Regularity Theory a Fourth Order PDE with Delta Right Hand Side

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Problem Formulation

We will consider the following PDE

$$\kappa \Delta^2 u - \sigma \Delta u = \delta_X \qquad \text{in} \qquad \Omega$$

where $\kappa > 0$, $\sigma \geq 0$ and $\Omega \subset \mathbb{R}^2$ is a bounded, Lipschitz domain.

The right hand side δ_X is the delta function centred at the point $X \in \Omega$.

We impose some appropriate homogeneous boundary conditions which will lead to the weak formulation.

Weak Formulation

We use integration by parts to derive a weak formulation and the possible homogeneous boundary conditions.

$$\begin{aligned}\int_\Omega (\kappa\Delta^2 u - \sigma\Delta u)v &= \int_\Omega -\kappa\nabla\Delta u\cdot\nabla v + \sigma\nabla u\cdot\nabla v\\ &+ \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial\nu}v + \frac{\partial u}{\partial\nu}v\\ &= \int_\Omega \kappa\Delta u\Delta v + \sigma\nabla u\cdot\nabla v\\ &+ \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial\nu}v + \frac{\partial u}{\partial\nu}v + \Delta u\frac{\partial v}{\partial\nu}\end{aligned}$$

Weak Formulation - Boundary Conditions

We thus pose the problem in $H^2(\Omega)$ and impose boundary conditions by considering the problem in a subspace $V \subset H^2(\Omega)$.

We take the first boundary condition to be $u|_{\partial\Omega} = 0$. The boundary integral now vanishes if we choose the second boundary condition as follows.

- Dirichlet boundary conditions: $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$
- Navier boundary conditions: $u = \Delta u = 0$ on $\partial\Omega$

Weak Formulation - Imposing Boundary Conditions

For Dirichlet boundary conditions we take the test space to be:

$$V = \left\{ v \in H^2(\Omega) \,\middle|\, v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} = H_0^2(\Omega).$$

For Navier boundary conditions we take the test space to be:

$$V = \left\{ v \in H^2(\Omega) \,\middle|\, v = 0 \text{ on } \partial\Omega \right\} = H^2 \cap H_0^1(\Omega).$$

Weak Formulation - Left Hand Side

We have now constructed the left hand side of the weak formulation:

$$a(u, v) := \int_\Omega \kappa \Delta u \Delta v + \sigma \nabla u \cdot \nabla v.$$

Notice that $a : V \times V \to \mathbb{R}$ is bilinear, bounded and coercive.

Weak Formulation - Right Hand Side

For the right hand side we follow the same process and 'integrate' the right side against a test function $v \in V$.

$$\int_\Omega \delta_X v = v(X)$$

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Weak Formulation - Right Hand Side

For the right hand side we follow the same process and 'integrate' the right side against a test function $v \in V$.

$$\int_\Omega \delta_X v = v(X)$$

- This doesn't make sense!
- We must interpret δ_X as an element of the dual space, V^*.

Weak Formulation - Right Hand Side

By the Sobolev embedding theorem we have $H^2(\Omega) \hookrightarrow C(\Omega)$.

We then interpret δ_X as $l \in V^*$ such that

$$l(v) = v(X).$$

Notice that $|l(v)| \leq \|v\|_{C(\Omega)} \leq C\|v\|_V$, so indeed $l \in V^*$.

Weak Form

We may thus state the weak form of our problem;

$$\text{Find } u \in V \text{ s.t. } a(u,v) = l(v) \quad \forall\, v \in V.$$

By the Lax-Milgram theorem we have the existence of a unique solution $u \in V$.

An Example - The Biharmonic Problem

To gain an idea of the regularity of solutions consider the problem

$$\Delta^2 u = \delta \text{ in } \Omega = B(0;1)$$

Beginning with Navier boundary conditions, we use the Green's function of the laplacian to construct a radial solution. The Green's function is given by:

$$\Phi(r) = \frac{1}{2\pi}\ln(r).$$

So we find $u(r)$ such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \Phi(r).$$

An Example - The Biharmonic Problem

We then conclude that u has the form

$$u(r) = \frac{r^2}{8\pi}\left[\ln(r) - 1\right] + a\ln(r) + b.$$

If $u \in H^2(\Omega)$ it is continuous and thus $a = 0$. We then enforce the zero boundary condition and so $b = 1/8\pi$.

