Bunnies, Stars And SuperForms

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Joint work with Mike Bennett, Samir Siksek and Samuele Anni
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BİLECİK ŞEYH EDEBALİ ÜNİ

ULUSLARARASI ÇA

BİLECİK Şeyh Edebali Üniversitesi Diophantine Denklemlerde Modüler Metotlar üzerine Temel Çalıştay isimli uluslararası bir çalıştay gerçekleştirdi.

Bilecik Şeyh Edebali Üniversitesi sahipliğinde, Fen Edebiyat Fakültesi Matematik bölümünün üstlendiği saat 09.30'da üniversite konferans salonunda başlayan, çalıştaya İngiltere, Kanada, Yunanistan, Japonya, Hollanda'nın yanı sıra matematik alanında uluslararası üne sahip olan bilim insanları katıldı. Konuşmacılar arasında Kanada British Colombia Üniversitesi'nden Michael A. Bennet, Hollanda VU Amsterdam Üniversitesi'nden Sander R. Dahmen, İngiltere Warwick Üniversite'nden Samir Siksek gibi bilim insanlarının yanında Bornova Anadolu Lisesi 3. Sınıf öğrencisi İbrahim Emre Kıvançı Uluslar arası Çalıştaya katılan en genç katılımcı olarak dikkat çekti. 3'te

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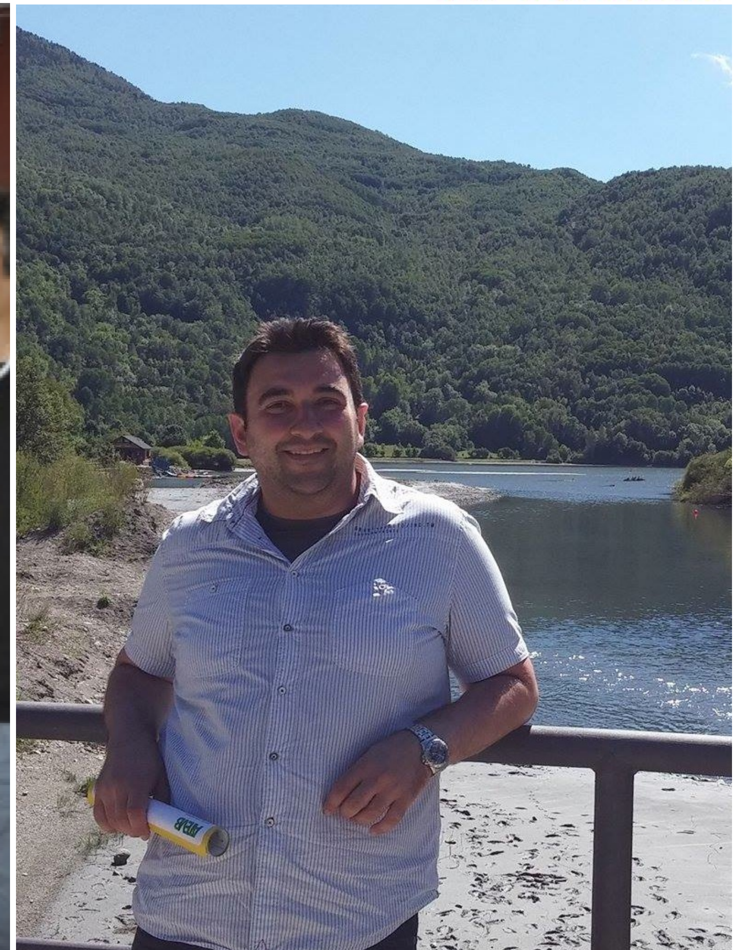
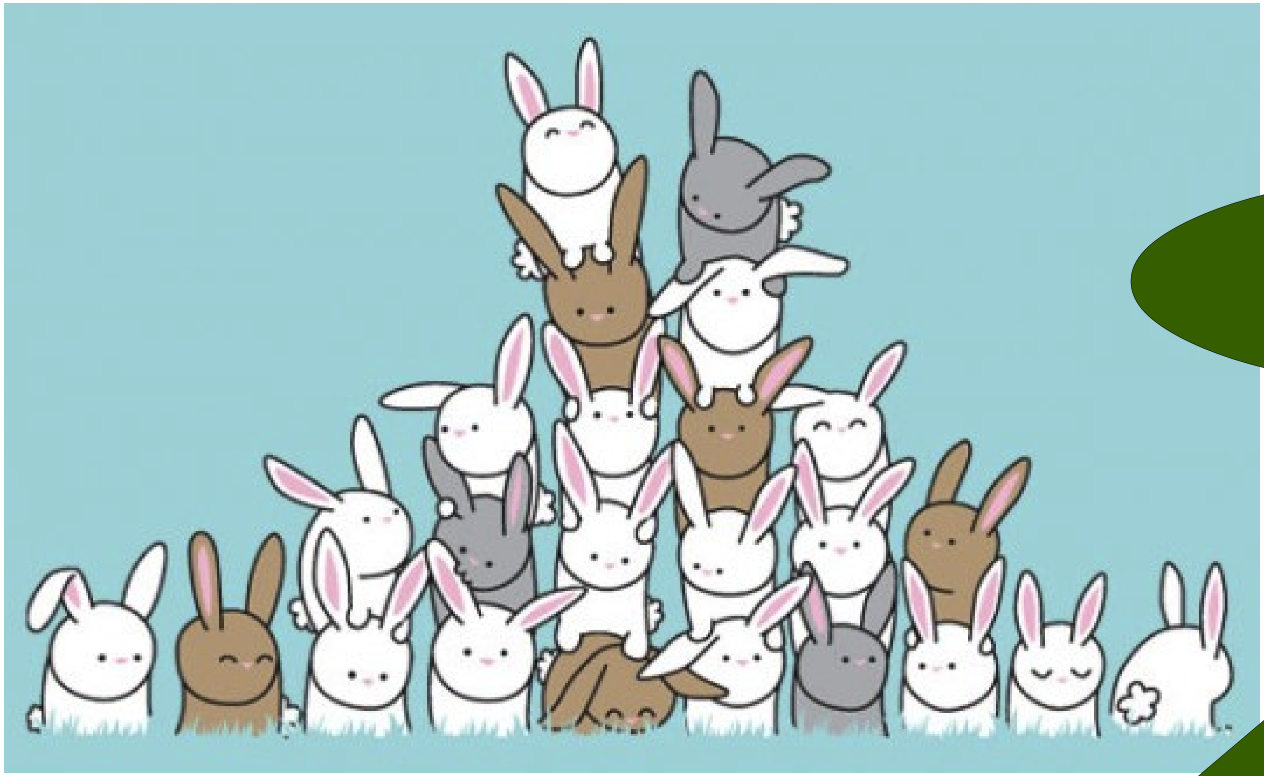



