1 Fundamentals

Vector identities for viscous flow

① $-\nu \nabla \times \underline{\omega} = -\nu \nabla \times (\nabla \times \underline{u})$
$= -\nu \nabla(\underbrace{\nabla \cdot \underline{u}}_{0}) + \nu\nabla^2 \underline{u}$

divergence free (incompressible)

GPK V2-1 More vector identities

a)

ii) $u_x = a x$
$u_y = -a y$

Vorticity $\underline{\omega} = \nabla \times \underline{u} = \det \begin{pmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{pmatrix}$

$= \underline{e_x}(\partial_y u_z - \partial_z u_y)$
$- \underline{e_y}(\partial_x u_z - \partial_z u_x)$
$+ \underline{e_z}(\partial_x u_y - \partial_y u_x)$

$= \underline{e_z}(\partial_x u_y - \partial_y u_x)$
$= (0,0,0)$

$\underline{u} \times \underline{\omega} = \begin{vmatrix} e_x & e_y & e_z \\ u_x & u_y & u_z \\ \omega_x & \omega_y & \omega_z \end{vmatrix} = (0,0,0)$

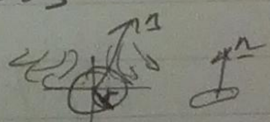
Cross product

$= \underline{e_x}(u_y w_z - u_z w_y)$
$- \underline{e_y}(u_x w_z - u_z w_x)$
$+ \underline{e_z}(u_x w_y - u_y w_x)$

$\nabla \cdot \underline{\omega} = \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} = 0$

Circulation $\Gamma = \oint_{C_1} \underline{u} \cdot d\underline{x} = \int_{S_1} \underline{\omega} \cdot \underline{n}\, dS = 0$

Line integral



2D Incompressible flow

1.1 $\underline{u} = (u(x,y,t), V(x,y,t), 0)$ incompressible

(a) Show $\underline{\omega}$ transverse to fluid motion plane:

$\underline{\omega} = \nabla \times \underline{u} = \underline{e_z}(\partial_x u_y - \partial_y u_x) = (0,0,\Omega)$

$\Omega = \partial_x u_y - \partial_y u_x$

$\Rightarrow \underline{\omega} \cdot \underline{u} = 0$

b) Show z-component of $\underline{\omega}$ is conserved along trajectories of fluid particles if $\nu = 0$ (inviscid).

Recall vorticity equation

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \underbrace{\cancel{\nu\nabla^2\underline{\omega}}}_{\nu=0}^{\neq 0}$$

$$(\underline{\omega}\cdot\nabla)\underline{u} = \left[(0,0,\Omega)\cdot(\partial_x,\partial_y,\partial_z)\right]\underline{u}$$

$$= \Omega(\partial_z u_x, \partial_z u_y, \partial_z u_z)$$

z-component : $\Omega\,\partial_z u_z = 0$ since $u_z = 0$.

So vortex stretching is zero, so equation in z-component is

$$\frac{\partial \Omega}{\partial t} + (\underline{u}\cdot\nabla)\Omega = \left(\frac{\partial}{\partial t} + (\underline{u}\cdot\nabla)\right)\Omega = 0$$

This means that Ω is conserved along fluid trajectories! - remember $\frac{\partial}{\partial t} + (\underline{u}\cdot\nabla) = D_t$ is the material ~~derivative~~ /Lagrangian derivative that follows the fluid.

1.2 <u>2D point vortex flow</u> Stationary, 2D, incomp flow has streamfunction $\psi(x,y)$ s.t. velocity field is

$$\underline{u} = (u,v) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$$

a) Find streamfunction for

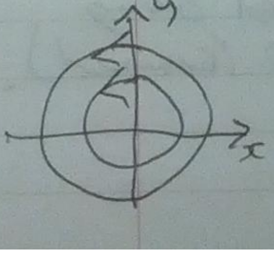
$$u = \frac{-y}{x^2+y^2}, \quad v = \frac{x}{x^2+y^2}$$

Sketch streamlines.

