On the Jacquet–Langlands correspondence for Hilbert modular forms.

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Contents

1 Introduction 2

2 Modular forms and Automorphic forms 2
   2.1 Modular Forms ................................................................. 2
   2.2 The Adele group of GL₂ over \( \mathbb{Q} \) ............................................. 4
   2.3 Automorphic Forms .............................................................. 5
   2.4 Hecke action ................................................................. 7
   2.5 Hilbert Modular Forms ..................................................... 7
   2.6 Hecke operators on Hilbert modular forms ................................ 11

3 Automorphic forms on quaternion algebras 12
   3.1 Quaternion algebras ............................................................. 12
   3.2 Automorphic forms for a quaternion algebra over \( \mathbb{Q} \) .................. 13
   3.3 Hecke operators ............................................................. 14
   3.4 Automorphic forms for a quaternion algebra over a totally real field .... 15
   3.5 Hecke operators on quaternionic modular forms over \( F \) .............. 16

4 Classical Jacquet–Langlands Correspondence 16

5 The p-adic Jacquet–Langlands correspondence. 17
1 Introduction

The study of modular forms is one of the central topics in number theory, due to its wide range of uses and connections to many other integral parts of number theory. The term modular form was first coined by Erich Hecke, but some of the theory had been developed some time before in relation to the theory of elliptic functions. Using modular forms it is possible to obtain many results about number theoretic functions like, for example, the sum of powers of divisors functions or the number of ways of writing an integer as a sum of $k$ squares. But most notably the theory is used in the proof of Fermat’s Last theorem. These deep connections with so many parts of number theory are the reason why modular forms are a subject of great study in modern number theory. In order to generalize this theory, the idea of automorphic forms comes into play, and using this it is possible to see the correct way to generalize a great deal of the theory of modular forms to more abstract situations. From these generalizations comes the theory of Hilbert modular forms, which can be seen as automorphic forms on totally real fields.

Using the theory of automorphic forms it is then possible to define analogues of modular forms over quaternion algebras, which turns out to be a very useful thing, since in 1970 Herve Jacquet and Robert Langlands in [1] proved what is now known as the Jacquet–Langlands correspondence, which is a result about automorphic representations, that when translated into the theory of automorphic forms, gives a correspondence between classical modular forms and modular forms on quaternion algebras (called quaternionic modular forms). It is this correspondence and its generalizations, that is the main focus of this work.

2 Modular forms and Automorphic forms

2.1 Modular Forms

In this section we will give a brief recap of the theory of modular forms, in order to later motivate the definition of Hilbert modular forms. For more details on this subject see [2].

Let $\mathcal{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$, we can define an action of $\text{GL}_2(\mathbb{Q})^+ = \{\gamma \in \text{GL}_2(\mathbb{Q}) | \det(\gamma) > 0\}$ on $\mathcal{H}$, by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}.$$

It is an easy exercise to check that this is an action on $\mathcal{H}$.

**Definition 2.1.** Let $f$ be a meromorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$, and let $k \geq 1$ be an integer. We say $f$ is weakly modular of weight $k$ (and level 1) if

$$f(\gamma z) = (cz + d)^k f(z)$$

for all $\gamma \in \text{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$.

For $f$ a meromorphic function on $\mathcal{H}$, we can define an action of $\text{GL}_2(\mathbb{Q})^+$ by setting

$$(f|k\gamma)(z) = \frac{\det(\gamma)^{k-1}}{j(\gamma, z)^k} f(\gamma z),$$

where $j(\gamma, z) = (cz + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is easily seen to be a action, and with this we see that the weakly modular functions defined above are the ones for which $f|k\gamma = f$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

**Remark 2.2.** Sometimes it is convenient to choose a different definition for this action, by using $\det(\gamma)^{k/2}$ in the action above.
One can show that, in fact, \( SL_2(\mathbb{Z}) \) is generated by \( T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) and \( S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \), so it is enough for \( f|_{k} T = f \) and \( f|_{k} S = f \), in order to determine that \( f \) is weakly modular for \( SL_2(\mathbb{Z}) \). Writing this out we see that we must have \( f(z + 1) = f(z) \) and \( f(-1/z) = z^k f(z) \) for all \( z \in \mathcal{H} \).

**Definition 2.3.** A modular form of weight \( k \) and level \( \Gamma \) is a holomorphic function on \( \mathcal{H} \) which is weakly modular of weight \( k \) and such that \( f(z) \) tends to a finite limit as \( z \) tends to \( \infty \) (we say \( f \) is holomorphic at \( \infty \)). We denote the space of modular forms by \( M_k(SL_2(\mathbb{Z})) \).

**Remark 2.4.** From the definition it is not hard to see that in fact \( M_k(SL_2(\mathbb{Z})) \) is a vector space over \( \mathbb{C} \).

