FIRST YEAR PHD REPORT

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ABSTRACT. We look at some potential links between totally real number fields and some theta expansions (these being modular forms). The literature related to modular forms is rich, and any links made to totally real number fields could help us to understand the number field better.

CONTENTS

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1. Introduction

The four squares problem, first posed in the *Arithmetica of Diophantus*, states that any positive integer n can be represented as the sum of four integer squares. In mathematical notation, we write that there exists integers w, x, y and z such that $n = w^2 + x^2 + y^2 + z^2$. In 1770, Lagrange gave a concrete proof for this conjecture. In 1834, Carl Gustav Jacobi looked further into the four squares problem and found an exact formula for the total number of ways a positive integer n can be represented as the sum of four squares. His formula can be seen below:

$$r_4(n) = 8 \sum_{m|n, 4\nmid m} m.$$

Naturally we are then led to asking the question: given positive integers n and k, in how many ways can n be written as a sum of k integer squares? In other words we are asked to find,

$$r_k(n) = \#\{(x_1, \dots, x_k) \in \mathbb{Z}^k : x_1^2 + \cdots + x_k^2 = n\},$$

where we call the $r_k(n)$ the *representation number of n.* Lagrange did not have access to the theory of modular forms during his breakthrough, and his proof arose through the use of classical methods. However, with the use of modular forms, one can find some very nice formulae for the representation numbers, $r_k(n)$.

The Jacobi Theta Function (named after Carl Gustav Jacobi since he was primarily the one investigating them), is defined as the following,

$$\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

where $q = e^{2\pi i z}$. We can extend this definition to associate a theta function to more generalised representation numbers, namely by the following construction,

$$\Theta_k(z) = \sum_{n=0}^{\infty} r_k(n) q^n$$

and it turns out that these constructions are indeed modular forms.

The next natural question to ask is whether we can generalise this further to any number field F, where we look at the representation numbers, $R_F(n)$ as the set of integral solutions to some positive definite quadratic form, which is constructed with respect to the number field F. In mathematical notation, we can write,

$$R_F(n) = \#\{(x_1, \dots, x_k) \in \mathbb{Z}^k : Q(x_1, \dots, x_k) = n\}$$

where n and k are positive integers, and k is defined to be the degree of the number field. We also have Q as a positive definite quadratic form in k variables, with integral coefficients.

Analogous to the constructions of Jacobi, we can construct further theta series by the following,

$$\Theta_F(z) = \sum_{n=0}^{\infty} R_F(n) q^n$$

where once again, we have $q = e^{2\pi i z}$.

The main question posed now is: given a number field, $F = \mathbb{Q}(\theta)$ for some algebraic integer θ, can we find a unique modular form associated to it? Moreover given a modular form, can we then find the unique number field associated to it?

The motivation behind such a question is that in forming such links, should they have a one-to-one correspondence, we can make use of the extensive literature and theory related to modular forms to then perhaps hope to understand generalised number fields a bit better. However, should one not be able to establish such links, then the question would then be whether any links between generalised number fields and modular forms can tell us anything at all about initial number field i.e. can we perhaps group certain number fields together, as if almost ‘classifying’ them in some sense.

In this report, we shall be looking primarily at three main topics in number theory. We cover some algebraic number theory, some theory about integral solutions to quadratic forms, and some theory about modular forms.

We primarily use [Stewart and Tall(1987)] as our main source to recall definitions and notation related to algebraic number theory.

We shall also be needing some theory related to modular forms. [Diamond and Shurman(2005)] is an excellent text for a beginner to the course, and [Hanke(2013)] provides us with some useful results to relate quadratic forms and modular forms. We shall summarise some of the main results from both of these sources in Section 4 - Background Material.

2. Construction of a generalised Theta expansion

We let K be a totally real number field of degree n. Then $K = \mathbb{Q}(\theta)$ for some algebraic integer θ. We denote the ring of integers of the field K by $\mathcal{O}_K$. Now let $\sigma_1, \dots, \sigma_n$ be the n distinct monomorphisms of K such that $\sigma_i : K \hookrightarrow \mathbb{R}$ for $i = 1, \dots, n$.