<u>Answer</u>

Solve $\frac{\partial\psi}{\partial y} = \frac{-y}{x^2+y^2}$ $\Rightarrow$ $\psi = -\frac{1}{2}\log(x^2+y^2)$

$-\frac{\partial\psi}{\partial x} = \frac{x}{x^2+y^2}$



Flow produced by a point vortex, ω concentrated at a point with +ve orientation that does not change with radius.

b) 2D inviscid fluid occupies region $x \geq 0, y \geq 0$ bounded by rigid boundaries, $x=0, y=0$.

Point vortex circulation Γ, image vortices across rigid boundary.

(1)

i)a) Note positions and circulation of each of the image vortices.

<u>Answer</u>: Primary vortex $\Gamma_1 = \Gamma$, $z_1 = x_1 + iy_1$

$\Gamma_2 = -\Gamma$, $z_2 = x_1 - iy_1 = z_1^*$

$\Gamma_3 = -\Gamma$, $z_3 = -x_1 + iy_1 = -z_1^*$

$\Gamma_4 = \Gamma$, $z_4 = -x_1 - iy_1 = -z_1$

Image vortices ensure no flux boundary condition $\underline{u} \cdot \underline{n}$

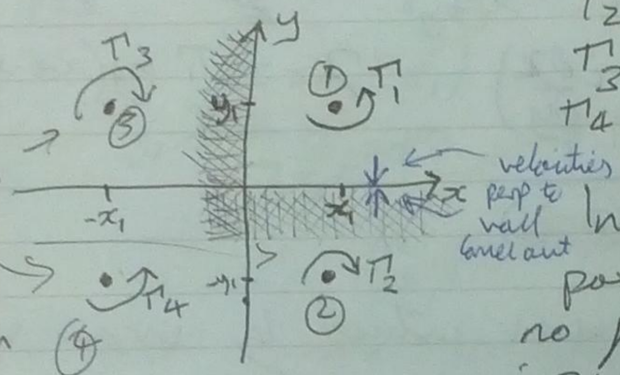


Image vortices are positioned to satisfy no flux boundary conditions $\therefore \underline{u} \cdot \underline{n} = 0$ (nothing can go through the walls).

i)b) Complex potential at $z = x+iy$ at primary vortex (x_1, y_1) ~~induced by image vortices~~? induced by vortex?

<u>Answer</u> Complex potential $\chi = \phi + i\psi$ where $u = \nabla\phi$, ψ = streamfunction

velocity potential (see sheet)

$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$, $v = \frac{\partial \phi}{\partial y}$ $\Rightarrow$ ~~ϕ~~ $u_\theta = \frac{\Gamma}{2\pi r}$

So $\phi = \frac{\Gamma \theta}{2\pi}$ and we know streamfunction

$u_\theta = -\frac{\partial \psi}{\partial r} \Rightarrow \psi = -\frac{\Gamma}{2\pi}\log(r)$

So complex potential $\chi = \phi + i\psi = -\frac{i\Gamma}{2\pi}(\log r + i\theta)$

So complex potential ~~at centre~~ of vortex at origin

$\chi = \frac{-i\Gamma}{2\pi}(\log r + i\theta)$

$$= -\frac{i\Gamma}{2\pi}\left(\log r + \log e^{i\theta}\right)$$

$$= -\frac{i\Gamma}{2\pi}\log\left(\underbrace{re^{i\theta}}_{z}\right)$$

At $z_1 = x_1 + iy_1$: $\chi = -\frac{i\Gamma_1}{2\pi}\log(z - z_1)$

(1) (ii) Complex velocity $\partial_z \chi = u - iv$

$$\cancel{= \frac{-y}{x^2+y^2} - \frac{ix}{x^2+y^2} = -\frac{z i}{z^2} = \frac{-i\Gamma}{z 2\pi}}$$

$$\partial_z \chi = \frac{1}{2}\left(\frac{\partial \chi}{\partial x} - i\frac{\partial \chi}{\partial y}\right) \qquad \chi = -\frac{i\Gamma_1}{2\pi}\log(x - x_1 + i(y - y_1))$$

$$\partial_z \chi = \frac{i\Gamma}{2\pi}\frac{1}{(z - z_1)}$$

c) Complex velocity $u - iv$ induced by image vortices at position $z = z_1$ of primary vortex

