MA3D1 Fluid Dynamics Support Class 9 - Turbulence

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1   Turbulence

Turbulence is a flow regime characterised by rapid property changes including rapod variation of pressure and velocity in space and time. Turbulence can be described as a state of continuous instability where random features of the flow are dominant. Turbulence occurs at high Reynolds numbers. Turbulence is characterised by *cascades*. Turbulence causes the formation of eddies
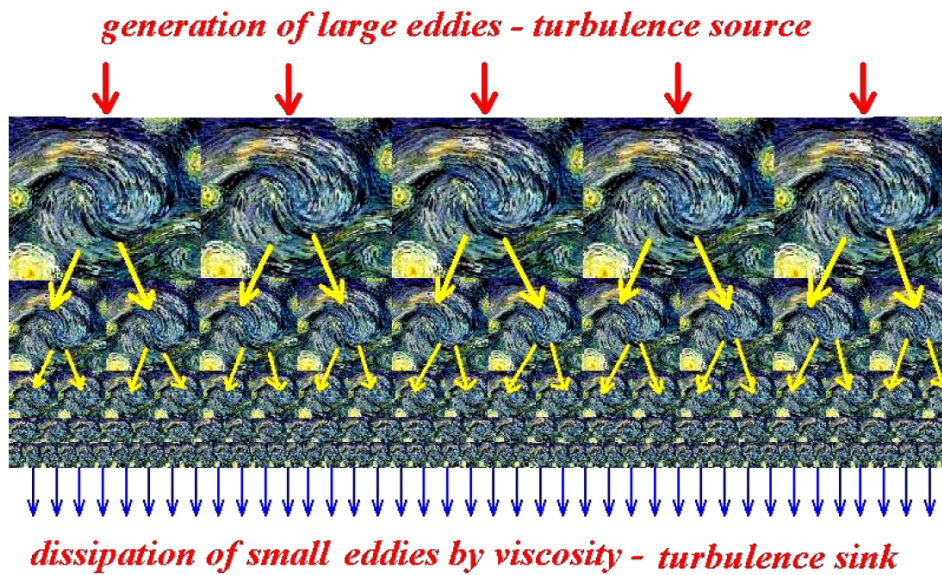

Figure 1: Turbulence characterised by transfer of energy to smaller scales.

of many different length scales. Energy cascades from large structures down to the smaller ones until the energy is dissipated by viscous effects. The scale at which this dissipation occurs is the *Kolmogorov length scale*.

The way in which energy is distributed over the scales is given by the energy spectrum function $E(k)$ with $k = |\boldsymbol{k}|$. The *Fourier transform* gives the Fourier representation of the velocity field $\boldsymbol{u}(\boldsymbol{x})$,

$$\hat{u}(\boldsymbol{k}) = \frac{1}{V} \int \boldsymbol{u}(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d^3x. \tag{1.1}$$

The energy for wavenumbers with $|\boldsymbol{k}| = k$ is

$$E(k) = \lim_{dk\to 0} \frac{1}{dk} \int_{k-\frac{dk}{2}}^{k+\frac{dk}{2}} \frac{1}{2}|\hat{u}(\boldsymbol{q})|^2|\boldsymbol{q}|^2 d\Omega\, d\boldsymbol{q}. \tag{1.2}$$

Kolmogorov predicted the form of this energy spectrum with a statistical/dimensional argument. Let ϵ be the energy dissipation rate, consider a length scale $L > l > \eta = (\nu^3/\epsilon)^{1/4}$ where $l = 1/k$, let $k_f \ll k \ll k_\eta$ with $k_f \sim L^{-1}$ and $k_\eta \sim \eta^{-1}$ (Kolmogorov wavenumber).
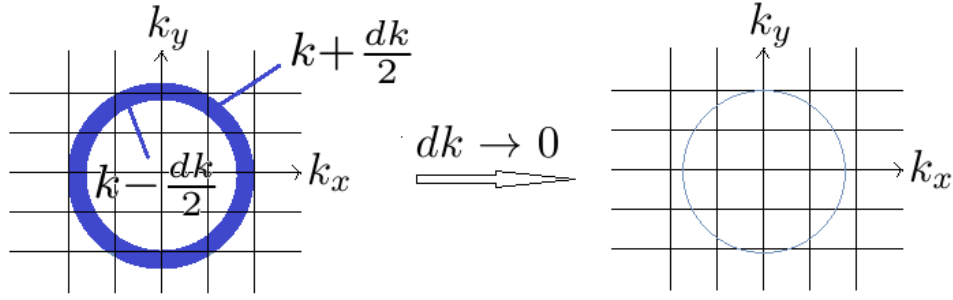
Figure 2: Visualisation of the definition of the energy spectrum. We take an annulus of wavenumbers of thickness dk and calculate the energy over this range of wavenumbers, then for wavenumbers $k = |\boldsymbol{k}|$ we take the limit of the energy as $dk \to 0$.
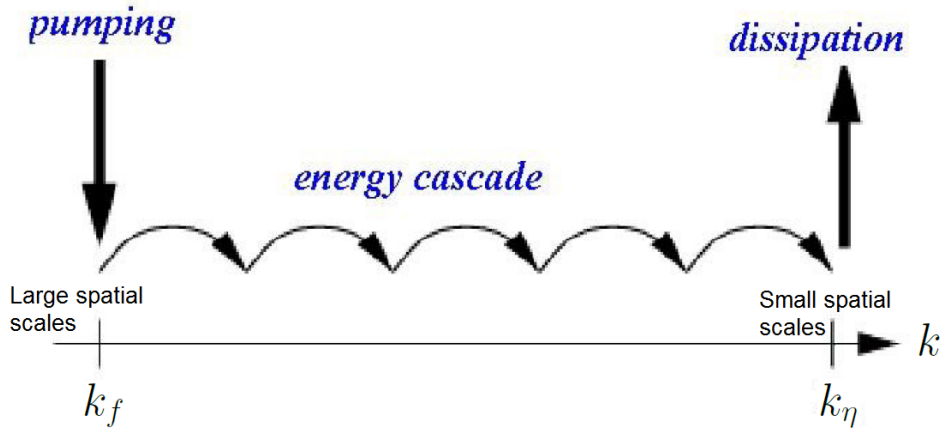