PIERRE DE FERMAT 1601 · 1665
RF
LA POSTE 2001
$x^n + y^n = z^n$
4,50 F
0,69 €
$x^n + y^n = z^n$
n'a pas de solution pour des entiers $n \geqslant 2$
ITVF
LAVERGNE



$F_n + 2 = y^p$

(n, y, p)
$=$
$(3, \pm 2, 2)$
$(-2, 1, p)$
???

$F_n = \{0, 1, 1, 2, 3, 5, \ldots\}$

Finding the Solutions to $F_n = y^p$

- THEOREM (Bugeaud, Mignotte and Siksek)

The only perfect powers of the Fibonacci sequence are, for $n \geq 0$:
$F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$

- Here, we find integer solutions (n, y, p) to the equation $F_n = y^p$

Finding the Solutions to $F_n \pm 1 = y^p$

- THEOREM (Bugeaud, Mignotte and Siksek)

The only perfect powers of the Fibonacci sequence are, for $n \geq 0$:

$F_1 - 1 = F_2 - 1 = 0,$ $F_0 + 1 = F_3 - 1 = 1,$

$F_4 + 1 = F_5 - 1 = 2^2,$ $F_6 + 1 = 3^2$

- Here, we find non-negative integer solutions (n, y, p) to the equation $F_n = y^p$

An overview: $F_n + 2 = y^p$

	Method and Steps	Result
1.	Equation	$F_n + 2 = y^p$
2.	Associate an Elliptic Curve	$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$
3.	Associate a Newform	Hilbert Newforms
4.	Corresponding Elliptic Curves	?
5.	Congrunces	?
6.	Lower Bound for Solutions	?
7.	Upper Bound for Solutions	?

Finding the Solutions to $F_n + 2 = y^p$

Preliminaries ...

Let $\epsilon = \frac{1+\sqrt{5}}{2}$ and $\bar{\epsilon} = \frac{1-\sqrt{5}}{2}$. By Binet's formula,

$$F_n = \frac{\epsilon^n - \bar{\epsilon}^n}{\sqrt{5}}.$$

$$\begin{aligned}
F_n + 2 &= y^p \\
\frac{\epsilon^n - \bar{\epsilon}^n}{\sqrt{5}} + 2 &= y^p \\
\epsilon^{2n} - (\epsilon\bar{\epsilon})^n + 2\epsilon^n\sqrt{5} &= \epsilon^n\sqrt{5}y^p \\
(\epsilon^n - \sqrt{5})^2 - 1 - 5 &= \epsilon^n\sqrt{5}y^p \\
\mu^2 - 6 &= \epsilon^n\sqrt{5}y^p
\end{aligned}$$

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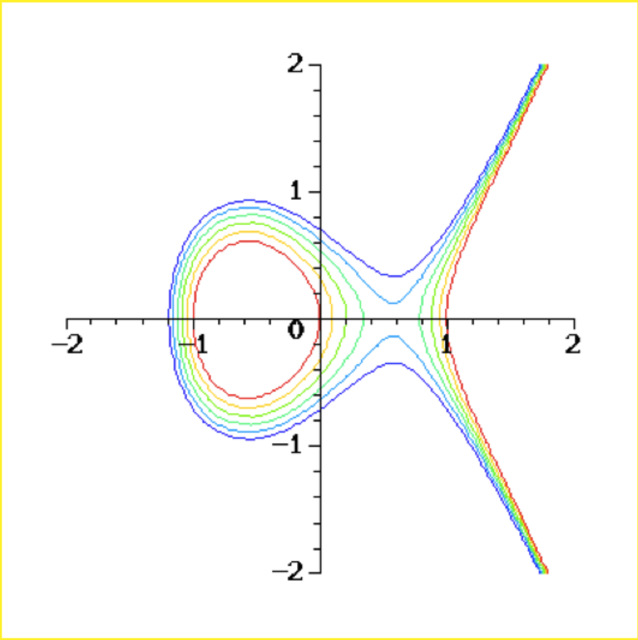
Finding the Solutions to $F_n + 2 = y^p$

Elliptic Curves ...



