Perfect Powers that are Sums of Consecutive like Powers

Vandita Patel
University of Warwick

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A Diophantine Equation

$$(x+1)^k + (x+2)^k + \cdots + (x+d)^k = y^n.$$

Question

Fix $k \geqslant 2$ and $d \geqslant 2$. Determine all of the integer solutions (x, y, n).

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$$x^k + (x+1)^k + \cdots + (x+d-1)^k = y^n.$$

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Fix $k \geqslant 2$ and $d \geqslant 2$. Determine all of the integer solutions (x, y, n).

Remark: We can let $n = p$ be a prime.

A Brief History: Sums of Consecutive Cubes

Euler:

$$6^3 = 3^3 + 4^3 + 5^3.$$

Dickson's *"History of the Theory of Numbers"*:

Catalan, Cunningham, Lucas and Gennochi.

Later contributions from:

1. Pagliani (1829): parametric solutions.
2. Uchiyama (1979): $d = 3, n = 2$ independently to Cassels.
3. Cassels (1985): $y^2 = x^3 + (x+1)^3 + (x+2)^3$.
4. Zhongfeng Zhang (2014): $y^n = x^3 + (x+1)^3 + (x+2)^3$.
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A Brief History

Well–Known:

$$\sum_{i=0}^{d} i^3 = \sum_{i=1}^{d} i^3 = \left(\frac{d(d+1)}{2}\right)^2 .$$

Pagliani:

$$\sum_{i=1}^{v^3} \left(\frac{v^4 - 3v^3 - 2v^2 - 2}{6} + i\right)^3 = \left(\frac{v^5 + v^3 - 2v}{6}\right)^3 .$$

where $v \equiv 2, 4 \pmod 6$.

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The Results

$$(x+1)^k + (x+2)^k + \cdots + (x+d)^k = y^n.$$

Theorem (M. A. Bennett, V. Patel, S. Siksek)

Let $k = 3$ and $2 \leqslant d \leqslant 50$. Then, any "non–trivial" integer solution (x, y, n) must have $n = 2$ or $n = 3$.

$$y \neq 0, \pm 1.$$

Without loss of any generality, we can let $x \geqslant 1$.

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Cubes that are Sums of Consecutive Cubes

$$3^3 + 4^3 + 5^3 = 6^3, \quad \text{attributed to Lucas}$$

$$11^3 + 12^3 + 13^3 + 14^3 = 20^3,$$

$$3^3 + 4^3 + 5^3 + \cdots + 22^3 = 40^3,$$

$$15^3 + 16^3 + 17^3 + \cdots + 34^3 = 70^3,$$

$$6^3 + 7^3 + 8^3 + \cdots + 30^3 = 60^3,$$

$$291^3 + 292^3 + 293^3 + \cdots + 339^3 = 1155^3.$$

Cubes that are Sums of Consecutive Cubes

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The Methodology

Step	Method	Number of Equations to Solve
1.	Useful equations and identities (p, p, p)	49 equations in (x, y, p)
2.	$p = 2$: Integer points on elliptic curves	49 equations in (x, y)
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4.	First descent: a factorisation for $p \geqslant 5$	906 equations in (x, y, p)
5.	Linear Forms in two logarithms: $p \geqslant 5$ Bounding $p < 3 \times 10^6$	$906 \times 216814 = 196,433,484$ equations in (x, y)
6.	Sophie-Germain type criterion (case $r \neq t$) $879 \times 216814 = 190,579,506$ in (x, y)	224 remain in (x, y)
7.	Modularity (case $r = t$) $27 \times 216814 = 5,853,978$ in (x, y)	53 remain in (x, y)
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	Equations remaining via 8., 6. and 7.	1219
9.	Local solubility tests	507
10.	A further descent	226
11.	Thue solver!	6 solutions found!

Step 1. (and 4.) is the Key!

