

MA3D1 Fluid Dynamics Support Class 3 - Boundary Layers

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1 Boundary Layers

In the inviscid ($\nu = 0$) case we have the free-slip boundary condition, that is the velocity normal to the boundary is zero, $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ but fluid is free to move along the boundary.

In reality there is a thin *boundary layer* where velocity undergoes a smooth but rapid adjustment to zero. This is because at the boundary the normal and tangential components of velocity must equal that of the boundary. If the boundary is at rest then $\mathbf{u} = 0$. The condition on the tangential component of velocity is known as the *no-slip* condition.

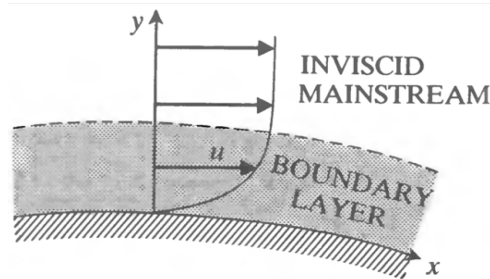


Figure 1: Boundary layers form near the boundary of a body in a flow, here viscous effects are very important. The mainstream can be considered as inviscid for low viscosity flows.

Velocity gradients in boundary layer much larger than in the main part of the flow, so viscous effect become significant no matter how small ν is.

In (Newtonian¹) viscous fluids, shear stress τ is proportional to the velocity gradient $\frac{\partial u}{\partial y}$,

$$\tau = \mu \frac{\partial u}{\partial y}, \quad (1)$$

where μ is called the coefficient of viscosity. We define the *kinematic viscosity* as $\nu = \frac{\mu}{\rho}$.

Boundary layers may separate from the boundary causing dramatic differences in the flow from that of inviscid theory. Wakes behind objects will form causing drag.

Referring to Figure 2, inviscid theory says the pressure p has a local maximum at A, a minimum at B and a local maximum at C. So between B and C there is a substantial increase in pressure along the boundary in the direction of the flow. This severe adverse pressure gradient along the boundary causes the boundary layer to separate. We can see how this adverse pressure gradient drives a reversed flow in Figure 3 on the right hand side. This pushes the boundary layer (typically of thickness $O(R^{-\frac{1}{2}})$) into the mainstream. Here $R = UL/\nu$ is the *Reynolds*

¹A fluid is said to be Newtonian if the viscous stresses that arise from its flow, at every point, are proportional to the local strain rate - the rate of change of its deformation over time. This course will not consider non-Newtonian fluids.

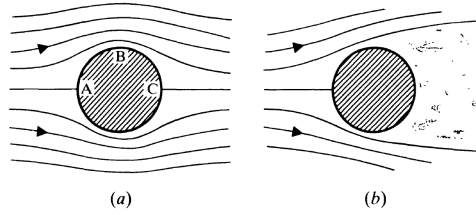


Figure 2: Flow past a cylinder for (a) inviscid fluid, (b) fluid of small viscosity.

number, where U is a typical flow speed and L is a characteristic length scale. This is a ratio of inertia/viscosity, a high Reynolds number means viscous effects are negligible. Note that near the boundary U becomes very small so viscous effects become important even for small ν . At high R reattachment can occur, this is where the boundary layer reattaches to the surface so the boundary layer doesn't fully separate until further towards the rear end of the object reducing the size of the wake, hence reducing drag. This reattachment principle is behind the design of aerofoils, the dimples on golf balls and the fur on tennis balls.

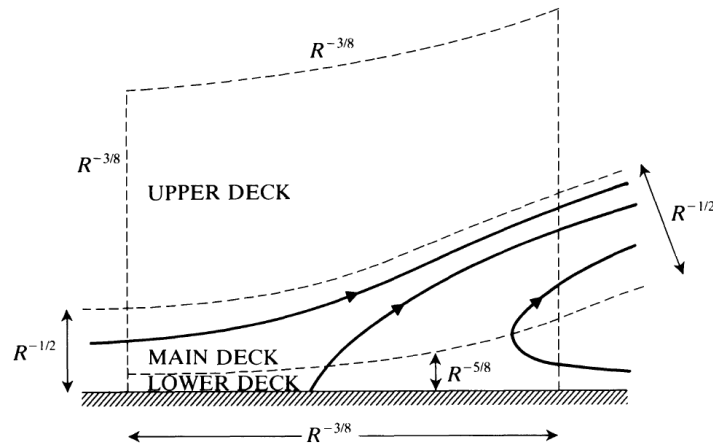


Figure 3: Separation point of a boundary layer on a rigid boundary.

2 Similarity Solutions

A *similarity solution* is a flow that looks the same at all time or at all length scales.

Example 1. (*Impulsively moved plate*) We consider a flow at rest in $y > 0$ with a plate at the bottom boundary that is suddenly moved at speed U . It was shown in lecture that Navier-Stokes reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

with initial condition $u(y, 0) = 0$ and boundary conditions $u(0, t) = U, u(\infty, t) = 0, \forall t > 0$. We seek a similarity solution, observe that the equation (2) is unchanged by the transformation $y \rightarrow \alpha y, t \rightarrow \alpha^2 t$ for a constant α . This suggests that there are solutions to (2) which are functions of y and t through the combination $y/t^{1/2}$, for this variable would also be unchanged by the transformation. In fact by looking at (2) we see it would be more convenient to take $y/(\nu t)^{1/2}$.

We try the solution $u = f(\eta)$ where $\eta = y/(\nu t)^{1/2}$ and as shown in lectures (or Acheson §2.3),

$$u = U \left[1 - \frac{1}{\pi^{1/2}} \int_0^\eta e^{-s^2/4} ds \right]. \quad (3)$$

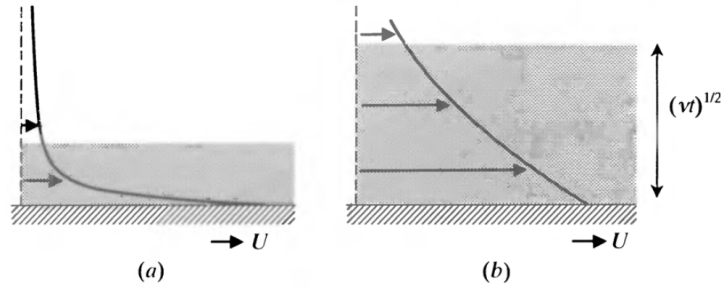


Figure 4: The diffusion of vorticity from a plane boundary suddenly moved with velocity U . The solid line indicate the velocity profile at some early time (a) and some later time (b); shading indicates the region of significant vorticity (boundary layer).

Figure 4 demonstrates how the velocity profiles $u(y)$ are geometrically similar at all times. The velocity at time t_1 is a function of $y/(\nu t_1)^{1/2}$ and at a later time it is the same function of $y/(\nu t_2)^{1/2}$. The velocity profile becomes stretched out in the y -direction. This is also a demonstration of the diffusion of vorticity,

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{(\pi\nu t)^{1/2}} e^{-\frac{y^2}{4\nu t}}. \quad (4)$$

The vorticity is concentrated in the layer near the boundary of thickness $(\nu t)^{1/2}$ which grows with time. The spreading of vorticity by viscous action thus smooths out what was, initially, a vortex sheet, .ie an infinite concentration of vorticity at the boundary ($y = 0, t \rightarrow 0$) and none elsewhere. In this way viscous effects gradually communicate the motion of the boundary to the whole fluid.