We can construct a positive definite quadratic form which we denote as Q as stated below using an element $\alpha \in \mathcal{O}_K$:

$$Q = [\sigma_1(\alpha)]^2 + [\sigma_2(\alpha)]^2 + \cdots + [\sigma_n(\alpha)]^2.$$

Note here that we write the element α in terms of a $\mathbb{Z}$-basis, i.e. we let $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ be a set of generators of $\mathcal{O}_K$ and so we have $\alpha = a_1\zeta_1 + a_2\zeta_2 + \cdots + a_n\zeta_n$ where $a_i \in \mathbb{Z}$ for all $i = 1, \dots, n$. Thus the quadratic form Q has variables $a_1, \dots, a_n$ and the coefficients of Q are in $\mathbb{Z}$.

We can count the number of integer solutions to the quadratic form $Q = m$ where $m \in \mathbb{N}$. We denote the number of integral solutions as $R_Q(m)$, i.e.

$$R_Q(m) = \#\{\underline{\mathbf{a}} \in \mathbb{Z}^m | Q(\underline{\mathbf{a}}) = m\}.$$

We now look at a series expansion where we let $R_Q(m)$ be the coefficients.

$$\Theta(z) = \sum_{m=0}^{\infty} R_Q(m) \cdot q^m$$

where $q = e^{2\pi i z}$. This is in fact a modular form, and we shall call it a theta expansion from here on.

3. A Worked Example - Quadratic Fields

In this section, we shall explicitly look at $K = \mathbb{Q}(\sqrt{d})$ where d is squarefree and a strict positive integer. Here, K is a totally real number field of degree 2. To do so, we follow the steps outlined in the introduction to attempt to arrive at the theta series for the totally real number field K.

We shall need a very useful theorem, as stated and proved in [Stewart and Tall(1987)], page 67.

Theorem 3.1. *Let d be a squarefree rational integer (i.e. $d \in \mathbb{Z}$). Then the integers of $K = \mathbb{Q}(\sqrt{d})$, which we denote as $\mathcal{O}_K$, are:*
(a) $\mathbb{Z}[\sqrt{d}]$ *if* $d \not\equiv 1 \pmod 4$
(b) $\mathbb{Z}\left[\frac{1}{2} + \frac{1}{2}\sqrt{d}\right]$ *if* $d \equiv 1 \pmod 4$.

We should remark at this stage that we shall call an element of $\mathbb{Z}$ a *rational integer*, and an element of $\mathcal{O}_K$ an *integer*.

3.1. Case $d \not\equiv 1 \pmod 4$.

First, we shall consider the totally real number field $K = \mathbb{Q}(\sqrt{d})$ in the case where $d \not\equiv 1 \pmod 4$. In this case, we have $\mathcal{O}_K = \langle 1, \sqrt{d} \rangle$ and so any element $\alpha \in \mathcal{O}_K$ can be written as

$$\alpha = a_1 + a_2\sqrt{d}$$

with $a_1, a_2 \in \mathbb{Z}$.

Recall that the distinct monomorphisms of K are:

- $\sigma_1 : K \longmapsto \mathbb{R}$
  $\sigma_1(a + b\sqrt{d}) = a + b\sqrt{d}$
- $\sigma_2 : K \longmapsto \mathbb{R}$
  $\sigma_1(a + b\sqrt{d}) = a - b\sqrt{d}$

where $a, b \in \mathbb{Q}$ and the embeddings map to $\mathbb{R}$ since we are looking at a totally real number field.