1. By a (p, p, p) equation, we mean $Ax^p + By^p = Cz^p$.
2. Roughly speaking we have
   (Linear Factor in x)(Quadratic Factor in x) $= y^p$.
3. Linear Factor $= \alpha y_1^p$.
4. Quadratic Factor $=$ (Linear Factor)^2 + Constant $= \beta y_2^p$.
5. Substitution should give $\alpha^2 (y_1^2)^p + \text{Constant} \cdot 1^p = \beta (y_2)^p$

Step 2.

1. $p = 2$ solved by Stroeker (1995).
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The Magic of Sophie Germain

After Step 4. We have equations of the form:

$$r y_2^p - s y_1^{2p} = t \tag{1}$$

where r, s, t are positive integers, and $\gcd(r, s, t) = 1$.
The linear forms in two logarithms bounds p. For each tuple (r, s, t) we can apply the methods of Sophie Germain to eliminate equations/tuples for a fixed value of p.

The Magic of Sophie Germain

Lemma

Let $p \geqslant 3$ be prime. Let r, s and t be positive integers satisfying $\gcd(r,s,t)=1$. *Let $q = 2kp+1$ be a prime that does not divide r. Define*

$$\mu(p,q) = \{\eta^{2p} \; : \; \eta \in \mathbb{F}_q\} = \{0\} \cup \{\zeta \in \mathbb{F}_q^* \; : \; \zeta^k = 1\} \qquad (2)$$

and

$$B(p,q) = \left\{\zeta \in \mu(p,q) \; : \; ((s\zeta + t)/r)^{2k} \in \{0,1\}\right\}.$$

If $B(p,q) = \emptyset$, then equation (1) *does not have integral solutions.*

The Magic of Sophie Germain

Proof.

Suppose $B(p,q)=\emptyset$. Let (y_1,y_2) be a solution to (1). Let $\zeta=\overline{y_1}^{2p}\in\mu(p,q)$. From equation (1) we have

$$(s\zeta+t)/r\equiv y_2^p \mod q.$$

Thus

$$((s\zeta+t)/r)^{2k}\equiv y_2^{q-1}\equiv 0 \text{ or } 1 \mod q.$$

This shows that $\zeta\in B(p,q)$ giving a contradiction. $\square$

The Magic of Sophie Germain - Why Does it Work?

1. If there are no solutions to $ry_2^p - sy_1^{2p} = t$,
2. and we take p to be large, then
3. notice that $\#\mu(p,q) = k+1$.
4. For $\zeta \in \mu(p,q)$, the element $((s\zeta + t)/r)^{2k} \in \mathbb{F}_q$ is either 0 or an p-th root of unity.
5. The "probability" that it belongs to the set $\{0,1\}$ is $2/(p+1)$.
6. The "expected size" of $B(p,q)$ is $2(k+1)/(p+1) \approx 2q/p^2$.
7. For large p we expect to find a prime $q = 2kp+1$ such that $2q/p^2$ is tiny and so we likewise expect that $\#B(p,q) = 0$.

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The Modular Way! $(r = t)$

$$ry_2^p - sy_1^{2p} = t$$

$$y_2^p - (s/r)y_1^{2p} = 1$$

Has solutions $(y_1, y_2) = (0, 1)$. This causes our previous lemma to fail.
However, the Modular Method does not see this solution. When constructing the Frey Curve, the discriminant is non-zero. Hence if $y_1 = 0$ then the discriminant is zero. (Similar to Fermat's Last Theorem).

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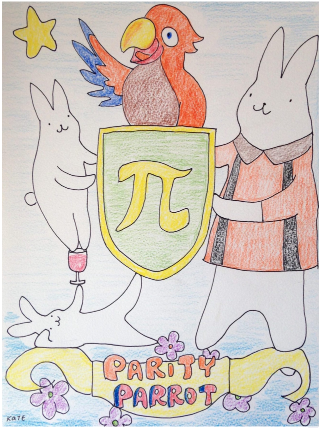
Pieter's Parity Parrot: Designed by Pieter Moree, Drawn by Kate Kattegat



THE CASE $k = 2$

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Dimensions of $S_2(N)$

When $k = 2$...

$$d = 22, \quad \dim = 5280$$

Dimension 200 is reasonable to compute with. We can push computations to dimension 2000 with some clever tricks.