Now since for a modular form of weight \( k \) and level \( \Gamma \), we have \( f(z+1) = f(z) \) and \( f \) is holomorphic at \( \infty \), so we get a Fourier expansion \( f(z) = \sum_{n=0}^{\infty} a_n q^n \) where \( q = e^{2\pi i z} \), we call this the \( q \)-expansion of \( f \). We say a modular form of weight \( k \) and level \( \Gamma \) is a cusp form if \( a_0 = 0 \) in this Fourier expansion. We denote the space of cusp forms by \( S_k(SL_2(\mathbb{Z})) \).

Now for \( SL_2(\mathbb{Z}) \) we have some important subgroups called congruence subgroups. To define them, we first look at the subgroup

\[ \Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid a - 1 \equiv b \equiv c \equiv d - 1 \equiv 0 \pmod{N} \right\} \]

for some positive integer \( N \).

**Definition 2.5.** A congruence subgroup of \( SL_2(\mathbb{Z}) \) is a subgroup, which contains \( \Gamma(N) \) for some positive integer \( N \).

**Remark 2.6.** Note that these congruence subgroups will have finite index in \( SL_2(\mathbb{Z}) \), as it is not hard to show that \( \Gamma(N) \) has finite index in \( SL_2(\mathbb{Z}) \).

The two main examples of congruence subgroups are

\[ \Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}, \] and

\[ \Gamma_1(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid a - 1 \equiv c \equiv d - 1 \equiv 0 \pmod{N} \right\}. \]

Now for a meromorphic function \( f : \mathcal{H} \to \mathbb{C} \), and a congruence subgroup \( \Gamma \), we say that it is weakly modular of weight \( k \) and level \( \Gamma \) if \( f|_{k \Gamma} = f \) for all \( \gamma \in \Gamma \). With this we can now extend the definition of modular forms as follows:

**Definition 2.7.** A modular form of weight \( k \) and level \( \Gamma \) is a weakly modular function of weight \( k \) and level \( \Gamma \), that is holomorphic on \( \mathcal{H} \) and \( f|_{k \alpha} \) is holomorphic at \( \infty \) for all \( \alpha \in SL_2(\mathbb{Z}) \). We denote this space by \( M_k(\Gamma) \).

Now note that we have a finite coset decomposition \( SL_2(\mathbb{Z}) = \bigcup_i \Gamma \alpha_i \), and that since \( f|_{k \gamma} \alpha_i = f|_{k} \alpha_i \) for all \( \gamma \in \Gamma \), then we only need to check the condition of \( f|_{k \alpha} \) being holomorphic at \( \infty \), only for the finite set of coset representatives. We say that a modular form \( f \) of weight \( k \) and level \( \Gamma \) is a cusp form if \( a_0 = 0 \) in the Fourier expansion of \( f|_{k \alpha} \) for all coset representatives \( \alpha \). We denote the space of such functions by \( S_k(\Gamma) \).

Next we define the Hecke operators on \( M_k(\Gamma_1(N)) \). Take \( \alpha \in GL_2(\mathbb{Q})^+ \) and consider the double coset \( \Gamma_1(N)\alpha \Gamma_1(N) \). We can write this as

\[ \Gamma_1(N)\alpha \Gamma_1(N) = \bigcup_i \Gamma_1(N)\gamma_i, \]

for \( \gamma_i \in GL_2(\mathbb{Q})^+ \). One can show that this union is finite. Now we define the double coset operator \( [\Gamma_1(N)\alpha \Gamma_1(N)] \) on \( M_k(\Gamma_1(N)) \) by

\[ [\Gamma_1(N)\alpha \Gamma_1(N)] f = \sum_i f|_{k \gamma_i}, \]
with the $\gamma_i$ as above. It is not hard to show that this is well defined.

Now for $p$ a prime and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we let $T_p = \Gamma_1(N)\alpha \Gamma_1(N)$, when $p \nmid N$ and for $p \mid N$ it is traditional to call this operator $U_p = \Gamma_1(N)\alpha \Gamma_1(N)$. Similarly if we take $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then it turns out that the action of $[\Gamma_1(N)\alpha \Gamma_1(N)]$ only depends on $d \mod N$, so we denote this operator by $\langle d \rangle$, and it is called the diamond operator. Now as $\langle d \rangle$ only depends on $d \mod N$ we can define a character $\chi : (\mathbb{Z}/NZ)^{\times} \to \mathbb{C}$ that sends $d$ to the eigenvalue of $\langle d \rangle$ on a modular form $f$ (this is easily seen to define a character). For a congruence subgroup $\Gamma$ we define $M_\chi(\Gamma) = \{ f \in M_k(\Gamma) | \langle d \rangle f = \chi(d)f \} \text{ for all } d \in (\mathbb{Z}/NZ)^{\times}$, and we define $S_\chi(\Gamma, \chi) = S_k(\Gamma) \cup M_k(\Gamma, \chi)$.