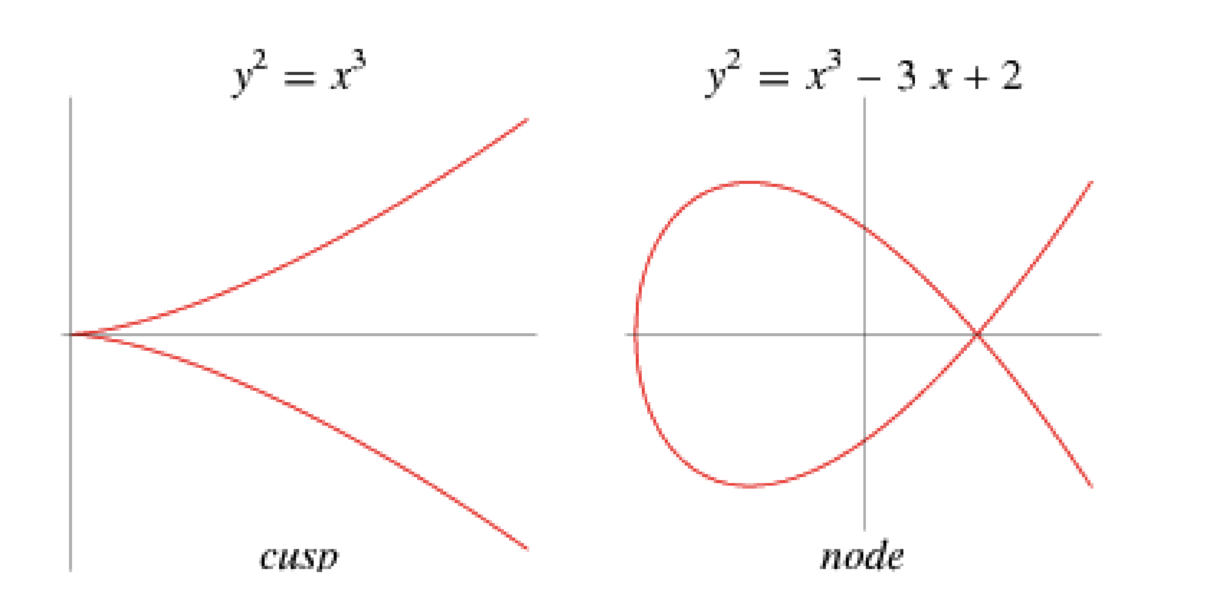
How do we find an elliptic curve?
What is an elliptic curve?

$$Y^2 = X^3 + aX + b$$

where the curve is non-singular (smooth) and $a, b \in \mathbb{R}$.

Finding the Solutions to $F_n + 2 = y^p$

Elliptic Curves ...



Finding the Solutions to $F_n + 2 = y^p$

The Frey Curve ...

$$\mu^2 - 6 = \epsilon^n \sqrt{5} y^p, \qquad \mu = \epsilon^n - \sqrt{5}, \qquad \epsilon = (1+\sqrt{5})/2$$

Model	Example
$E : Y^2 = X^3 + AX^2 + BX$	$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$
$\Delta_E = -16 \cdot B^2(A^2 - 4B)$	$\Delta_{E_n} = 2^8 \cdot 3^2 \cdot \epsilon^n \cdot \sqrt{5} \cdot y^p$
$\mathcal{N}_E$ - Tate's Algorithm	$\mathcal{N}_{E_n} = (2)^7 \cdot (3) \cdot (\sqrt{5}) \cdot \prod_{\mathfrak{q} \mid y, \mathfrak{q} \neq (\sqrt{5})} \mathfrak{q}.$

An overview: $F_n + 2 = y^p$

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3.	Associate a Newform	Hilbert Newforms
4.	Corresponding Elliptic Curves	?
5.	Congruences	?
6.	Lower Bound for Solutions	?
7.	Upper Bound for Solutions	?

Finding the Solutions to $F_n + 2 = y^p$
Newforms ...

- **DEFINITION**

A Newform lives in a finite dimensional space, namely $S_k(N)$.

$$f(z) = q + \sum_{n \geq 2} a_n q^n, \quad a_n \in \mathbb{C}, \; q = e^{2\pi i z}$$

A Hilbert Newform is a generalisation of newforms to functions of 2 or more variables.

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Finding the Solutions to $F_n + 2 = y^p$

Ribet's Level Lowering ...

$$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$$

$$\mathcal{N}_{E_n} = (2)^7 \cdot (3) \cdot (\sqrt{5}) \cdot \prod_{\mathfrak{q} \mid y, \mathfrak{q} \neq (\sqrt{5})} \mathfrak{q}$$

- Hilbert Cuspform that is new with level $N = (2)^7(3)(5^{1/2})$. This space has 6144 Newforms!!!

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An overview: $F_n + 2 = y^p$

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3.	Associate a Newform	Hilbert Newforms
4.	Corresponding Elliptic Curves	? E_a
5.	Congruences	? Points mod m on E_a
6.	Lower Bound for Solutions	? If $n > 1$ then $n > 10^{9000}$
7.	Upper Bound for Solutions	? $n < 10^{9000}$

$$a_m(E_n) \equiv a_m(E) \mod p$$

Computational Number Theory ...

$$f_1 := q - 84q^3 - 82q^5 - 456q^7 + 4869q^9 - 2524q^{11} + O(q^{12})$$

$$f_2 := q + 44q^3 + 430q^5 - 1224q^7 - 251q^9 - 3164q^{11} + O(q^{12})$$

Computational Number Theory ...

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$$f_1 := q - 84q^3 - 82q^5 - 456q^7 + 4869q^9 - 2524q^{11} + O(q^{12})$$

$1 - 1 = 0$

$-2524 + 3164 = 640$

$$f_2 := q + 44q^3 + 430q^5 - 1224q^7 - 251q^9 - 3164q^{11} + O(q^{12})$$

Computational Number Theory ...