When $k = 4$...

$$d = 21, \quad \dim \approx 1,500,000$$

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Linear Forms in Three Logarithms

If I try... naively

$$\approx 10^{20}$$

If Mike Bennett tries... naively

$$\approx 10^{14}$$

If we manage to locate Mike Bennett and then get him to work...

$$\approx 10^{10}$$

which also needs a lot of luck!! ♣♣♣♣♣♣♣

Linear Forms in Three Logarithms

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$$\approx 10^{20}$$

If Mike Bennett tries... naively

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If we manage to locate Mike Bennett and then get him to work...

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which also needs a lot of luck!! 🍀🍀🍀🍀🍀🍀🍀

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THE CASE $k = 2$

Step	Method	Number of Equations to Solve	$k = 2$
1.	Useful equations and identities (p, p, p)	49 equations in (x, y, p)	$(p, p, 2)$ ✓
2.	$p = 2$: Integer points on elliptic curves	49 equations in (x, y)	∞ ✓
3.	$d = 2$: Results of Nagell	2 equations (x, y, p)	
4.	First descent: a factorisation for $p \geqslant 5$	906 equations in (x, y, p)	✗
5.	Linear Forms in two logarithms: $p \geqslant 5$ Bounding $p < 3 \times 10^6$	$906 \times 216814 = 196,433,484$ equations in (x, y)	☠
6.	Sophie-Germain type criterion (case $r \neq t$) $879 \times 216814 = 190,579,506$ in (x, y)	224 remain in (x, y)	✝
7.	Modularity (case $r = t$) $27 \times 216814 = 5,853,978$ in (x, y)	53 remain in (x, y)	Levels too big!! ✗
8.	First descent when $p = 3$	942 in (x, y)	
	Equations remaining via 8., 6. and 7.	1219	
9.	Local solubility tests	507	
10.	A further descent	226	
11.	Thue solver!	6 solutions found!	

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Pythagoras

$$3^2 + 4^2 = 5^2$$

$$20^2 + 21^2 = 29^2$$

An infinite family of solutions - can be given parametrically!

Even k and Towards Densities

Theorem (Zhang and Bai, 2013)

Let q be a prime such that $q \equiv 5, 7 \pmod{12}$. Suppose $q \parallel d$. Then the equation $x^2 + (x+1)^2 + \cdots + (x+d-1)^2 = y^n$ has no integer solutions.

Corollary (Use Dirichlet's Theorem)

Let $\mathcal{A}_2$ be the set of integers $d \geqslant 2$ such that the equation

$$x^2 + (x+1)^2 + \cdots + (x+d-1)^2 = y^n$$

has a solution (x, y, n). Then $\mathcal{A}_2$ has natural density zero.

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The Result

Theorem (V. Patel, S. Siksek)

Let $k \geqslant 2$ be an even integer. Let $\mathcal{A}_k$ be the set of integers $d \geqslant 2$ such that the equation

$$x^k + (x+1)^k + \cdots (x+d-1)^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geqslant 2$$

has a solution (x, y, n). Then $\mathcal{A}_k$ has natural density zero. In other words we have

$$\lim_{X \to \infty} \frac{\#\{d \in \mathcal{A}_k : d \leqslant X\}}{X} = 0.$$

The Result

Theorem (V. Patel, S. Siksek)

