MA4H7 Atmospheric Dynamics Support Handout 2 - Rotation and Kelvin Waves

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1 Rotation

We consider the Euler equations in a rotating frame,

$$\frac{D\boldsymbol{u}_R}{Dt} = -\frac{1}{\rho}\nabla p - \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})}_{\text{Centrifugal Force}} - \underbrace{2\boldsymbol{\Omega} \times \boldsymbol{u}_R}_{\text{Coriolis Force}} - g\hat{\boldsymbol{z}} \tag{1}$$

where $\boldsymbol{u}_R$ is the velocity in the rotating frame given by $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}$ where $\boldsymbol{u}_I$ is the velocity in the inertial frame (as normal) and $\boldsymbol{r}$ is the coordinate in the rotating frame. We will drop the subscript R notation from now on. The centrifugal force is negligible and can generally be ignored.

The Coriolis force is a fictitious force observed on moving objects in a rotating frame causing a deflection in the motion. If the frame is rotating clockwise then the deflection is to the left of the motion of the object. For anticlockwise rotation the deflection is to the right. This explains why cyclones in the Northern Hemisphere are anticlockwise (and clockwise in the Southern Hemisphere).

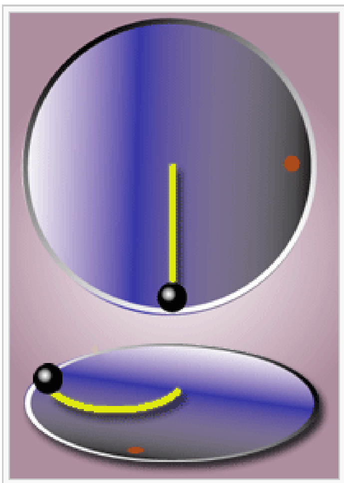

Figure 1: In the inertial frame of reference (upper part of the picture), the black ball moves in a straight line. However, the observer (red dot) who is standing in the rotating/non-inertial frame of reference (lower part of the picture) sees the object as following a curved path due to the Coriolis and centrifugal forces present in this frame.

The *Rossby Number* is a ration of inertia $|(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}|$ forces to Coriolis forces,

$$Ro = \frac{U}{\Omega L}. \tag{2}$$

The *Ekman Number* is a ration of viscous $|\nu \Delta \boldsymbol{u}|$ forces to Coriolis forces,

$$Ek = \frac{\nu}{\Omega L^2}. \tag{3}$$

2 Geostrophic Wind

Suppose we have a rotating fluid, like Earth's atmosphere with rotation $\boldsymbol{\Omega} = (0, 0, \Omega)$. Consider small Rossby number $Ro = U/\Omega L$ and Ekman number $Ek = \nu/\Omega L^2$, that is Coriolis forces are dominant over inertial and viscous forces, $|\boldsymbol{u} \cdot \nabla \boldsymbol{u}| \leq |\boldsymbol{\Omega} \times \boldsymbol{u}|$ and $|\nu \Delta \boldsymbol{u}| \leq |\boldsymbol{\Omega} \times \boldsymbol{u}|$. Then the Euler equation becomes a balance between Coriolis and pressure forces,

$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho} \nabla p. \tag{4}$$

This is called a *geostrophic flow*. Note that the Coriolis force $2\boldsymbol{\Omega} \times \boldsymbol{u}$ is always perpendicular to the flow $\boldsymbol{u}$. Assuming this geostrophic balance we have the *geostrophic wind*

$$(u_g, v_g) = \frac{1}{2\Omega\rho} \left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x} \right), \tag{5}$$

which is perpendicular to the pressure gradient, therefore blows parallel to isobars (lines of constant pressure). In the Northern Hemisphere the geostrophic wind blows anticlockwise (cyclonic - same rotation direction as the Earth) around low pressure.