We say $f \in M_k(\Gamma(1)(N))$ is an eigenform, if it is an eigenvector for the operators $T_p, U_p$ and $\langle d \rangle$ for all primes $p$ and all $d \mod N$. Lastly we say $f \in S_k(\Gamma(1)(N))$ is an oldform if $f(z) = g(dz)$ for some $g \in S_k(\Gamma_1(M))$ with $M \mid N$ and $d \mod N/M$ a proper divisor. Now on the space of cusp forms we can define an inner product called the Petersson inner product (See [2], p. 182, for a definition of the Petersson inner product). With this we define the space of newforms in $S_k(\Gamma_1(N))$ to be the orthogonal complement of the space of oldforms, with respect to the Petersson inner product.

### 2.2 The Adele group of $GL_2$ over $\mathbb{Q}$

We begin by first recalling the definition of the adeles of a number field $F$.

**Definition 2.8.** Let $F$ be a global number field, and let $F_v$ denote the completion of $F$ with respect to a valuation $v$. The adele ring $\mathbb{A}_F$ of $F$, is a subring of $\prod_v F_v$, where $v$ runs through all the places of $F$ and for $\alpha = (\alpha_v) \in \mathbb{A}_F$, we have $\alpha_v \in \mathcal{O}_v$ for all but finitely many $v$ (here $\mathcal{O}_v$ denotes the ring of integers of $F_v$, for $v$ non-archimidean).

We make $\mathbb{A}_F$ into a ring by defining the addition and multiplication componentwise. The construction of $\mathbb{A}_F$ from $F_v$ as above, is called the *restricted* product of $F_v$ with respect to the subrings $\mathcal{O}_v$.

If instead we take the restricted topological product of the $F_v^{\times}$ with respect to $\mathcal{O}_v^{\times}$, then we get the idele group $\mathbb{A}_F^{\times}$ of $F$, which is just the unit group of the adeles. With this definition, $\mathbb{A}_F$ will be a locally compact topological group, since the $F_v$ are locally compact and the $\mathcal{O}_v$ are compact. Also since for each place $v$ of $F$ we have a natural inclusion $F \subset F_v$, so we can embed $K$ diagonally into $\mathbb{A}_F$, and hence we get a diagonal embedding $F^{\times} \to \mathbb{A}_F^{\times}$.

Lastly if in the definition of adele ring we only take the product over $v$ discrete, then we call these the finite adeles and we denote it by $\mathbb{A}_{F,f}$ and when $F = \mathbb{Q}$, we write simply $\mathbb{A}_f$.

Now we seek to do something similar for $GL_2/\mathbb{Q}$. For this we first need some results. For $G = GL_2$, we define $G_p = GL_2(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ are the usual $p$-adic numbers, and we let $G_\infty = GL_2(\mathbb{R})$. Also we write $G^{\pm}_\infty = \{ \gamma \in G_\infty | \det(\gamma) > 0 \}$. In order to make notation more consistent we allow ourselves to write things like $p = \infty$. As usual we let $\mathcal{O}_p$ denote the ring of integers of $\mathbb{Q}_p$, and we define $K_p = GL_2(\mathcal{O}_p)$ (if $p = \infty$ then $K_\infty = \mathcal{O}_2(\mathbb{R})$, the orthogonal matrices).

**Proposition 2.9.** Each $K_p$ is a maximal compact subgroup of $G_p$, and when $p$ is finite then it is also an open subgroup.

**Proof.** If $p = \infty$, then we want to show first that the orthogonal group $\mathcal{O}_2(\mathbb{R})$ is compact. But this is easy to see, since we can view $\mathcal{O}_2(\mathbb{R})$ as a closed subspace of $\mathbb{R}^4$, whose points satisfy some polynomial equations, so it is compact. We won’t prove it is maximal here, since this is a standard result, but we will note that to prove this one can show that any compact subgroup of $GL_n(\mathbb{R})$ is contained in the orthogonal group of a definite quadratic form, from which one can get the result.

If $p$ is finite, then it is easy to see that $\mathcal{O}_p$ is a compact subgroup of $\mathbb{Q}_p$, and therefore since $K_p$ can be viewed as the subset of $\mathcal{O}_p^{2+1} \times \mathcal{O}_p$, which satisfies $det(A)a = 1$, we see that $K_p$ is compact. Also note that $K_p$ is an open and closed subset of $GL_2(\mathbb{Q}_p)$. To show that it is maximal see [3], Appendix 1, p. 121. \[\Box\]

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1. By place we mean an equivalence class of valuations.
2. To see it is open, we just need to use that the determinant map is continuous, the result then follows at once.
With this we can define $G_{\mathbb{A}} = \text{GL}_2(\mathbb{A})$, which is the restricted topological product (over all $p$ including $\infty$) of $G_p$ with respect to $K_p$ (here $\mathbb{A}$ are the adeles over $\mathbb{Q}$). The center of $G_{\mathbb{A}}$ is $Z_{\mathbb{A}} = \{ t I | t \in \mathbb{A}^\times, I$ the identity matrix $\}$.