Let $k \geqslant 2$ be an even integer and let r be a non-zero integer. Let $\mathcal{A}_{k,r}$ be the set of integers $d \geqslant 2$ such that the equation

$$x^k + (x+r)^k + \cdots (x + r(d-1))^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geqslant 2$$

has a solution (x, y, n). Then $\mathcal{A}_{k,r}$ has natural density zero. In other words we have

$$\lim_{X \to \infty} \frac{\#\{d \in \mathcal{A}_{k,r} : d \leqslant X\}}{X} = 0.$$

Bernoulli polynomials and relation to sums of consecutive powers

Definition (Bernoulli Numbers, b_k)

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}.$$

$b_0 = 1, b_1 = -1/2, b_2 = 1/6, b_3 = 0, b_4 = -1/30, b_5 = 0, b_6 = 1/42.$

Lemma

$b_{2k+1} = 0$ *for* $k \geqslant 1$.

Bernoulli polynomials and relation to sums of consecutive powers

Definition (Bernoulli Polynomial, B_k)

$$B_k(x) := \sum_{m=0}^{k} \binom{k}{m} b_m x^{k-m}.$$

Lemma

$$x^k + (x+1)^k + \cdots + (x+d-1)^k = \frac{1}{k+1}\left(B_{k+1}(x+d) - B_k(x)\right).$$

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Apply Taylor's Theorem and use $B'_{k+1}(x) = (k+1) \cdot B_k(x)$.

Lemma

Let $q \geqslant k+3$ be a prime. Let $d \geqslant 2$. Suppose that $q \mid d$. Then

$$x^k + (x+1)^k + \cdots + (x+d-1)^k \equiv d \cdot B_k(x) \pmod{q^2}.$$

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Bernoulli polynomials and relation to sums of consecutive powers

$$x^k + (x+1)^k + \cdots + (x+d-1)^k = y^n.$$

Proposition (Criterion)

Let $k \geqslant 2$. Let $q \geqslant k+3$ be a prime such that the congruence $B_k(x) \equiv 0 \pmod{q}$ has no solutions. Let d be a positive integer such that $\mathrm{ord}_q(d) = 1$. Then the equation has no solutions. (i.e. $d \notin \mathcal{A}_k$).

Remark: Computationally we checked $k \leqslant 75{,}000$ and we could always find such a q.

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Relation to Densities?

We need to use Chebotarev's density theorem, which can be seen as "a generalisation of Dirichlet's theorem" on primes in arithmetic progression.

Proposition

Let $k \geqslant 2$ be even and let G be the Galois group of $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

Assuming the proposition, we may then use Chebotarev's density theorem to find a set of primes q_i with positive Dirichlet density such that $\mathrm{Frob}_{q_i} \in G$ is conjugate to μ. Then we can apply Niven's results to deduce our Theorem.

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Niven's Results (Flash!)

The setup:

1. Let $\mathcal{A}$ be a set of positive integers.
2. Define: $\mathcal{A}(X) = \#\{d \in \mathcal{A} : d \leqslant X\}$ for positive X.
3. Natural Density: $\delta(\mathcal{A}) = \lim_{X\to\infty} \mathcal{A}(X)/X$.
4. Given a prime q, define: $\mathcal{A}^{(q)} = \{d \in \mathcal{A} : \mathrm{ord}_q(d) = 1\}$.

Theorem (Niven)

Let $\{q_i\}$ be a set of primes such that $\delta(\mathcal{A}^{(q_i)}) = 0$ and $\sum q_i^{-1} = \infty$. Then $\delta(\mathcal{A}) = 0$.

A Legendre Symbol analogue

Proposition

Let $k \geqslant 2$ be even and let G be the Galois group $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

Conjecture

For any even integer k, $B_k(x)$ is irreducible over $\mathbb{Q}$.

Remark: The conjecture implies the Proposition. This then proves our Theorem.

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Tough Stuff

A sketch of an unconditional proof!