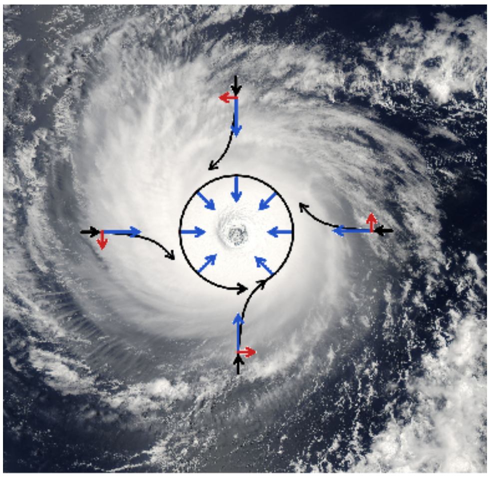

Figure 2: Deflection effects of the Coriolis force on flow towards a low pressure cell causing a cyclone. The aegeostrophic velocity component $\boldsymbol{u}_a$ follows the pressure gradient, shown in blue arrows. The Coriolis effect is shown in red arrows, the geostrophic velocity component $\boldsymbol{u}_g$ is in this direction. The resulting deflected flow is shown in black arrows.

3 Waves

Fluids flows and in particular geophysical flows can exhibit a wide range of different types of waves. Making various simplifying assumptions and approximations about a flow we can isolate different wave solutions. First we define some basic wave terminology.

The *wavenumber* k is a measure of the number of times a wave has the same phase per unit of space.

The *wavelength* λ is the distance between repeating units of a propagating wave of a given frequency, it is related to the wavenumber by

$$\lambda = \frac{2\pi}{k}. \tag{6}$$

The *phase speed* c_{ph} describes the motion within a wave packet. Velocity at which a phase of any one frequency component of the wave will propagate within the packet,

$$c_{ph} = \frac{\omega}{k}. \tag{7}$$

Here ω is the frequency and the *dispersion relation* is the relation $\omega(k)$ between ω and the wavenumber k. In higher dimensions this is

$$c_{ph} = \hat{\boldsymbol{k}} \frac{\omega}{|\boldsymbol{k}|}, \tag{8}$$

where $\boldsymbol{k} = (k, l, m)$ is a vector of the wavenumbers in each direction and $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$.

The *group velocity* c_g describes the motion of the whole wave packet,

$$c_g = \frac{\partial \omega}{\partial k} (= \nabla_{\boldsymbol{k}} \omega \text{ in higher dimensions}). \tag{9}$$

If $c_g \neq c_{ph}$ then the waves are *dispersive*. Note that for a non-dispersive wave we have

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k} \tag{10}$$

which implies that the dispersion relation is of the form $\omega = ak$ for some constant a. Therefore if the group velocity $c_g = \partial\omega/\partial k$ is some function of k then the waves must be dispersive.

4 Kelvin Waves

Kelvin waves are travelling disturbances that require the support of a lateral boundary. Therefore it most often occurs in the ocean where it travels along coastlines. For these waves we use the linearised shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{11}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{12}$$

$$\frac{\partial \eta}{\partial t} = -b\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{13}$$

where the depth has been linearised $h = b + \eta$ with b the average depth. Now considering a wave with the coast at $x = 0$ and the domain $x > 0$, we assume the velocity in the x-direction to be zero $u = 0$. Then taking a second time derivative of (12) and then using (13) we get

$$\frac{\partial^2 v}{\partial t^2} = -g\frac{\partial}{\partial y}\frac{\partial \eta}{\partial t} = c^2\frac{\partial^2 v}{\partial y^2} \tag{14}$$

where $c = \sqrt{gb} = c_g = c_{ph}$, which is identified as the speed of surface gravity waves in non-rotating shallow water. Equation (14) is the one dimension wave equation and has general solution

$$v = V_+(x, y + ct) + V_-(x, y - ct) \tag{15}$$

and using (12) or (13) we get the surface displacement,

$$\eta = -\sqrt{\frac{b}{g}}V_+(x, y + ct) + \sqrt{\frac{b}{g}}V_-(x, y - ct) \tag{16}$$