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$$f_1 := q - 84q^3 - 82q^5 - 456q^7 + 4869q^9 - 2524q^{11} + O(q^{12})$$

$1 - 1 = 0$

$-84 - 44 = -128$

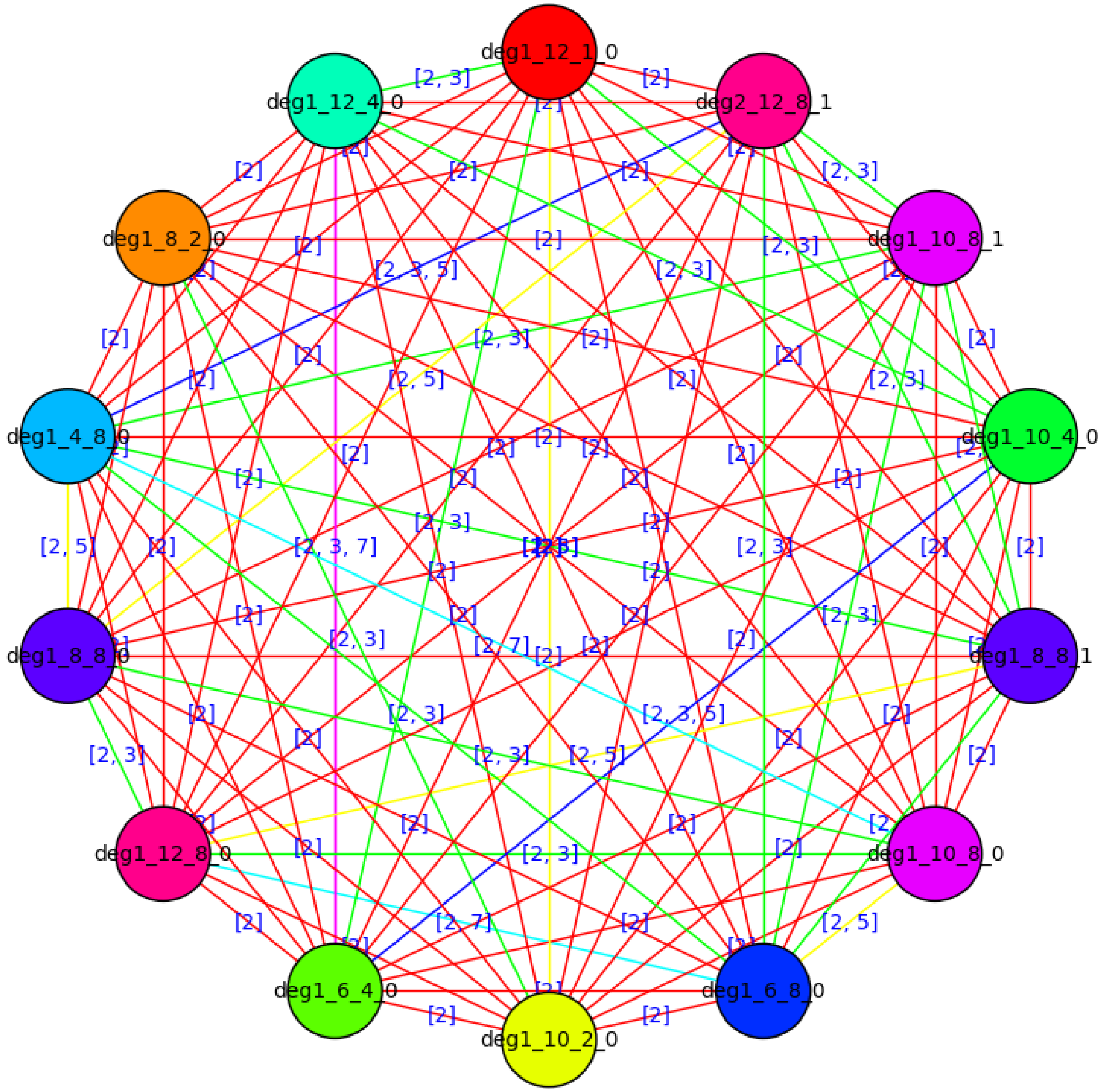
$-82 - 430 = -512$

$-456 + 1224 = 768$

