12pm 6th May 2014
B3·03

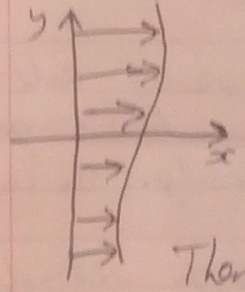
Fluids Revision 1

Class 1: laminar pipe and shear flows.
Class 2: Monday 12th 10-11 B3·03 – overview/question EMAIL.
Class 3: Friday 23rd 11-12 B3·02 – Kerr

Section 6 questions is revision question.
Rain fall on roof questions not examinable.

Shear Flow

We consider a velocity field $\underline{u} = (u(y,t), 0, 0)$ with only one velocity in streamwise direction x, which depends on y.



Incompressibility satisfied $\partial_x u = 0 = \nabla \cdot \underline{u}$
Advection is zero $(\underline{u}\cdot\nabla)\underline{u} = u\partial_x u = 0$
Pressure gradient is $\nabla P = (\partial_x P, 0, 0)$
Viscosity depends on y: $\nu \Delta u = \frac{\partial^2}{\partial y^2} u$.

Then equation is: $\partial_t u = -\frac{1}{\rho}\partial_x P + \nu \partial_{yy}^2 u$

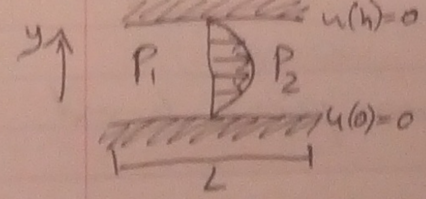
Assume $\partial_x P = P_{,x}$ constant and assume steady state then solve $\partial_{yy}^2 u = \frac{P}{\rho\nu}$

Integrate w.r.t y twice: $u(y) = \frac{P}{2\rho\nu} y^2 + Ay + B$

to be found from boundary conditions.

Poiseuille Flow

Flow through pipe



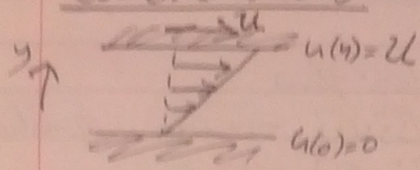
Pressure gradient is $\partial_x p = \frac{P_2 - P_1}{L} = P_{,x}$

$u(0) = 0 \Rightarrow B = 0$
$u(h) = 0 \Rightarrow A = -\frac{P_{,x} h}{2\rho\nu}$

$$\Rightarrow u(y) = \frac{P_{,x}}{2\rho\nu} y(y-h)$$

Couette Flow

Flow between moving plates (etc)



Not pressure driven

$u(0)=0 \Rightarrow B=0$

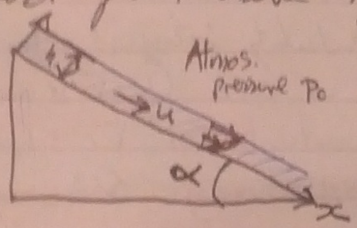
$u(h)=\mathcal{U} \Rightarrow A = -\frac{\mathcal{U}}{h}$

$\Rightarrow u(y) = \frac{\mathcal{U}}{h} y$

For large Reynolds numbers, Poiseuille / Couette flow turbulent.

Flow over an inclined plane

Layer of incompressible fluid bounded below by a plane fixed at an angle α to the horizontal and from above by a free surface.



Consider steady state due to balance between gravity and viscosity. $\underline{u} = (u(y), 0, 0)$

a) Formulate no-slip boundary condition for velocity at the bottom plane, and pressure BC. at free surface.

Answer $u = v = \partial_y u = 0$ at $y=0$

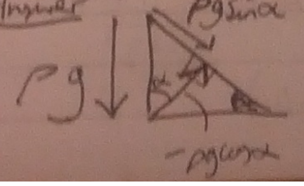
no flow through surface

b) Why is $\frac{\partial u}{\partial y} = 0$ at $y=h$

Answer Shear stress $\tau = \mu \nabla u$ must be zero at the free surface (no fluid above), ~~by assumption~~ therefore $\partial_y u = 0$.