(where the integration constant can be removed my a redefinition of the mean depth). Using (11) gives the form of V_+ and V_- as

$$V_+ = V_{0+}(y + ct)e^{-x/R}, \;\; V_- = V_{0-}(y - ct)e^{+x/R} \tag{17}$$

where $R = c/f$ is the *Rossby radius of deformation* (distance covered by a wave travelling at speed c during one inertial period $2\pi/f$) and V_{0+}, V_{0-} are arbitrary functions determined by initial conditions. For the Northern Hemisphere (where $f > 0$) the solution V_- explodes as $x \to \infty$, hence we can only use V_+ for a physical solution. We therefore get the general solution

$$u = 0 \tag{18}$$

$$v = \sqrt{gb}F(y + ct)e^{-x/R} \tag{19}$$

$$\eta = -bF(y + ct)e^{-x/R} \tag{20}$$

where F is an arbitrary function.

Figure 3: Direction of Kelvin waves depicted by arrows along coastlines. Kelvin waves will travel south on the east coast of Britain, north along the west coast and east along the north coast of France coming in off of the Atlantic ocean. A Kelvin wave travelling south along the east coast of Great Britain could travel around the coasts of the North sea in an anticlockwise direction and reach the west coast of Norway.

Kelvin waves are *trapped* due to the exponential decay away from the boundary. Observing that writing the arbitrary function $F(y + ct) = e^{ik(y+ct)} = e^{i(ky-\omega t)}$ we see that $-\omega = ck$. Then $c = c_{ph} = c_g = -\omega/k < 0$ (pointing south), Kelvin waves are non-dispersive. The waves propagate without distortion at the speed of surface gravity waves. In the Northern Hemisphere the waves travel with the coast on its right; in the Southern Hemisphere with the coast on its left. Surface Kelvin waves are usually generated by the ocean tides and by local wind effects in coastal areas.

5 Poincaré Waves

These waves are gravity waves that are slow enough to be affected by Coriolis forces, that is they are a combination of gravity waves and inertial waves. They occur in open ocean, so we assume an infinite domain and we align the crests of the waves with the y-axis so that $\partial/\partial y = 0$. Then the linearised shallow water equations (11), (12) and (13) become

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = gb\frac{\partial^2 u}{\partial x^2}. \tag{21}$$

Now we try a wave solution

$$u = u_0 \exp[i(kx - \omega t)]. \tag{22}$$

Substituting this into the above equation gives the dispersion relation

$$\omega^2 = f^2 + ghk^2. \tag{23}$$

Note that not using the assumption that $\partial/\partial y = 0$ yields the dispersion relation (after some messy rearranging)

$$\omega[\omega^2 - f^2 - gb(k^2 + l^2)] = 0 \tag{24}$$

which reveals a trivial $\omega = 0$ geostrophic solution, that is a zero frequency wave that flows perpendicular to pressure gradients.

The phase velocity is

$$c_{ph} = \frac{\omega}{k} = \frac{\pm\sqrt{f^2 + gbk^2}}{k} \tag{25}$$

and the group velocity is

$$c_g = \frac{\partial \omega}{\partial k} = \frac{gbk}{\pm\sqrt{f^2 + gbk^2}} \neq c_{ph} \tag{26}$$

therefore these waves are dispersive.

For short waves, ie. k large (or $f \to 0$) the phase and group speeds tend to $\sqrt{gb}$ and we get *surface gravity waves* (waves induced by buoyancy) in non-rotating shallow water.

For long waves, ie. k small, then $\omega \sim f$ and particles move in circles: taking real parts of velocity we have $u = u_0 \cos(kx - \omega t)$ and $v = (fu_0/\omega)\sin(kx - \omega t)$. (Otherwise they move in an ellipse). These are called *inertial waves* (waves induced by rotation). These waves are not visible on the surface and are interior to the fluid, therefore they are sometimes called *internal waves.*