$4869 + 251 = 5120$

$-2524 + 3164 = 640$

$$f_2 := q + 44q^3 + 430q^5 - 1224q^7 - 251q^9 - 3164q^{11} + O(q^{12})$$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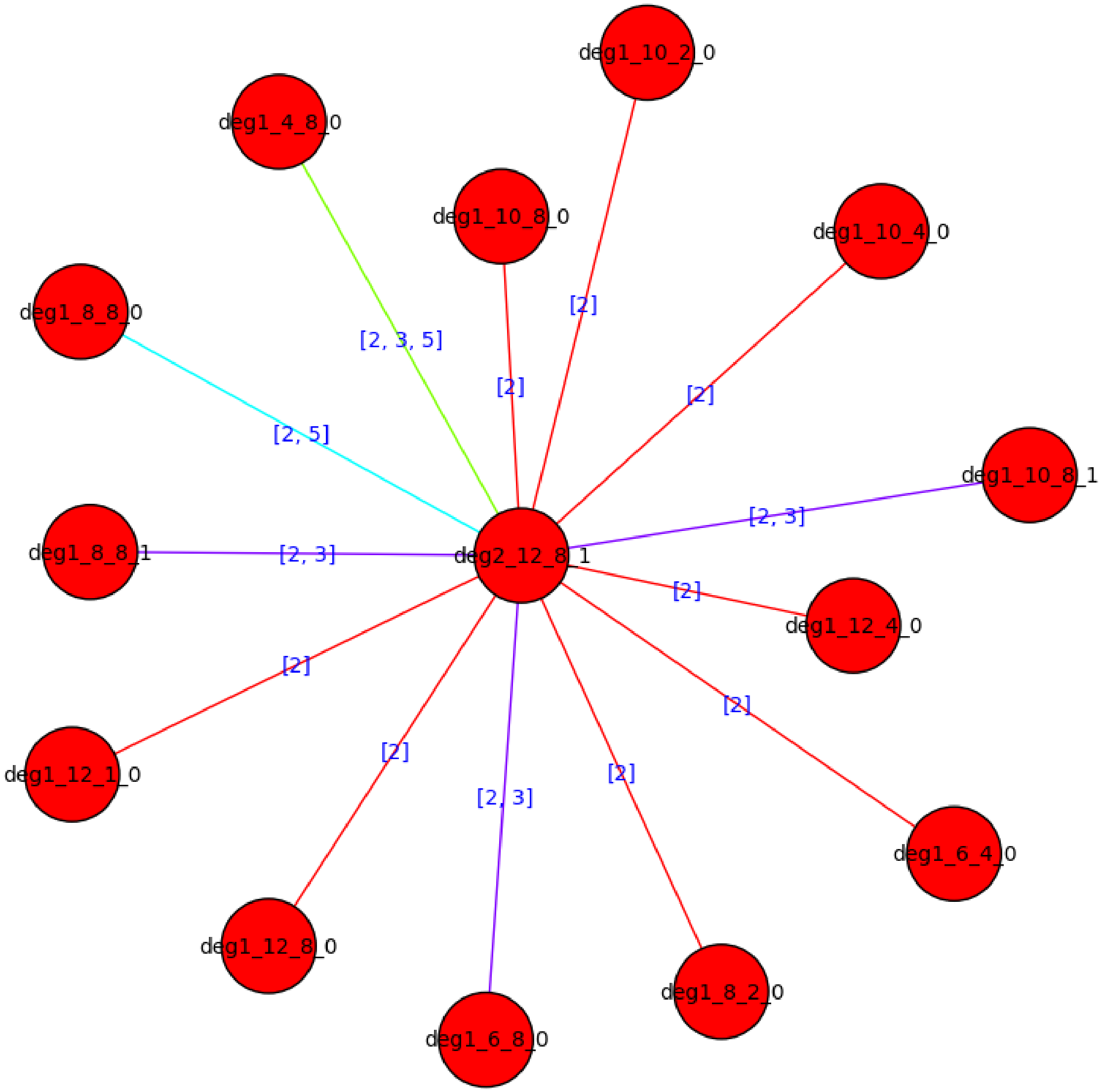
deg1_10_2_0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