MA3D1 Fluid Dynamics Support Class 2 - Conservation and Circulation

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1 Kelvin's Circulation Theorem

Theorem 1. *In an ideal flow with a conservative force, let $C(s,t)$ be a closed material contour. Then the* circulation

$$\Gamma = \oint_{C(s,t)} \boldsymbol{u} \cdot dx = \int_S \boldsymbol{\omega} \cdot dS, \tag{1.1}$$

is independent of time.

This is an important theorem in fluid dynamics. Note that this only holds for non-viscous fluids.

2 Complex Potential

If a flow is 2D, incompressible and irrotational then the velocity field can be represented as

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \tag{2.1}$$

where ϕ is the velocity potential and ψ is the streamfunction. These are the Cauchy-Riemann equation, which in Complex Analysis implies there is an analytic function called the *complex potential*,

$$\chi = \phi + i\psi, \tag{2.2}$$

which is a function of $z = x + iy$, then $\partial_z \chi = u - iv$ is the complex velocity. Examples of the complex potential in the lecture notes include uniform flow at an angle, stagnation point and point vortex.

3 Irrotational Flow Around a Cylinder ***2013/14 Problem***

We are given that the complex potential of uniform flow of speed U_0 in the x-direction around a cylinder of radius a is

$$\chi = U_0 \left(z + \frac{a^2}{z} \right). \tag{3.1}$$

This comes from something called Milne-Thomson's circle Theorem, (Acheson §4.4, §4.5). If the cylinder has circulation Γ then

$$\chi = U_0 \left(z + \frac{a^2}{z} \right) - \frac{i\Gamma}{2\pi} \log z, \tag{3.2}$$

(see complex potential of point vortex). To find velocities u_r and u_θ we use the polar form $z = re^{i\theta}$ to get

$$\chi = U_0 \left(re^{i\theta} + \frac{a^2 e^{-i\theta}}{r} \right) - \frac{i\Gamma}{2\pi} (\log r + i\theta),$$

and using the identity $e^{i\theta} = \cos\theta + i \sin\theta$,

$$\chi = \underbrace{U_0 \left(r + \frac{a^2}{r} \right) \cos\theta + \frac{\Gamma\theta}{2\pi}}_{\phi} + i \underbrace{\left(U_0 \left(r - \frac{a^2}{r} \right) \sin\theta - \frac{\Gamma}{2\pi} \log r \right)}_{\psi}. \tag{3.3}$$

Then using

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, u_\theta = -\frac{\partial \psi}{\partial r} \text{ or } u_r = \frac{\partial \phi}{\partial r}, u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

we get

$$u_r = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta, u_\theta = -U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.4}$$

Example 1. *Rankine Vortex in a strain field*
Consider a vortex with vorticity which has only z-component, $\omega = (0, 0, \omega_z)$, and which is uniformly distributed in a circle of radius R:

$$\omega_z = \Omega = const \text{ for } r^2 = x^2 + y^2 < R^2, \text{ and } \omega_z = 0 \text{ for } r^2 \geq R^2.$$

The flow with such vorticity and with velocity field $\boldsymbol{u}_R$ which decays at infinity ($\boldsymbol{u}_R \to 0$ for $r \to \infty$) is called a Rankine vortex*; in this case the vortex radius R is constant.*

In this example we will consider a case when the velocity field does not decay at infinity. Namely, we will consider a flow with velocity

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{u}_\sigma \tag{3.5}$$

where $\boldsymbol{u}_R$ is the Rankine vortex and $\boldsymbol{u}_\sigma$ is a uniform strain field of the form

$$\boldsymbol{u}_\sigma = (-\sigma x, -\sigma y, 2\sigma z). \tag{3.6}$$

In this case the vortex radius R and the vorticity will be time dependent, $R = R(t), \Omega = \Omega(t)$, because of the vortex stretching produced by the strain.

1. *Prove that the Rankine vortex is a solution to the Euler equation for an inviscid fluid. Find the incompressible velocity field $\boldsymbol{u}_R$ of the Rankine vortex.*

2. *Prove that the uniform strain field $\boldsymbol{u}_\sigma$ given by expression (3.6) satisfies the ideal flow equations.*

3. *Now consider the combination of the Rankine vortex and the strain field as in expression (3.5) and prove that it satisfies the ideal flow equations. Find dependencies $R(t)$ and $\Omega(t)$. Interpret your results in terms of the vortex stretching mechanism.*

4. *The Burgers vortex is a generalisation of the considered solution to viscous flows. This solution is stationary because the vortex stretching is stabilised by the vorticity diffusion due to viscosity. The stain field in this vortex is the same as in (3.6), but the vorticity profile now is*

$$\omega_z = \Omega_0 e^{-\lambda r^2}$$

where $\Omega_0 = const$. Find λ in terms of σ and ν.

Example 2. *Vortex Lift*

Some aeroplanes have sharply swept leading edge of the wing which generate vortices on the the upper sides of both wings. Examples include a delta winged F-106 military jet and a commercial one (no longer in use) - Concorde. Each vortex is trapped by the air flow and remains fixed to the upper surface of the wing. The major advantage of vortex lift is that it allows angles of attack that would stall a normal wing. The vortices also produce high drag which can help to slow down the aircraft. This is why the vortex lift is used during (high angle of attack) landing of most supersonic jets.

In this problem we will aim to understand how vortices produce lift. For this, we will consider a simplified situation in which an infinite straight vortex with circulation Γ is placed parallel to an infinite flat plate (an idealised wing). For simplicity, we will assume that the flow is inviscid and incompressible.

1. *Formulate the free-slip boundary conditions on the plate.*

2. *Find the velocity field produced by the vortex of circulation Γ on the top surface of the plate. (**Hint:** use the vortex image method to satisfy the free-slip boundary conditions.)*

3. *Find the pressure distribution on the top surface of the plate assuming that the surrounding pressure (i.e. far away from the vortex) is atmospheric, p_0.*

4. *Assuming that the pressure at the lower side of the plate is uniform and equal to the atmospheric value, find the total force on the plate per unit length in the vortex direction.*

Example 3. ***Water Clock***

A water clock is an axisymmetric vessel with a small exit hole of radius a in the bottom. Find the vessel shape for which the water level falls equal heights in equal intervals of time. (Hint: The hole is so small that the water passes through it very slowly and its velocity can be found from Bernoulli's theorem for stationary flows).