Now it is well known that $G_{\mathbb{A}}$ is unimodular (its left and right Haar measures coincide), so we get a natural $G_{\mathbb{A}}$-invariant measure on $G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$, which will be used later when defining automorphic cusp forms. See [4] Section 7.6.

Now we have the following structure theorem for $G_{\mathbb{A}}$.

**Theorem 2.10.** (Strong Approximation for $G_{\mathbb{A}}$). Let $U$ be an open compact subgroup of $\text{GL}_2(\mathbb{A}_f)$, such that $\det(U) = \widehat{\mathbb{Z}}^\times$. Then $\text{GL}_2(\mathbb{A}_f) = G_{\mathbb{Q}}U$ and $\text{GL}_2(\mathbb{A}) = G_{\mathbb{Q}}G^+_{\infty}U$.

**Proof.** See [4], Theorem 6.8, p. 65. \qed

**Remark 2.11.** There seems to be some abuse of the term strong approximation in the above result. Technically the result above is not strong approximation for $\text{GL}_2$, since it can be shown that the usual notion of strong approximation fails, which basically means that $\text{GL}_2(\mathbb{Q})$ is not dense in $\text{GL}_2(\mathbb{A}_f)$. The result above is in fact a consequence of strong approximation for $\text{SL}_2$ which does hold.

Now, for $K_p(N) = \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in K_p | c \equiv 0 \mod NZ_p \}$, with $N$ any positive integer. It is easy to see that when $p \nmid N$ we have $K_p = K_p(N)$. So if we let $U = K_0(N) = \prod_{p<\infty} K_p(N)$, then as $\det : K_p(N) \to \mathbb{O}_p^\times$, we have $\det(K_0(N)) = \widehat{\mathbb{Z}}^\times$, so by strong approximation we have $G_{\mathbb{A}} = G_{\mathbb{Q}}G^+_{\infty}K_0(N)$.

**Definition 2.12.** A grossencharacter of $\mathbb{Q}$ is a unitary character of $\mathbb{A}^\times$ that is trivial on $\mathbb{Q}^\times$.

We can define a grossencharacter on $\mathbb{Q}$ by using a character $\psi$ of $(\mathbb{Z}/NZ)^\times$, as follows: compose the natural homomorphism from $\mathbb{O}_p^\times$ to $(\mathbb{Z}/NZ)^\times$ with $\psi$ (for $p \neq \infty$), then take the product over all places and since we have $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^\times \prod_{p<\infty} \mathbb{Q}_p$ (which is equivalent to saying $\mathbb{Q}$ has class number 1) we get a character of $\prod_{p<\infty} \mathbb{O}_p^\times$ and hence a character of $\mathbb{A}^\times$ (trivial on $\mathbb{Q}^\times \mathbb{R}^\times$)

Now by identifying the idele class group of $\mathbb{Q}$ with $Z_{\mathbb{Q}} \setminus Z_{\mathbb{A}}$ we get a grossencharacter on $Z_{\mathbb{Q}} \setminus Z_{\mathbb{A}}$.

### 2.3 Automorphic Forms

We are now in a position to give a general definition of an automorphic form for $\text{GL}_2/\mathbb{Q}$. Here we are following [5].

**Definition 2.13.** An Automorphic form on $\text{GL}_2/\mathbb{Q}$ with grossencharacter $\psi$, is any function $\phi$ on $G_{\mathbb{A}}$, satisfying the following:

1. $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G_{\mathbb{Q}}$;
2. We have $\phi(gz) = \phi(zg) = \psi(z)\phi(g)$, for all $z \in Z_{\mathbb{A}}$;
3. Let $R = K_{\infty} \prod_{p<\infty} K_p$. Then we have $\phi$ is right $R$-finite. This means that the space of functions on $G_{\mathbb{A}}$ spanned by the right translates of $\phi(g)$ by $R$ is finite dimensional;
4. As a function on \( G_{\infty} \) alone, \( \phi \) is smooth and \( Z(\mathfrak{g}) \)-finite (here \( \mathfrak{g} \) is the universal enveloping algebra of \( G_{\infty} \));

5. \( \phi \) is slowly increasing, meaning that for every \( c > 0 \) and compact subset \( T \subset G_{\mathbb{A}} \), there exist constants \( C, N \) such that

\[
\phi \left( \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} g \right) \leq C|a|^N,
\]

for all \( g \in T \) and \( a \in \mathbb{A}^\times \) with \( |a| > c \).