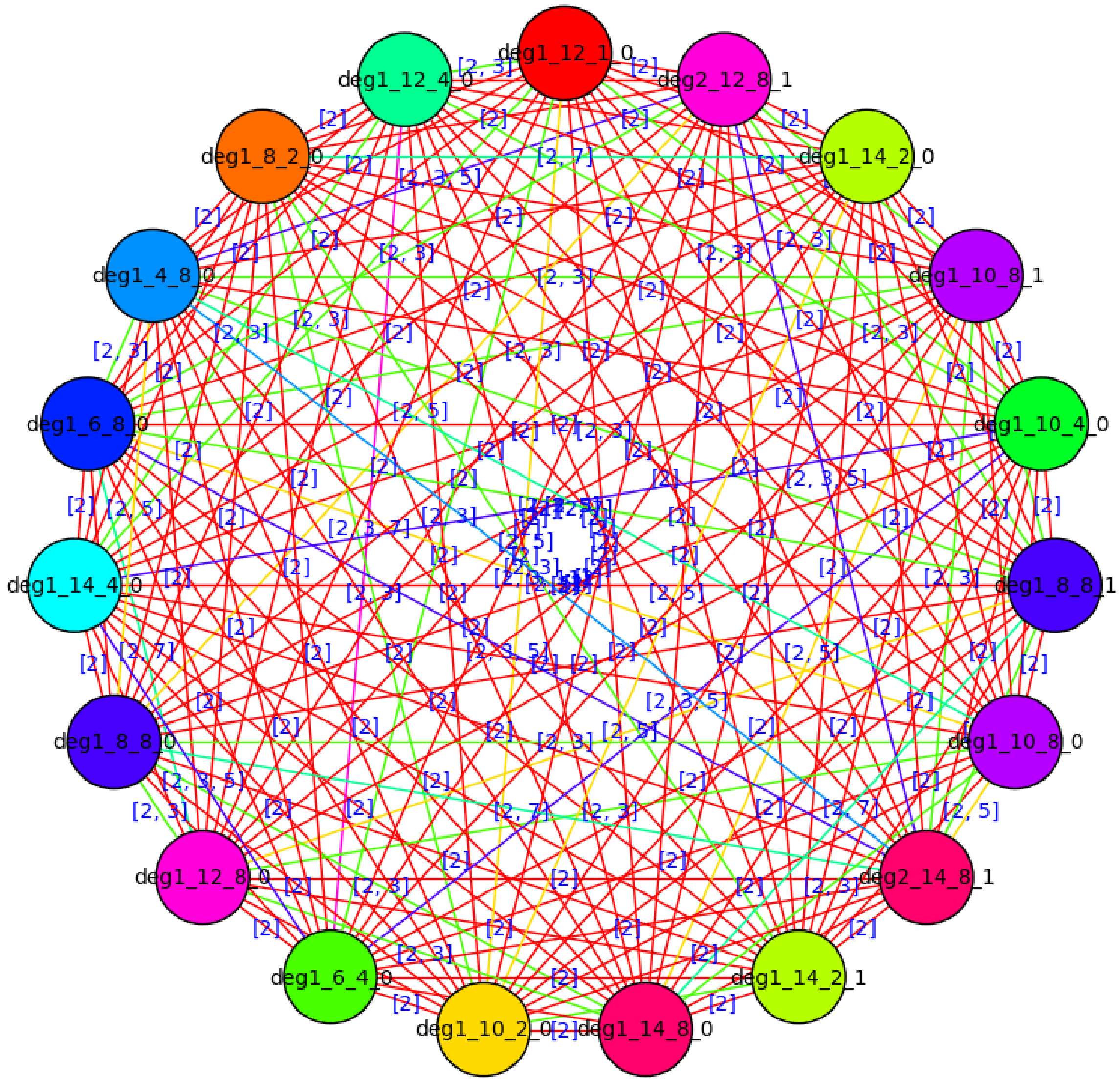
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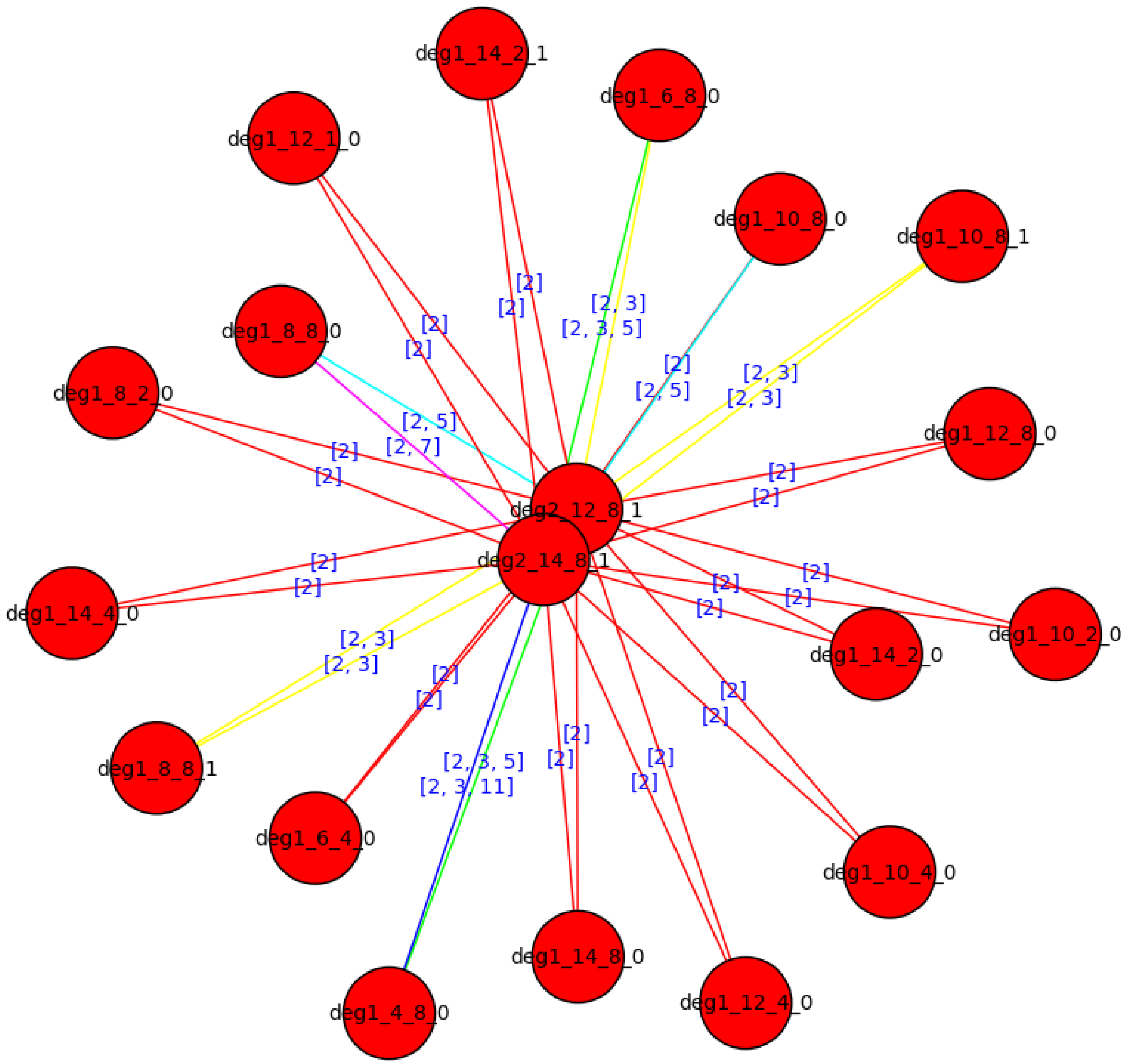
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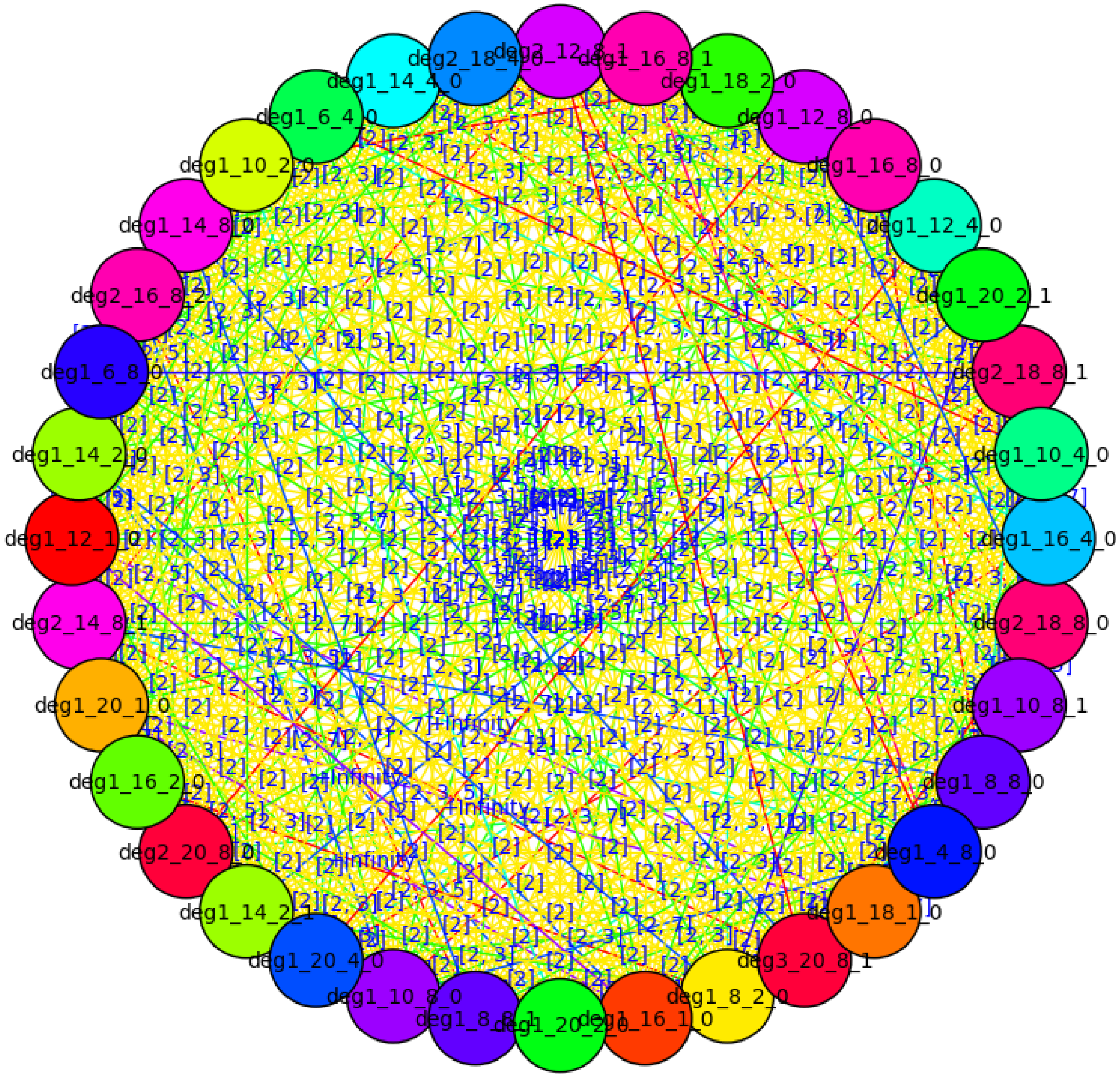