Proposition

Let $k \geqslant 2$ be even and let G be the Galois group $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

Theorem (von Staudt-Clausen)

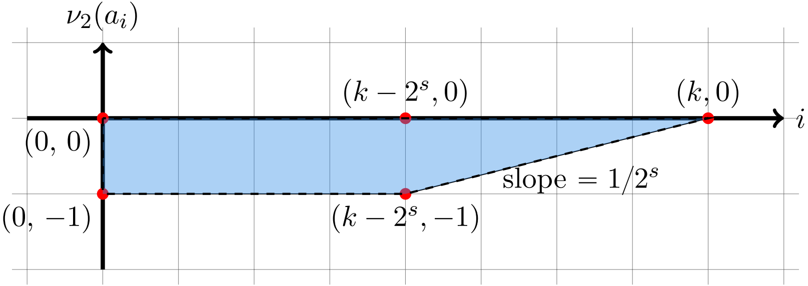
Let $n \geqslant 2$ be even. Then

$$b_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z}.$$

2 is the Oddest Prime

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geqslant 1$.

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{k-i} b_{k-i} x^i = \sum_{i=0}^{k} a_i x^i$$



Another nice result

1. Sloping part corresponds to irreducible factor over $\mathbb{Q}_2$.
2. Root in $\mathbb{Q}_2$ must have valuation zero.
3. Root belongs to $\mathbb{Z}_2$ and is odd.
4. Symmetry $(-1)^k B_k(x) = B_k(1-x)$ gives a contradiction.

Theorem (V. Patel, S. Siksek)

Let $k \geqslant 2$ be an even integer. Then $B_k(x)$ has no roots in $\mathbb{Q}_2$.

Theorem (K. Inkeri, 1959)

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What is Going On?

$$
\begin{tikzcd}
L = \text{Splitting Field of } B_k(x) \arrow[d, dash, "\text{G} = \text{Galois Group}"'] & L_{\mathfrak{P}} \arrow[d, dash, "H \subset G"'] & \mathbb{F}_{\mathfrak{P}} \arrow[d, dash, "\text{C} = \text{Cyclic}"'] \\
\mathbb{Q} & \mathbb{Q}_2 & \mathbb{F}_2 = \text{Residue Field}
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μ lives here!

A sketch proof of the Proposition

The Setup:

- $k \geqslant 2$ is even.
- L is the splitting field of $B_k(x)$.
- G is the Galois group of $B_k(x)$.
- $\mathfrak{P}$ be a prime above 2.
- ν_2 on $\mathbb{Q}_2$ which we extend uniquely to $L_{\mathfrak{P}}$ (also call it ν_2).
- $H = \mathrm{Gal}(L_{\mathfrak{P}}/\mathbb{Q}_2) \subset G$ be the decomposition subgroup corresponding to $\mathfrak{P}$.

A sketch proof of the Proposition

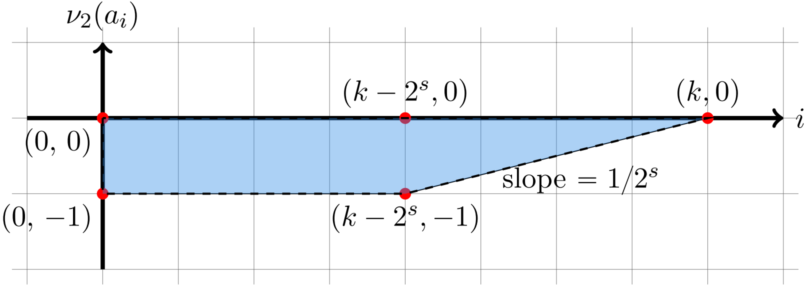
$$B_k(x) = g(x)h(x)$$

where $g(x)$ has degree $k - 2^s$. Label the roots $\{\alpha_1, \ldots, \alpha_{k-2^s}\}$, and $h(x)$ has degree 2^s. Label the roots $\{\beta_1, \ldots, \beta_{2^s}\}$.

