \quad \text{(G.7.10)}$$

Consider (7.7.37)

$$\begin{aligned} \frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(T^3+Y) &= \frac{gg'}{\sqrt{g^2+g'^2}}\frac{1}{\sqrt{g^2+g'^2}}(gB_\mu^3 + g'a_\mu)(T^3+Y) \\ &= \frac{1}{g^2+g'^2}g^2g'B_\mu^3T^3 + \frac{1}{g^2+g'^2}gg'^2a_\mu Y \\ &+ \frac{1}{g^2+g'^2}(g'^2ga_\mu T^3 + g'g^2B_\mu^3Y) \end{aligned} \quad \text{(G.7.11)}$$

So adding (G.7.10) and (G.7.11) we get:

$$\begin{aligned}
\frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g^2T^3-g'^2Y) \quad &+ \quad \frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(T^3+Y)=\frac{1}{g^2+g'^2}g'(g'^2+g^2)B^3_\mu \\
&+ \quad \frac{1}{g^2+g'^2}g'(g'^2+g^2)a_\mu \\
&= \quad g'B^3_\mu T^3+ga_\mu Y
\end{aligned} \tag{G.7.12}$$

Using this in (G.7.6) with(G.7.9) gives (7.7.3)
we know (7.7.47)

$$e=\frac{gg'}{\sqrt{g^2+g'^2}} \tag{G.7.13}$$

we defined

$$Q=T^3+Y \tag{G.7.14}$$

So (G.7.6) becomes:

$$\begin{aligned}
D_\mu \quad &= \quad \partial_\mu-i\frac{g'}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-)-i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g'^2T^3+g^2T^3-g^2Q) \\
&- \quad ieA_\mu Q
\end{aligned} \tag{G.7.15}$$

our coefficient of T^3Z_μ is

$$-\sqrt{g^2+g'^2}i \tag{G.7.16}$$

we defined (7.7.44)

$$cos\theta=\frac{g'}{\sqrt{g^2+g'^2}} \tag{G.7.17}$$

so our coefficient is :

$$\frac{g'}{cos\theta_W}i \tag{G.7.18}$$

we can re-write the T^3Z_μ term as:

$$-i\sqrt{g^2+g'^2}Z_\mu(T^3-\frac{g^2}{g^2+g'^2}Q) \tag{G.7.19}$$

we defined (7.7.45) :

$$sin\theta=\frac{g}{\sqrt{g^2+g'^2}} \tag{G.7.20}$$

so we can re-write the $Z_\mu T^3$ term (G.7.19) as:

$$-i\frac{g'}{cos\theta_W}(T^3-sin^\theta_W Q) \tag{G.7.21}$$

and thus (G.7.15) becomes:

$$D_\mu=\partial_\mu-i\frac{g'}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-)-i\frac{g'}{cos\theta_W}(T^3-sin^\theta_W Q)-ieA_\mu Q \tag{G.7.22}$$

which is (7.7.49)