Now we can construct a generalised quadratic form Q as follows:

$$\begin{aligned} Q(a_1, a_2) &= [\sigma_1(\alpha)]^2 + [\sigma_2(\alpha)]^2 \\ &= [a_1 + a_2\sqrt{d}]^2 + [a_1 - a_2\sqrt{d}]^2 \\ &= 2a_1^2 + 2da_2^2 \\ &= [\mathrm{Tr}(\alpha)]^2 - 2\mathrm{Nm}(\alpha). \end{aligned} \tag{3.1}$$

where we define the Trace (Tr) and Norm (Nm) as follows:

Definition 3.1. *Let $\alpha \in K$ where K is a number field of degree n. Let σ_i for $i = 1, \ldots, n$ be the n distinct monomorphisms of K. Then we define the Norm and Trace as follows:*

$$\mathrm{Tr}(\alpha) = \sum_i \sigma_i(\alpha)$$

$$\mathrm{Nm}(\alpha) = \prod_i \sigma_i(\alpha).$$

In this specific case, we can calculate the norm and trace of an arbitrary element $\alpha \in \mathcal{O}_K$ to deduce that the identity given in Equation (3.1) is true.

$$\begin{aligned}\mathrm{Tr}(\alpha) &= \sum_{i=1}^{2} \sigma_i(\alpha) \\ &= a_1 + a_2\sqrt{d} + a_1 - a_2\sqrt{d} \\ &= 2a_1.\end{aligned}$$

$$\begin{aligned}\mathrm{Nm}(\alpha) &= \prod_{i=1}^{2} \sigma_i(\alpha) \\ &= (a_1 + a_2\sqrt{d})(a_1 - a_2\sqrt{d}) \\ &= 2a_1^2 - 2da_2^2.\end{aligned}$$

Next, we are to calculate the number of integer solutions to $Q(a_1, a_2) = m$ for all rational integers m, i.e. we need to calculate the $R_Q(m)$.

Let us look at a specific example now. Let us choose $d = 3$, then we are looking for the number of integer solutions to the equation:

$$2a_1^2 + 6a_2^2 = m$$

which we can rewrite as:

$$a_1^2 + 3a_2^2 = \mu.$$

We show the first few terms for $R_Q(\mu)$ in the table below.

TABLE 1. The first few terms for $R_Q(\mu)$

$Q(a_1, a_2) = \mu$			$R_Q(\mu)$
$a_1^2 + 3a_2^2$	$=$	0	1
$a_1^2 + 3a_2^2$	$=$	1	2
$a_1^2 + 3a_2^2$	$=$	2	0
$a_1^2 + 3a_2^2$	$=$	3	2
$a_1^2 + 3a_2^2$	$=$	4	6
$a_1^2 + 3a_2^2$	$=$	5	0
$a_1^2 + 3a_2^2$	$=$	6	0
$a_1^2 + 3a_2^2$	$=$	7	4
$a_1^2 + 3a_2^2$	$=$	8	0
$a_1^2 + 3a_2^2$	$=$	9	2
$a_1^2 + 3a_2^2$	$=$	10	0

Notice that we have the relation $R_Q(2\mu) = R_Q(m)$, and of course $R_Q(m) = 0$ when m is odd.

We can construct the function:

$$\begin{aligned}\Theta_Q(z) &:= \sum_{m=0}^{\infty} R_Q(m)q^m = \sum_{\mu=0}^{\infty} R_Q(2\mu)q^{2\mu} \\ &\approx 1 + 2q^2 + 0q^4 + 2q^6 + 6q^8 + 0q^{10} + 0q^{12} + 4q^{14} + 0q^{16} + 2q^{18} + 0q^{20} + \cdots\end{aligned}$$

This is indeed a modular form, of weight $k = 1$ and Level N. We shall see a rigorous argument as to why this is in Section 4 - Background Material.

3.2. **Case $d \equiv 1 \pmod 4$.** We now move on to consider the totally real number field $K = \mathbb{Q}(\sqrt{d})$ in the case where $d \equiv 1 \pmod 4$. In this case, we have $\mathcal{O}_K = \langle 1, \frac{1}{2} + \frac{1}{2}\sqrt{d}\rangle$ and so any element $\alpha \in \mathcal{O}_K$ can be written as

$$\alpha = a_1 + a_2\left(\frac{1}{2} + \frac{1}{2}\sqrt{d}\right)$$

with $a_1, a_2 \in \mathbb{Z}$.

The distinct monomorphisms of K remain the same as the case where $d \not\equiv 1 \pmod 4$ and we shall still call them σ_1 and σ_2.