Figure 3: Visualisation of the energy cascade in Fourier space. Forcing occurs at the large spatial scales with small wavesnumbers k_f and the energy cascades down to the small spatial scales with wavenumber k_η where dissipation occurs.

The dimensions are of the energy dissipation rate and energy are

$$[\epsilon] = \left[\frac{U^2}{T}\right], \quad [E(k)] = \left[\frac{U^2}{k}\right] = \left[\frac{L^3}{T^2}\right]. \tag{1.3}$$

Assuming $E(k) \sim \epsilon^a k^b$ then we have

$$\left[\frac{L^3}{T^2}\right] = \left[\frac{L^2}{T^3}\right]^a \left[\frac{1}{b}\right]^b.$$

Matching powers of L and T fives $b = -5/3$ and $a = 2/3$. Therefore the only dimensionally consistent energy spectrum is

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{1.4}$$

where we hope C is a universal constant. Plotting the energy spectrum of a flow over several different times will help to identify the regime that the energy spectrum follows.

Plotting a compensated energy spectrum $E(k)k^{5/3}\epsilon^{-2/3}$ will give a strong indication to a Kolmogorov energy spectrum if it flattens out with zero slope. Early times will not reach a turbulent structure hence will not display a Kolmogorov energy cascade. Note that a $k^{-5/3}$ spectrum does not necessarily indicate turbulence.
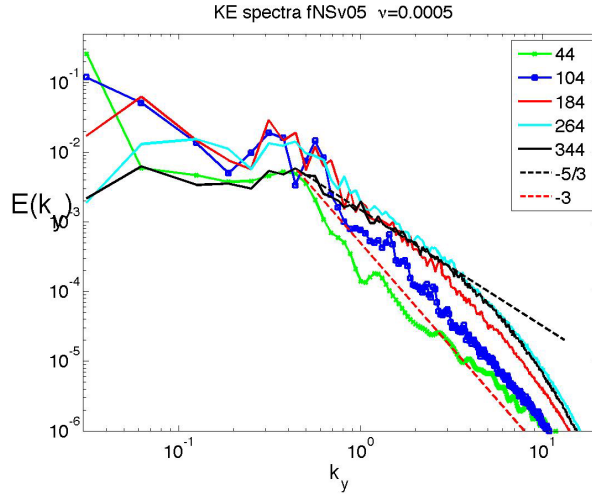
Figure 4: Energy spectrum for 5 different times in a flow. Plotting the k^{-3} and $k^{-5/3}$ slopes suggests that early times follow a k^{-3} regime whereas late times follow a $k^{-5/3}$ regime.
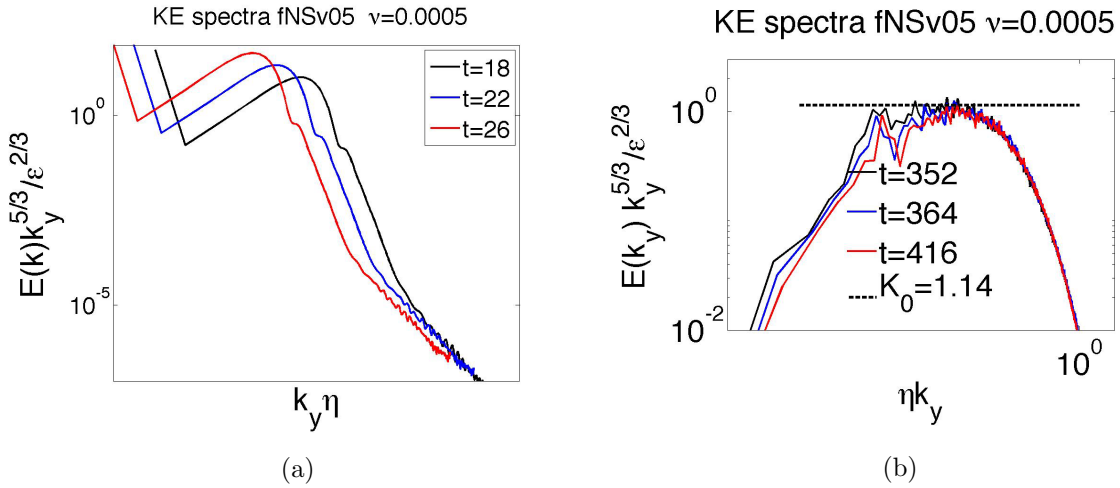

(a)

(b)

Figure 5: Compensated energy spectra of (a) early times where a turbulent structure has not formed so a Kolmogorov $k^{-5/3}$ regime is not evident, and (b) late times where there is a flattening to zero slope indicating a Kolmogorov energy cascade.

3D turbulence displays a forward energy cascade with small scale structures forming as energy moves from large to small scales followed by strong dissipation. There is also strong enstrophy production.