(i) $$\partial_z \chi = u - iv = \frac{i\Gamma}{2\pi}\left[-\underbrace{\frac{1}{z_1 - z_1^*}}_{2} - \underbrace{\frac{1}{z_1 + z_1^*}}_{3} + \underbrace{\frac{1}{2z_1}}_{4}\right]$$

$$u - iv = \frac{i\Gamma}{2\pi}\left[-\frac{1}{2iy_1} - \frac{1}{2x_1} + \frac{1}{2x_1 + 2iy_1}\right]$$

$$= \frac{\Gamma}{2\pi}\left[-\frac{1}{2y_1} - \frac{i}{2x_1} + \frac{(x_1 - iy_1)i}{2(x_1^2 + y_1^2)}\right]$$

(ii) $$u = \frac{\Gamma}{4\pi}\left[-\frac{1}{y_1} + \frac{y_1}{x_1^2 + y_1^2}\right] = \frac{\Gamma}{4\pi}\left[-\frac{x_1^2}{y_1(x_1^2 + y_1^2)}\right]$$

$$v = \frac{\Gamma}{4\pi}\left[\frac{1}{x_1} - \frac{x_1}{x_1^2 + y_1^2}\right] = \frac{\Gamma}{4\pi}\left[\frac{y_1^2}{x_1(x_1^2 + y_1^2)}\right]$$

(iii) Deduce trajectory $\frac{\partial y_1}{\partial x_1}$:

<u>Answer</u> $\frac{\partial y_1}{\partial x_1}$: $\frac{\partial x_1}{\partial t} = u$, $\frac{\partial y_1}{\partial t} = v$

$$\Rightarrow \frac{\partial y_1}{\partial x_1} = \frac{v}{u} = -\frac{y_1^3}{x_1^3} = -\left(\frac{y_1}{x_1}\right)^3$$

(iv) Show path taken by primary vortex is $\frac{1}{x_1^2} + \frac{1}{y_1^2} = \text{const.}$

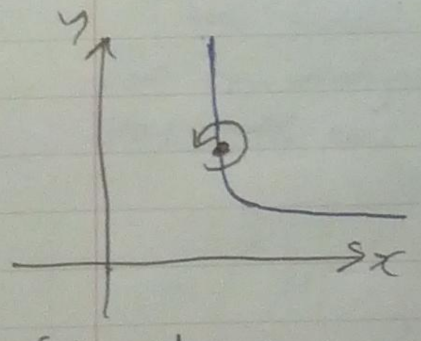
<u>Answer</u> $\frac{\partial y_1}{\partial x_1} = -\left(\frac{y_1}{x_1}\right)^3$ Integrate ~~w.r.t. x_1~~:

$$y_1 = \cancel{-\frac{y_1^3}{2x_1^2} + \text{const} \Rightarrow \frac{1}{y_1^2} = \frac{1}{2x_1^2} + \text{const}}$$

$$\Rightarrow \cancel{\frac{1}{y_1^2} - \frac{1}{2x_1^2} = \text{const}}$$

~~(v) Why does a smoke ring expand as it approaches a wall~~

$$\frac{1}{y_1^3}\frac{\partial y_1}{\partial x_1} = -\frac{1}{x_1^3} \Rightarrow \int \frac{1}{y_1^3}\,\partial y_1 = \int -\frac{1}{x_1^3}\,\partial x_1$$

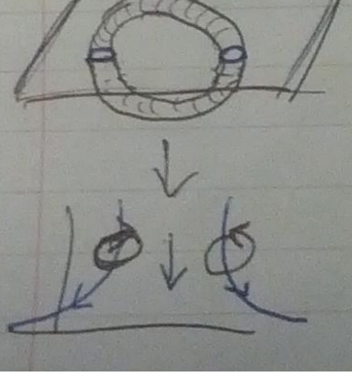


$$\Rightarrow -\frac{1}{2y_1^2} = \frac{1}{2x_1^2} + \text{const}$$

$$\Rightarrow \frac{1}{x_1^2} + \frac{1}{y_1^2} = \text{const}$$

(v) Why does a smoke ring expand as it approaches a wall?

<u>Answer</u>



Smoke ring can be idealised as a vortex pair, therefore will separate at the wall in the same way.