Now we can construct a generalised quadratic form Q as follows:

$$
\begin{aligned}
Q(a_1, a_2) &= [\sigma_1(\alpha)]^2 + [\sigma_2(\alpha)]^2 \\
&= \left[a_1 + a_2\left(\frac{1}{2} + \frac{1}{2}\sqrt{d}\right)\right]^2 + \left[a_1 + a_2\left(\frac{1}{2} - \frac{1}{2}\sqrt{d}\right)\right]^2 \\
&= 2a_1^2 + 2a_1a_2 + \frac{(1+d)}{2}a_2^2 \\
&= [Tr]^2 - 2Nm. \qquad (3.2)
\end{aligned}
$$

where we note that $Q(a_1, a_2)$ is a quadratic form with integer coefficients. Since $d \equiv 1 \pmod 4$, this implies that $(1+d)/2 \in \mathbb{Z}$.

Again, we can calculate the norm and trace of an arbitrary element $\alpha \in \mathcal{O}_K$ to deduce that the identity given in Equation (3.2) is true.

$$
\begin{aligned}
\mathrm{Tr}(\alpha) &= \sum_{i=1}^{2} \sigma_i(\alpha) \\
&= a_1 + a_2\left(\frac{1+\sqrt{d}}{2}\right) + a_1 + a_2\left(\frac{1-\sqrt{d}}{2}\right) \\
&= 2a_1 + a_2.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Nm}(\alpha) &= \prod_{i=1}^{2} \sigma_i(\alpha) \\
&= \left(a_1 + a_2\left(\frac{1+\sqrt{d}}{2}\right)\right)\left(a_1 + a_2\left(\frac{1-\sqrt{d}}{2}\right)\right) \\
&= a_1^2 + a_1a_2 + \frac{(1-d)}{4}a_2^2.
\end{aligned}
$$

To summarise, we have the following: Let K be a totally real number field, with $K = \mathbb{Q}(\sqrt{d})$ with $d > 0$ being a squarefree rational integer. Then the quadratic form associated to K is

$$Q(a_1, a_2) = [Tr]^2 - 2Nm.$$

Next, we are to calculate the number of integer solutions to $Q(a_1, a_2) = m$ for all rational integers m, i.e. we need to calculate the $R_Q(m)$.

Let us look at a specific example now. Let us choose $d = 5$, then we are looking for the number of integer solutions to the equation:

$$2a_1^2 + 2a_1a_2 + 3a_2^2 = m.$$

We show the first few terms for $R_Q(m)$ in the table below.

TABLE 2. The first few terms for $R_Q(m)$

$Q(a_1, a_2) = m$			$R_Q(m)$
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	0	1
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	1	0
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	2	2
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	3	4
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	4	0
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	5	0
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	6	0
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	7	4
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	8	2
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	9	0
$2a_1^2 + 2a_1a_2 + 3a_2^2$	$=$	10	2

We can construct the function:

$$\begin{aligned}\Theta_Q(z) &:= \sum_{m=0}^{\infty} R_Q(m)q^m \\ &\approx 1 + 0q + 2q^2 + 4q^3 + 0q^4 + 0q^5 + 0q^6 + 4q^7 + 2q^8 + 0q^9 + 2q^{10} + \cdots\end{aligned}$$

Again, we see that this is a modular form of weight $k = 1$ and level N. We shall see a rigorous argument as to why this is in Section 4 - Background Material.

Some further questions which arise through this worked example are:

- A proof which shows that using the representation numbers as the coefficients of some type of Fourier expansion is indeed a modular form.
- Generalisation to other totally real number fields?

4. Background Material

To make further progress with the problem we shall be needing some theory related to modular forms. We primarily use [Diamond and Shurman(2005)] for basic definitions, and [Hanke(2013)] for details on the relationship between quadratic forms and modular forms.

4.1. Some Basic Definitions and Notation.

We begin by outlining some basic definitions, notation and theorems to form a foundation for us to then understand and define a modular form.

Definition 4.1. *Let* $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ *denote the complex upper half plane.*

Definition 4.2. *Let* $\gamma \in \mathrm{SL}_2(\mathbb{R})$ *where* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *We define the map*

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} &\mapsto \mathbb{H} \\ (\gamma, z) &\mapsto \gamma \cdot z = \frac{az+b}{cz+d} \end{aligned}$$

which is the group action of $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$.