Furthermore if we have

\[
\int_{\mathbb{Q}\backslash \mathbb{A}} \phi \left( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g \right) dx = 0
\]

for almost all \( g \) we call it a cusp form. Here we are using the Haar measure we mentioned at the beginning.

Remark 2.14. Condition (4) above looks complicated, but this is just a generalization of the holomorphicity condition in the standard definition of a modular form.

Next we are going to see how to take a cusp form in \( S_k(\Gamma_0(N), \psi) \), with \( \psi \) a grossencharacter as above, and get an automorphic form.

Definition 2.15. If \( f \in S_k(\Gamma_0(N), \psi) \), then define

\[
\phi_f(g) = f(g_{\infty}(i))j(g_{\infty}, i)^{-k} \det(g_{\infty})^{k/2} \psi(k_0),
\]

where \( g = \gamma g_{\infty} k_0 \) with \( \gamma \in G_{\mathbb{Q}}, g_{\infty} \in G_{\infty} \) and \( k_0 \in K_0(N) \) as above, and

\[
j(g, z) = (cz + d) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Now it is not immediately clear that this is well defined. If we have \( \gamma' g_{\infty}' k_0' = \gamma g_{\infty} k_0 \), then \( t = \gamma'^{-1} \gamma = k_0' k_0^{-1} g_{\infty}' g_{\infty}^{-1} \) is in \( G_{\mathbb{Q}} \cap G_{\infty}^+ K_0(N) = \Gamma_0(N) \), so as \( f \) is a modular form, we see immediately that it is in fact well defined.

Remark 2.16. Sometimes it is useful to replace \( \det(g_{\infty})^{k/2} \) by \( \det(g_{\infty})^s \) in the above definition, for some other number \( s \).

Proposition 2.17. The map sending \( f \) to \( \phi_f \) is an isomorphism between \( S_k(\Gamma_0(N), \psi) \) and the space of functions \( \phi \) on \( G_{\mathbb{A}} \) satisfying the following:

1. \( \phi(\gamma g) = \phi(g) \) for all \( \gamma \in G_{\mathbb{Q}} \);
2. \( \phi(gz) = \phi(zg) = \psi(z)\phi(g) \) for all \( z \in Z_{\mathbb{A}} \);
3. \( \phi(gk_0) = \phi(g)\psi(k_0) \) and \( \phi(gr(\sigma)) = e^{-i\pi}\phi(g) \) for \( k_0 \in \prod_{p<\infty} K_p(N) \) and \( r(\sigma) = \begin{pmatrix} \cos(\sigma) & -\sin(\sigma) \\ \sin(\sigma) & \cos(\sigma) \end{pmatrix} \);
4. As a function on \( G_{\infty}^+ \), \( \phi \) satisfies the differential equation

\[
\Delta \phi = -(k/2)(k/2 - 1) \phi,
\]

where \( \Delta \) is the Laplace operator;
5. \( \phi \) is slowly increasing and cuspidal (as before).

Proof. (Sketch) First we note that condition (1) holds at once since on the right hand side of (†) \( \gamma \) does not appear (and as we noted before this is well defined). For condition (2) we need to look more closely at how the character is defined, but this is not very hard (just a computation) to see the result (See [11] p.197-198). Similarly (3),(4) are
just computations. For (5) we use that \( f \) is a cusp form (so there is some bound on its growth) to get the result (See [4] p.203).

Now we can check that these functions \( \phi_f \) are in fact automorphic forms. Conditions (1), (2), (5) are clearly the same as in the definition of automorphic forms. Condition (3) is the same as saying that \( \phi \) is right \( K_\infty \prod_{p<\infty} K_p \)-finite. To see this, note that for any \( x \in SO_2(\mathbb{R})K_0(N) \) we get that the space of right translates of \( \phi(g) \) by \( x \) is one dimensional, by the above. Also from the definitions it is easy to see that \( SO_2(\mathbb{R})K_0(N) \) has finite index in \( K_\infty \prod_{p} K_p \), so we can see that in fact \( \phi_f \) is \( K_\infty \prod_{p} K_p \)-finite. Lastly condition (4) above implies that \( \phi_f \) is \( Z(\mathfrak{g}) \)-finite since the center of the universal algebra \( \mathfrak{g} \) is generated by \( \Delta \) and \( I \) (see [6] p. 279).

With this we see that the above gives us a way to get an automorphic form from a cusp form. Now, this also works for any modular form of character \( \psi \) but in this case the growth condition is harder to prove, and clearly it won’t be cuspidal.

Remark 2.18. If in condition (4) above, we change \( \Delta \) for the Laplace–Beltrami operator, we can see Maass wave forms as automorphic forms.