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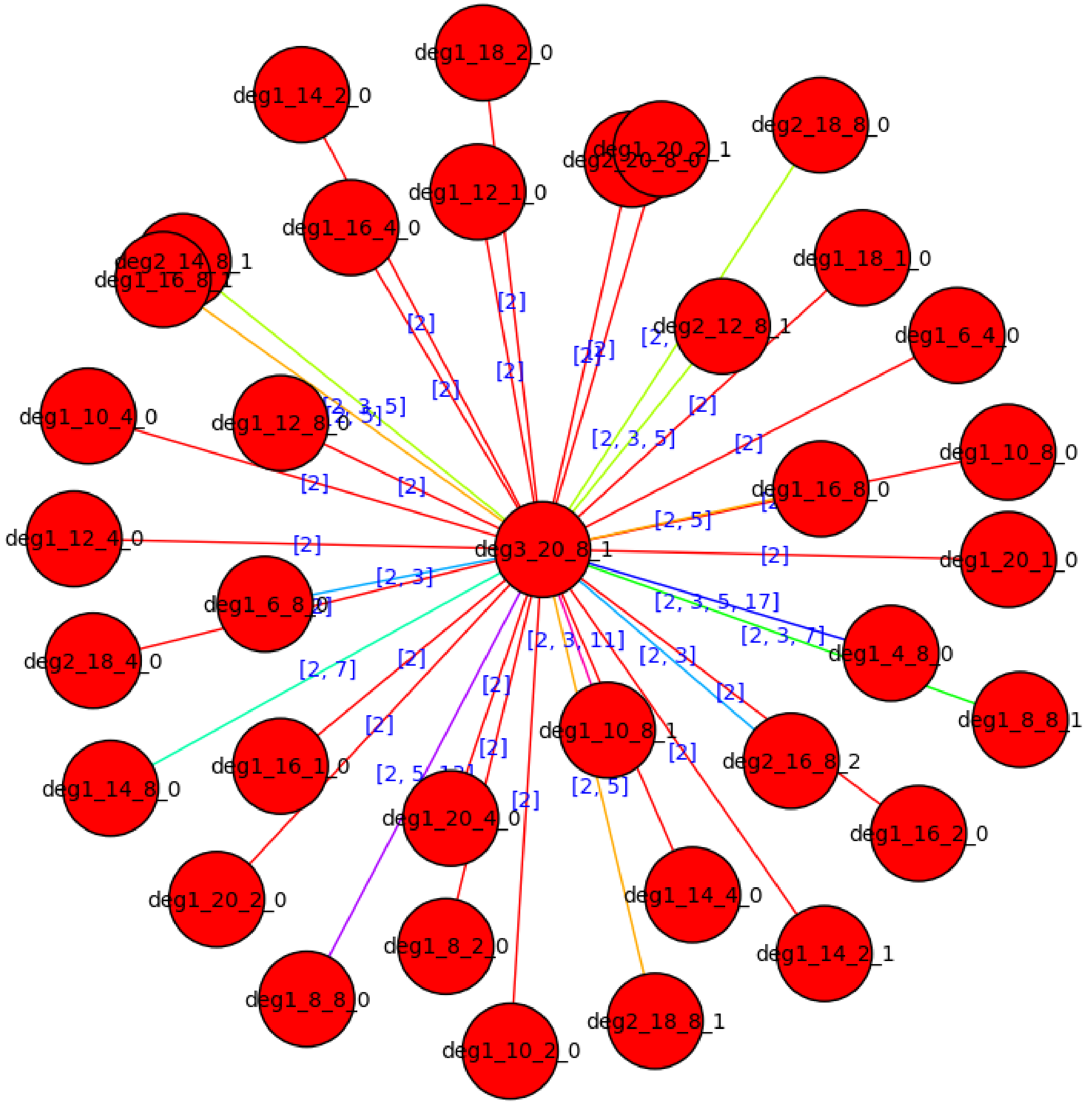