Definition 4.3. *Let f be a meromorphic function on $\mathbb{H}$, and let $k \in \mathbb{Z}$. We say that f is weakly modular of weight k if:*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Some notation: We have the slash action,

$$(f|_k\gamma)(z) = (cz+d)^{-k} \cdot f\left(\frac{az+b}{cz+d}\right).$$

Definition 4.4. *A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which is weakly modular of weight k and holomorphic at infinity.*

A modular form can be expressed as a convergent power series (Laurent Series) which is usually called the q-expansion of f,

$$f(z) := \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$.

Definition 4.5. *Let N be a positive integer. The principal congruence subgroup of level N is the group:*

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \middle| \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

A congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$ for some $N \geq 1$. The least such N is called the LEVEL of Γ.

Definition 4.6. *Let Γ be a congruence subgroup. The set of CUSPS of Γ is the set $Cusps(\Gamma) := \Gamma \setminus \mathbb{P}^1(\mathbb{Q})$ of Γ-orbits in $\mathbb{P}^1(\mathbb{Q})$. Some notation: $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ is called the projective line over $\mathbb{Q}$.*

Definition 4.7. *Let f be a meromorphic function on $\mathbb{H}$, let Γ be a congruence subgroup and let k be an integer. We say that f is weakly modular of weight k for the group Γ (of level Γ or level N) if f satisfies $f|_k\gamma = f$ for all $\gamma \in \Gamma$.*

A modular form of weight k for the group Γ is a holomorphic function, $f : \mathbb{H} \to \mathbb{C}$, weakly modular of weight k for Γ and holomorphic at all cusps of Γ. If f vanishes at all cusps of Γ then it is called a CUSP FORM (of weight k for Γ).

4.2. **Relating Quadratic Forms and Modular Forms.** Recall that Q is a positive definite quadratic form by construction. Then we know that $R_Q(m) < \infty$, which we shall need to construct meaningful modular forms.

Then, we can define the Theta Expansion of Q as a series expansion:

$$\Theta_Q(z) := \sum_{m=0}^{\infty} R_Q(m) e^{2\pi i z m} = \sum_{m=0}^{\infty} R_Q(m) q^m.$$

Firstly, we shall need to show that this series has some properties with respect to convergence, and the following Lemma helps us to establish this.

Lemma 4.1. *The Fourier Series $f(z) := \sum_{m=0}^{\infty} a_m q^m$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$ to a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ if all of the coefficients, $a_m \in \mathbb{C}$ satisfy $|a_m| \leq Cm^r$ for some constant $C > 0$ and some $r > 0$.*

For the proof, one can use [Hanke(2013)] which in turn references [Miyake and Maeda(1989)] - Lemma 4.3.3 page 117.

Theorem 4.1. *The Theta series, $\Theta_Q(z)$ of a positive definite integer-valued quadratic form Q converges absolutely and uniformly to a holomorphic function $f : \mathbb{H} \to \mathbb{C}$.*

Proof. Since Q is a positive definite quadratic form, we know that the number of integer solutions to Q are bounded. The solutions correspond to the number of lattice points in a smooth bounded region of $\mathbb{R}^n$ and so we have $\sum_{i=0}^{M} R_Q(i) < CM^n$ for some constant C. Therefore, we have $R_Q(m) < C_1 M^{n-1}$ for some constant C_1, and by Lemma 4.1, we have $\Theta_Q(z)$ converges absolutely and uniformly to a holomorphic function when $z \in \mathbb{H}$. $\square$

Clearly, by the definition of $\Theta_Q(z)$, we have $\Theta_Q(z) = \Theta_Q(z+1)$.

In the special case where $Q = x^2$, we have $\Theta_Q(-1/4z) = \sqrt{-2iz}\Theta_Q(z)$ and $\Theta_Q(z) = \Theta_Q(z+1)$. ($\Theta$ in this case is famously known as Jacobi's Theta Series). A proof of this can be found on page 25 of [Bruinier et al.(2008)Bruinier, van der Geer, Harder, and Zagier].