2.4 Hecke action.

Now, we look to get some analogues for the standard Hecke operators \( T_p \) in the theory of modular forms for the more general theory of automorphic forms. The above shows that we can view the elements of \( S_k(\Gamma_0(N)) \) as automorphic forms, so for \( p \) a prime not dividing \( N \) we want to define an analogue of the Hecke operator \( T_p \).

Using the notation from section 2.1 we have \( K_p(N) = GL_2(\mathcal{O}_p) \) (as \( p \nmid N \)). So consider the following double coset in \( G_p \)

\[
H_p = K_p \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) K_p.
\]

Now for an automorphic form \( \phi \) on \( GL_2/\mathbb{Q} \) (as in the previous section), we define

\[
\tilde{T}_p \phi(g) = \int_{H_p} \phi(gh) dh.
\]

Note that this is just the convolution over \( G_p \) of \( \phi_f \) with the characteristic function of \( H_p \). At first this might look a bit strange compared to the classical definition of the \( T_p \) operators, but note that we have the following result.

Lemma 2.19. If we let \( g = T_p f \), for \( f \in S_k(\Gamma_0(N)) \), and \( p \) does not divide \( N \), then

\[
p^{k/2-1} \tilde{T}_p \phi_f = \phi_g
\]

Proof. See [5], Lemma 3.7, p. 48.

2.5 Hilbert Modular Forms

One of the motivations for working with automorphic forms, is that it gives us a general framework. Using this framework we can relate different types of objects and also understand how we should define analogues of modular forms over general number fields. This is what we will be doing in this section. Here we begin by naturally extending the definition of automorphic forms to totally real fields\(^3\) and then we see how these new objects describe Hilbert modular forms.

\(^3\)A totally real field is a number field \( F \), such that every embedding of \( F \) into the complex numbers, has image in the reals.
In this section we look to work not over \( \mathbb{Q} \), but over some fixed totally real field \( F \). In this case we get analogous results to the ones above, and it also allows us to make the connection with Hilbert modular forms. Throughout this section we have \( F \) being a fixed totally real number field, and \( [F : \mathbb{Q}] = n > 1 \).

We begin by giving the definition of an automorphic form for \( \text{GL}_2 \) over \( F \). We let \( \mathbb{A}_F \) denote the adeles over \( F \) and if \( v \) is a place of \( F \) we define \( G_v = \text{GL}_2(F_v) \), and we let \( G_\infty = \prod_{v \in J_F} G_v \), where \( J_F \) is the set of infinite places of \( F \). For \( v \in J_F \) we denote the corresponding embedding by \( a \mapsto a^v \) (which we extend element-wise to matrices when needed). To ease notation we write \( G_{\mathbb{A}_F} = \text{GL}_2(\mathbb{A}_F) \) and \( G_F = \text{GL}_2(F) \).

**Definition 2.20.** An Automorphic form on \( \text{GL}_2/F \) with grossencharacter \( \psi \) for \( F \) is any function \( \phi \) on \( G_{\mathbb{A}_F} \), satisfying the following:

1. \( \phi(\gamma g) = \phi(g) \) for all \( \gamma \in G_F \);
2. We have \( \phi(gz) = \phi(zg) = \psi(z)\phi(g) \) for all \( z \in Z_{\mathbb{A}_F} \), where \( Z_{\mathbb{A}_F} \) is the center of \( G_{\mathbb{A}_F} \);
3. Let \( R = K_\infty \prod_{p < \infty} K_v \), then we have that \( \phi \) is right \( R \)-finite. Where \( K_v = \text{GL}_2(\mathcal{O}_v) \) and \( K_\infty = \mathcal{O}_2(\mathbb{R})^J_F \), where, once again, \( \mathcal{O}_2(\mathbb{R}) \) are the orthogonal matrices.
4. As a function on \( G_\infty \) alone, \( \phi \) is smooth and \( Z(\mathfrak{g}) \)-finite (here \( \mathfrak{g} \) is the universal enveloping algebra of the complexification of \( G_\infty \));
5. \( \phi \) is slowly increasing;

Furthermore if we have

\[
\int_{F \setminus \mathbb{A}_F} \phi \left( \begin{bmatrix} 1 & \gamma \xi \xi^T \end{bmatrix} g \right) \, dx = 0
\]

for almost all \( g \) we call it a cusp form.

Next we are going to define a subset of this which will correspond to the adelic Hilbert modular forms.