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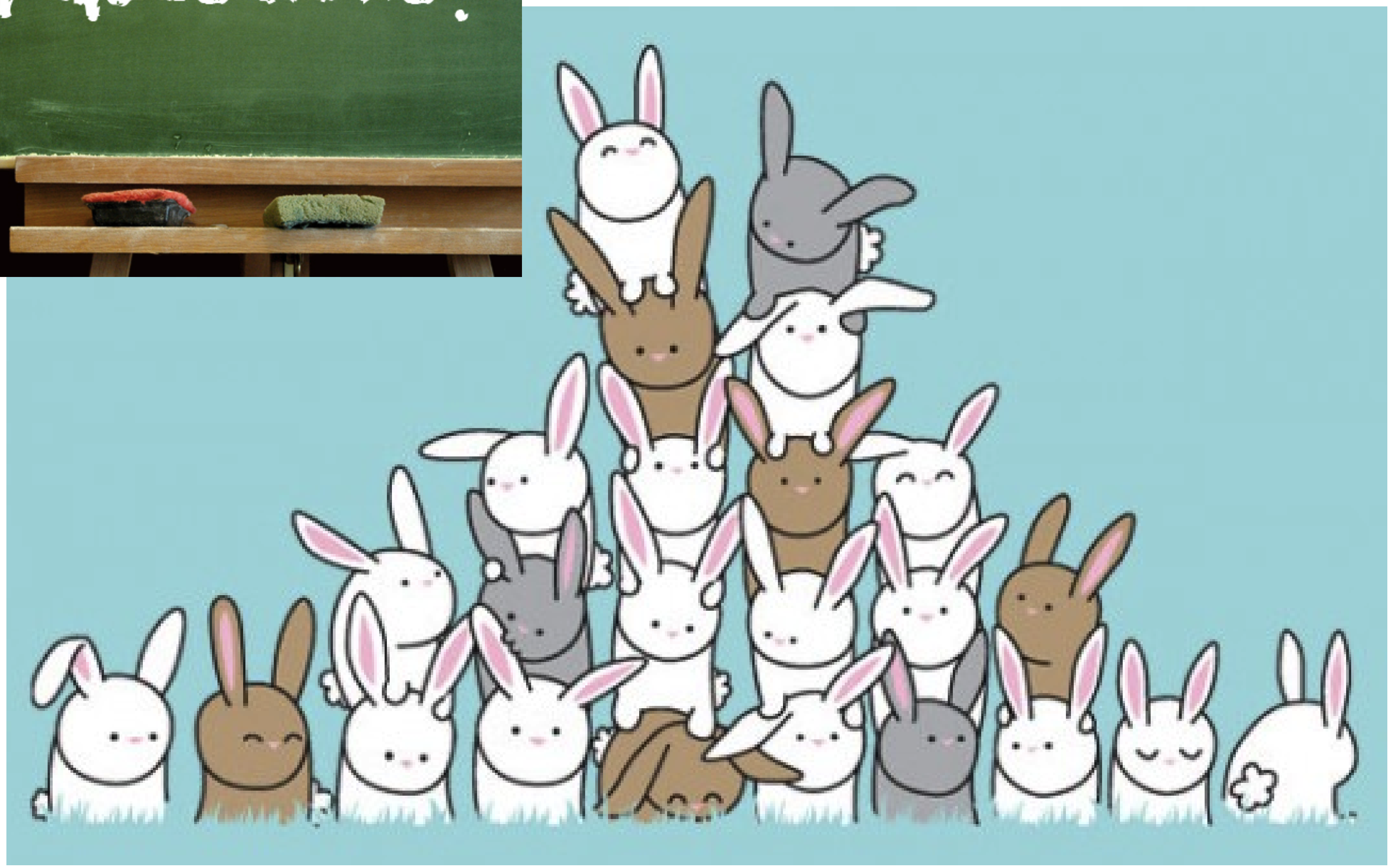

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