(c) What are the gravitational forces perpendicular and parallel to the surface.

Answer



d) Give the steady state Navier-Stokes equations for this system.

Answer If $\underline{u} = (u(y), v(y), 0)$ then by incompressibility

$$\underbrace{\partial_x u}_{0} + \partial_y \mathbf{v} = 0 \Rightarrow \partial_y \mathbf{v} = 0 \text{ and } v(0) = 0 \Rightarrow v = 0$$ everywhere.

So NS are:

x direction → $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g\sin\alpha \quad (1)$$

y direction → $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - g\cos\alpha \quad (2)$$

e) Solve for pressure.

Answer Integrate (2): $p(x,y) = -\rho g y\cos\alpha + f(x)$

Also $P = p_0$ at $y = h$ $\forall x$.

subtract $p(x,h) = p_0 = -\rho g h\cos\alpha + f_0$ ($f(x)$ constant)

$\Rightarrow P - P_0 = \rho g(h-y)\cos\alpha$ (3)

f) Solve velocity

Answer From (3) we see $\frac{\partial P}{\partial x} = 0$ so (1) reduces to $\nu\frac{\partial^2 u}{\partial y^2} = -g\sin\alpha$. Integrate and use BCs $u = 0$ at $y = 0$, $\frac{\partial u}{\partial y} = 0$ at $y = h$

to get $u = \frac{g}{2\nu}y(2h-y)\sin\alpha$

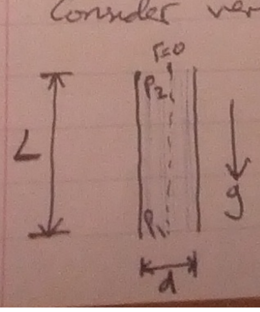
g) Find volume of fluid Q passing through cross-section of the layer per unit length in the x-direction per unit time.

Answer $$Q = \int_0^h u\,dy = \int_0^h\left[\frac{g}{\nu}yh - \frac{g}{2\nu}y^2\right]\sin\alpha$$

$$= \left[\frac{g}{2\nu}h^2h - \frac{g}{6\nu}h^3\right]\sin\alpha = \frac{g}{3\nu}h^3$$

Flow Through Pipes - Viscometer

Consider vertical pipe, assume cylindrically symmetric, steady laminar Poiseuille flow.



Aim: show flow rate through pipes is $Q = \frac{\pi d^4}{128\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)$

$$\left(\Delta = \frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial z^2}\right)$$

N-S simplification

(a) Navier-Stokes simplifies to (neglect time deriv, advection)

$$(4)\quad \frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{r}{\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)$$

using cylindrical coord.

Find velocity profile.

(b) Flow is symmetric at $r=0$, BCs: $\frac{dw}{dr}=0$ at $r=0$

also note that $w=0$ at $r=\frac{d}{2}$.

Integrate (4) to get

$$r\frac{dw}{dr} = \frac{-r^2}{2\nu}\left(\left(\frac{P_2-P_1}{\rho L}\right)+g\right)+C_1$$

use $\frac{dw}{dr}=0$ at $r=0$ to get $C_1=0$.

Integrate again $w = \frac{-r^2}{4\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)+C_2$

use $w=0$ at $r=d/2$ to get $C_2 = \frac{d^2}{16\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)$

So that $w = \frac{1}{4\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)\left(\frac{d^2}{4}-r^2\right)$

Find flow rate.

(c) Flow rate $Q = 2\pi\int_0^{d/2} w\,r\,dr$

$$= 2\pi\int_0^{d/2}\frac{1}{4\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)\left(\frac{d^2 r}{4}-r^3\right)dr$$

$$= \left[\frac{2\pi}{4\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)\left(\frac{d^2r^2}{8}-\frac{r^4}{4}\right)\right]_0^{d/2}$$

$$= \frac{\pi}{2\nu}\left(\frac{P_2-P_1}{\rho L}+g\right)\frac{d^4}{64}.$$