Let \( n \subset \mathcal{O}_F \) be an integral ideal of \( F \). For \( v \) a finite place of \( F \), define

\[
K_v(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v \mid c \equiv 0 \pmod{n} \right\},
\]

and we let \( K_0(n) = \prod_{v \notin J_F} K_v(n) \). Now as \( \prod_{v \in J_F} \text{GL}_2(\mathbb{R})^+ \) acts transitively on \( \mathcal{H}_F = \mathcal{H}_s \) (where \( \text{Card}(J_F) = s \)) by fractional linear transformations (here \( \mathcal{H} \) is the usual complex upper half-plane). We see (analogously to the classical case) that the stabilizer of \((i, \ldots, i)\) is \( K_\infty^+ = (\mathbb{R}^\times \text{SO}_2(\mathbb{R}))^{J_F} \). If on each copy of \( \mathcal{H} \) we let \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) act by \( z \mapsto -\bar{z} \), we can extend (uniquely) the action of \( \prod_{v \in J_F} \text{GL}_2(\mathbb{R})^+ \) on \( \mathcal{H}_F \), to get an action of \( \prod_{v \in J_F} \text{GL}_2(\mathbb{R}) \) on \( \mathcal{H}_F \).

Now we define the factor of automorphy \( j(\gamma, g) \) for \( \gamma \in \prod_{v \in J_F} \text{GL}_2(\mathbb{R}) \) by

\[
j(\gamma, g) = \prod_{v \in J_F} j(\gamma_v, g_v) \quad \text{for } g = (g_v) \in \mathcal{H}_F,
\]

where the factors \( j(\gamma, g_v) \) on the right are the factors of automorphy defined previously. In what follows we will abuse notation and simply write \( j(\gamma, g) \) for \( j(\gamma, g) \).

An element \( \mathbf{k} = (k_v)_v \in \mathbb{Z}^{J_F} \), is called a weight vector. We will always assume that \( k_v \geq 2 \) and that they all have the same parity. Also let \( k_0 = \max_k k_v \), \( t_v = (k_0 - k_v)/2 \), \( \mathbf{1} = (1, \ldots, 1) \) and \( \mathbf{g} = \mathbf{k} + \mathbf{t} - \mathbf{1} \).

Now for \( g = (g_v) \in \mathcal{H}_F \) we let \( j(\gamma, g)_{\mathbf{k}} = \prod_{v \in J_F} j(\gamma_v, g_v)^{k_v} \), and \( \det(\gamma)^{k} = \prod_v \det(\gamma_v)^{k_v} \).

**Definition 2.21.** An adelic Hilbert modular form of weight \( \mathbf{k} \) and level \( \mathfrak{n} \), is a function \( f : G_{\mathbb{A}_F} \rightarrow \mathbb{C} \) satisfying the following:

1. \( f(\gamma g u) = f(g) \) for all \( \gamma \in G_F \), \( g \in G_{\mathbb{A}_F} \) and all \( u \in K_0(\mathfrak{n}) \).
2. \( f(ga) = \det(a)^2 j(a, i)^{-2} f(g) \) for all \( a \in K_F^\times \) and \( g \in \GL_2(A_F) \).

For \( x \in \GL_2(A_{F,j}) \) we define \( f_x : H_F \to \C \) (here \( A_{F,j} \) are the finite adeles of \( A_F \)), by
\[
 f_x(z) = \det(g)^{-2} j(g, i)^{k} f(xg),
\]
where \( g \in \prod_{v \in J_F} \GL_2(\mathbb{R})^+ \) is chosen such that \( gi = z \) (this is well-defined, by 2).

3. \( f_x \) is holomorphic for all \( x \in \GL_2(A_{F,j}) \).

4. Lastly, if we have
\[
 \int_{U(A_F)/U(\mathbb{Q})} f(ux) du = 0
\]
for all \( x \in \GL_2(A_F) \) and all additive Haar measures \( du \) on \( U(A_F) \) (the unipotent radical of \( \GL_2(F) \)) we call it an adelic cusp form.

We denote this space by \( M_k(n) \), and the subspace of cusp forms by \( S_k(n) \).

Now from this definition one can check that the elements of \( M_k(n) \) are in fact automorphic forms for \( \GL_2(F) \) (even though some of the conditions look slightly different!).

Next we give the definition of a Hilbert modular form and show how these are related the adelic Hilbert modular forms just defined. First take some fractional ideal \( m \) of \( F \), and set
\[
 \Gamma_0(m, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \frac{O_F}{mn} \right) \left( \frac{O_F^{-1}}{O_F} \right) \mid ad - bc \in O_F^\times \right\}. 
\]

Here \( O_F^\times \) denotes the elements of \( O_F^\times \), whose image under the embeddings of \( J_F \) into \( \mathbb{R} \) are positive. Here \( \Gamma_0(m, n) \) is an example of a congruence subgroup of \( G_F^+ = \{ \gamma \in G_F \mid \det(\gamma) > 0, \forall v \in J_F \} \).

**Remark** 2.22. We don’t give a precise definition of congruence subgroup, simply because we will not need it, for what we are interested, the example of a congruence subgroup defined above is enough.