- All roots $\subset L_\beta$.
- $h(x)$ is irreducible.
- Therefore H acts transitively on β_j.
- Pick $\mu \in H$ such that μ acts freely on the roots of $h(x)$.
- Check it doesn't end up fixing a root of $g(x)$.

"Bad Prime = Extremely Useful Prime!"

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geqslant 1$.



Finding μ

Lemma

Let H be a finite group acting transitively on a finite set $\{\beta_1, \ldots, \beta_n\}$. Let $H_i \subset H$ be the stabiliser of β_i and suppose $H_1 = H_2$. Let $\pi : H \to C$ be a surjective homomorphism from H onto a cyclic group C. Then there exists some $\mu \in H$ acting freely on $\{\beta_1, \ldots, \beta_n\}$ such that $\pi(\mu)$ is a generator of C.

1. Let $\mathbb{F}_{\mathfrak{P}}$ be the residue field of $\mathfrak{P}$.
2. Let $C = \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_2)$.
3. C is cyclic generated by the Frobenius map: $\bar{\gamma} \to \bar{\gamma}^2$.
4. Let $\pi : H \to C$ be the induced surjection.
5. Finally use the Lemma.

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4. Let $\pi : H \to C$ be the induced surjection.
5. Finally use the Lemma.

Check $g(x)$

$$B_k(x) = g(x)h(x)$$

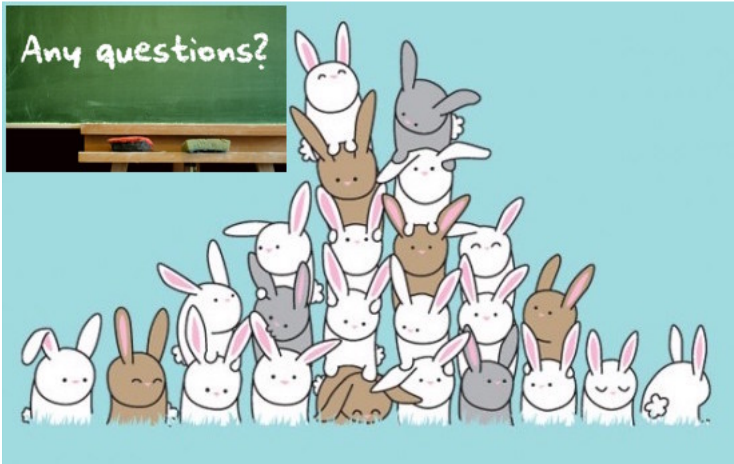
where $g(x)$ has degree $k - 2^s$. Label the roots $\{\alpha_1, \dots, \alpha_{k-2^s}\}$, and $h(x)$ has degree 2^s. Label the roots $\{\beta_1, \dots, \beta_{2^s}\}$.

Lemma

μ acts freely on the α_i.

1. Suppose not. Let α be a root that is fixed by μ.
2. $\nu_2(\alpha) = 0$ so let $\bar{\alpha} = \alpha \pmod{\mathfrak{P}}$, $\bar{\alpha} \in \mathbb{F}_{\mathfrak{P}}$.
3. α fixed by μ hence $\bar{\alpha}$ fixed by $\langle \pi(\mu) \rangle = C$.
4. Hence $\bar{\alpha} \in \mathbb{F}_2$. $f(x) = 2B_k(x) \in \mathbb{Z}_2[x]$.
5. $f(\bar{1}) = f(\bar{0}) = \bar{1}$. A contradiction!

Thank you for Listening!