Notice that $u \in C^{1,\gamma}(\Omega)$ for all $\gamma \in [0, 1)$. The same result holds for Dirichlet boundary conditions.

Regularity for General Problem

In fact the same regularity result holds for the general problem.

Theorem

Suppose Ω has C^3 boundary and $u \in V$ is the weak solution of

$$\kappa \Delta^2 u - \sigma \Delta u = \delta_X$$

Then $u \in W^{3,p}(\Omega)$ for all $p \in (1,2)$ and hence $u \in C^{1,\gamma}(\bar{\Omega})$ for all $\gamma \in [0,1)$.

Proof

Define $T : L^2(\Omega) \to C(\Omega)$ by $Tf = v_f$ such that

$$a(v_f, v) = (f, v)_{L^2(\Omega)} \, \forall v \in V \tag{1}$$

And thus define the adjoint operator $T^* : C(\Omega)^* \to L^2(\Omega)$. Hence, for any $f \in L^2(\Omega)$

$$\begin{aligned} (T^*(\delta_X), f)_{L^2(\Omega)} &= \delta_X[Tf] \\ &= v_f(X) \\ &= a(u, v_f) \\ &= (u, f)_{L^2(\Omega)} \end{aligned}$$

Thus $u = T^*(\delta_X)$.

Proof

Now let $\psi \in C_0^\infty(\Omega)$ and find $v \in H_0^1(\Omega)$ such that $\kappa \Delta v - \sigma v = \psi$. By elliptic regularity, v is smooth, we will only require $v \in H^4(\Omega)$.

$$
\begin{aligned}
\int_\Omega \Delta u \psi &= \int_\Omega u \Delta \psi \\
&= \int_\Omega u[\kappa \Delta^2 v - \sigma \Delta v] \\
&= (T^* \delta_X, \kappa \Delta^2 v - \sigma \Delta v) \\
&= \delta_X[v] \\
&= v(x)
\end{aligned}
$$

Proof

$$\implies \left| \int_\Omega \Delta u \psi \right| \leq \|v\|_\infty$$
$$\leq C(\Omega, p) \|\psi\|_{W^{1,p}(\Omega)^*} \text{ for } p \in (1,2)$$

Where $\psi \in W^{1,p}(\Omega)^*$ is given by

$$\psi[w] = \int_\Omega \psi w.$$

$C_0^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)^*$ thus $\Delta u \in W^{1,p}(\Omega)$.
By elliptic regularity $u \in W^{3,p}(\Omega)$, hence $u \in C^{1,\gamma}(\bar{\Omega})$.

Application - Biomembrane Deformation

If we consider an elastic membrane which is deformed by point forces we are lead to the energy functional:

$$\mathcal{E}(u, \mathbf{X}^{\pm}) = \int_{\Omega} \underbrace{\tfrac{\kappa}{2}|\Delta u|^2 + \tfrac{\sigma}{2}|\nabla u|^2}_{\text{Elastic Energy}} + \underbrace{\alpha \sum_{j=1}^{N^+} u(X_j^+) - \beta \sum_{j=1}^{N^-} u(X_j^-)}_{\text{Coupling Energy}}$$

- Energy minimisation produces our fourth order equation.
- Gradient flow produces particle movement $\dot{X}_j^{\pm} \propto \nabla u(X_j^{\pm})$.

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Further Work - Eighth Order Equations

Another model for biomembrane deformation considers the coupling energy:

$$\mathcal{E}_C(u, \mathbf{X}^{\pm}) = \alpha \sum_{j=1}^{N^+} \mathrm{D}u(X_j^+) - \beta \sum_{j=1}^{N^-} \mathrm{D}u(X_j^-).$$

We add higher order terms to the elastic energy and arrive at an equation of the form:

$$\kappa_8 \Delta^4 u - \kappa_6 \Delta^3 u + \kappa \Delta^2 u - \sigma \Delta u = \mathrm{D}\delta_X$$