Theorem 4.2. *For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $4 \mid c$, *we have*

$$\Theta_Q\left(\frac{az+b}{cz+d}\right) = \epsilon_d^{-1}\left(\frac{c}{d}\right)\sqrt{cz+d}\,\Theta_Q(z)$$

where $Q = x^2$, $-\pi/2 < \arg(\sqrt{z}) \leq \pi/2$,

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4; \\ i & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

and

$$\left(\frac{c}{d}\right) = \begin{cases} \left(\frac{c}{d}\right) & \text{if } c > 0 \text{ or } d > 0; \\ -\left(\frac{c}{d}\right) & \text{if both } c < 0, d < 0. \end{cases}$$

We can now generalise for a generic quadratic form Q.

Theorem 4.3. *Suppose that Q is a non-degenerate positive definite quadratic form over $\mathbb{Z}$ in n variables, with level N. Then for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $N \mid c$, *we have*

$$\Theta_Q\left(\frac{az+b}{cz+d}\right) = \left(\frac{\det(Q)}{d}\right)\left[\epsilon_d^{-1}\left(\frac{c}{d}\right)\sqrt{cz+d}\right]^n \Theta_Q(z)$$

where $\sqrt{(}z), \epsilon_d, \left(\frac{c}{d}\right)$ *are defined as in the previous theorem.*

Corollary 4.1. *$\Theta_Q(z)$ is a modular form of weight $n/2$ where n is the degree of the number field we began with (and hence also the number of variables in the constructed quadratic form). $\Theta_Q(z)$ has level N*

and character $\chi(\cdot) = \left(\frac{(-1)^{\lfloor n/2 \rfloor}\det(Q)}{\cdot}\right)$, *with respect to the trivial multiplier system* $\epsilon(\gamma, k) := 1$ *when* n *is even, and with respect to the theta multiplier system* $\epsilon(\gamma, k) := \epsilon_d^{-1}\left(\dfrac{c}{d}\right)$ *when* n *is odd.*

We make a final remark here to say that the level N of the modular form is the same as the level of the quadratic form Q.

References

[Bruinier et al.(2008)Bruinier, van der Geer, Harder, and Zagier] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. The 1-2-3 of modular forms. *AMC*, 10:12, 2008.

[Diamond and Shurman(2005)] Fred Diamond and Jerry Michael Shurman. *A first course in modular forms*, volume 228. Springer, 2005.

[Hanke(2013)] Jonathan Hanke. Quadratic forms and automorphic forms. In *Quadratic and Higher Degree Forms*, pages 109–168. Springer, 2013.

[Miyake and Maeda(1989)] Toshitsune Miyake and Yoshitaka Maeda. *Modular forms*. Springer, 1989.

[Stewart and Tall(1987)] Ian N Stewart and David O Tall. Algebraic number theory, 1987.

5. Part [B] - description of academic activities

The aim of this section is to outline and describe my academic activities for the 2013-2014 academic year.

First we shall list all of the courses and seminars/study groups attended, marking out those which were examined.

- TCC - Local Fields (University of Bristol), examined
- MA4H9 - Modular Forms, examined
- MA426 - Elliptic Curves, examined
- Algebraic Geometry for Number Theory (Study Group)
- TCC - Modular Curves
- Mumford Curves (Study Group)
- Galois Cohomology (Study Group)
- attendance at Number Theory Seminar, every Monday during term time
- attendance at Number Theory group meetings - held weekly during term time

Below is a list of books and papers read during the first year of the PhD.

- Algebraic Number Theory - Ian Stewart
- Galois Theory - Ian Stewart
- The Collision of Quadratic Fields, Binary Quadratic Forms, and Modular Forms - Karen Smith - can be found at the following page: http://www.math.oregonstate.edu/ swisherh/KarenSmith.pdf
- A First Course in Modular Forms - Diamond and Shurman (first couple of chapters)
- Quadratic Forms and Automorphic Forms, Arizona Winter School notes, Jonathan Hanke - can be found at the follwing page: http://swc.math.arizona.edu/aws/2009/09HankeNotes.pdf