Solving the equations for $k = 2$

$$d\left(\left(x+\frac{d+1}{2}\right)^2+\frac{(d-1)(d+1)}{12}\right)=y^p.$$

$$X^2 + C \cdot 1^p = (1/d)y^p$$

Solving the equations for $k = 2$

d	Equation	Level	Dimension
6	$2y^p - 5 \times 7 = 3(2x+7)^2$	$2^7 \times 3^2 \times 5 \times 7$	480
11	$11^{p-1}y^p - 2 \times 5 = (x+6)^2$	$2^7 \times 5 \times 11$	160
13	$13^{p-1}y^p - 2 \times 7 = (x+7)^2$	$2^7 \times 7 \times 13$	288
22	$2 \times 11^{p-1}y^p - 7 \times 23 = (2x+23)^2$	$2^7 \times 7 \times 11 \times 23$	5,280
23	$23^{p-1}y^p - 2^2 \times 11 = (x+12)^2$	$2^3 \times 11 \times 23$	54
26	$2 \times 13^{p-1}y^p - 3^2 \times 5^2 = (2x+27)^2$	$2^7 \times 3 \times 5 \times 13$	384
33	$11^{p-1}y^p - 2^4 \times 17 = 3(x+17)^2$	$2^3 \times 3^2 \times 11 \times 17$	200
37	$37^{p-1}y^p - 2 \times 3 \times 19 = (x+19)^2$	$2^7 \times 3 \times 19 \times 37$	5,184
39	$13^{p-1}y^p - 2^2 \times 5 \times 19 = 3(x+20)^2$	$2^3 \times 3^2 \times 5 \times 13 \times 19$	1,080
46	$2 \times 23^{p-1}y^p - 3^2 \times 5 \times 47 = (2x+47)^2$	$2^7 \times 3 \times 5 \times 23 \times 47$	32,384
47	$47^{p-1}y^p - 2^3 \times 23 = (x+24)^2$	$2^5 \times 23 \times 47$	1,012
59	$59^{p-1}y^p - 2 \times 5 \times 29 = (x+30)^2$	$2^7 \times 5 \times 29 \times 59$	25,984

Solving the equations for $k = 4$

d	Equation	Level	Dimension
5	$y^p + 2 \times 73 = 5(X)^2$	$2^7 \times 5^2 \times 73$	5,472
6	$y^p + 7 \times 53 = 6(X)^2$	$2^8 \times 3^2 \times 7 \times 53$	12,480
7	$7^{p-1} y^p + 2^2 \times 29 = (X)^2$	$2^3 \times 7 \times 29$	42
10	$y^p + 3 \times 11 \times 149 = 10(X)^2$	$2^8 \times 5^2 \times 3 \times 11 \times 149$	449,920
13	$13^{p-1} y^p + 2 \times 7 \times 101 = (X)^2$	$2^7 \times 7 \times 13 \times 101$	28,800
14	$7^{p-1} y^p + 13 \times 293 = 2(X)^2$	$2^8 \times 7 \times 13 \times 293$	168,192
15	$y^p + 2^3 \times 7 \times 673 = 15(X)^2$	$2^5 \times 3^2 \times 5^2 \times 7 \times 673$	383,040
17	$17^{p-1} y^p + 2^3 \times 3 \times 173 = (X)^2$	$2^5 \times 3 \times 17 \times 173$	5,504
19	$19^{p-1} y^p + 2 \times 3 \times 23 \times 47 = (X)^2$	$2^7 \times 3 \times 19 \times 23 \times 47$	145,728
21	$7^{p-1} y^p + 2 \times 11 \times 1321 = 3(X)^2$	$2^7 \times 3^2 \times 7 \times 11 \times 1321$	1,584,000
26	$13^{p-1} y^p + 3^2 \times 5 \times 1013 = 2(X)^2$	$2^8 \times 3 \times 5 \times 13 \times 1013$	777,216
29	$29^{p-1} y^p + 2 \times 7 \times 2521 = (X)^2$	$2^7 \times 7 \times 29 \times 2521$	1,693,440
30	$y^p + 19 \times 29 \times 31 \times 71 = 30(X)^2$	$2^8 \times 3^2 \times 5^2 \times 19 \times 29 \times 31 \times 71$	804,384,000

Where X is a quadratic in the original variable x.