Now we can define an action of \( \Gamma_0(m, n) \) on the space of functions \( f : H_F \to \C \) by
\[
 f|_k \gamma(z) = \left( \prod_{v \in J_F} \det(\gamma)_{\sigma_v}^{-2} j(\gamma, z)^{k_v} \right) f(\gamma z) \quad \text{for all } \gamma \in \Gamma_0(m, n). 
\]

**Definition 2.23.** The space \( M_k(m, n) \) of classical Hilbert modular forms of weight \( k \) and level \( \Gamma_0(m, n) \) consists of holomorphic functions \( f : H_F \to \C \) such that \( f|_k \gamma = f \).

Now since for any Hilbert modular form \( f \), we have \( f(z + u) = f(z) \) for all \( z \in H_F \) and \( u \in \mathfrak{d}^{-1} \), where \( \mathfrak{d} \) is the different ideal of \( F \). We get a Fourier expansion
\[
 f(z) = \sum_{u \in \mathfrak{d}^{-1}} a_u e^{2\pi i \Tr(uz)}. 
\]

**Definition 2.24.** If \( \Gamma \) is a congruence subgroup of \( G_F^+ \), then \( \Gamma \backslash \mathbb{P}^1(F) \) (the set of \( \Gamma \)-orbits) is called the set of cusps of \( \Gamma \).

Notice that in the definition of Hilbert modular form, we did not give a holomorphy condition at the cusps. This is due to the following result:

\[ \text{Recall that } \mathfrak{d}^{-1} = \{ x \in F \mid \Tr_{F/Q}(x) \in \Z, \exists x \in O_F \}. \]

9
Lemma 2.25. (Koecher’s Principle). For \([F : \mathbb{Q}] > 1\), every Hilbert modular form \(f \in \mathcal{M}(m, n)\) is holomorphic at the cusp \(\infty\) (and hence at all cusps), in the sense that for any \(a_n\) in the above Fourier expansion, we have \(a_n \neq 0 \Rightarrow u = 0\) or \(u\) is totally positive (i.e. its image under all of the embeddings of \(J_F\) is always positive).

Proof. See [7], Theorem 3.3, p. 64.

We can now define the set of cusp forms \(S_k(m, n)\), to be the subset of \(\mathcal{M}_k(m, n)\) consisting of elements \(f\), whose constant term \(a_0\) in the Fourier expansion of \(f|_k \gamma\) is zero for all \(\gamma \in G_F^+\) (or for \(\gamma\) a representative of a cusp).

Remark 2.26. One can show that \(\mathcal{M}_k(m, n) = S_k(m, n)\) unless we have \(k_v = k_{v'}\) for all \(v, v' \in J_F\), see [7], Proposition 3.6, p. 66.

We recall here the definition of the narrow class group.

Definition 2.27. The narrow class group for \(F\), is defined as \(C_F^+ = I_F/P_F^+\), where \(I_F\) is the set of all fractional ideals of \(\mathcal{O}_F\) and \(P_F^+\) is the group of principal fractional ideals \(a\mathcal{O}_F\), where \(a\) is totally positive.

In order to relate the classical Hilbert modular forms and the adelic Hilbert modular forms, we need to first see how to restate Theorem 2.10 in this setting.

Lemma 2.28. If \(K\) is a compact open subgroup of \(GL_2(k_{F,f})\), then there is the following bijection

\[G_F \backslash G_{A_F} / KG_{\infty}^+ = F \backslash \mathbb{A}_F / \det(K) \mathbb{R}^{J_F}.\]

Furthermore, if \(\det(K) = \mathcal{O}_F^\times = \prod_p \mathcal{O}_{F,p}^\times\), then the double coset above has the same cardinality as the narrow class group of \(F\).

Proof. (Sketch) This proof is essentially the same as the proof of Theorem 2.10 but in more generality.

Now let \(m_i\) for \(i = 1, \ldots, r\) be a complete set of representatives for the narrow ideal classes of \(F\), and take \(x_i \in G_{A_F,f}\) be such that \(\det(x_i)\) generates \(m_i\). Then using the fact that \(\det(K_0(n)) = \mathcal{O}_F^\times\), we have by the above Lemma, that since the \(x_i\) represent the double coset defined in the lemma, we get

\[G_{A_F} = \prod_{i=1}^{r} G_F x_i K_0(n) G_{\infty}^+.\]

With this we now have the following result:

Proposition 2.29. There is an isomorphism of complex vector spaces

\[\mathcal{M}_k(n) \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathcal{M}_k(m_i, n),\]

given by associating to each adelic Hilbert modular form \(f\), the \(r\)-tuple \((f_{x_1}, \ldots, f_{x_r})\), where the \(f_{x_i}\) are defined as in 2.21. Furthermore this isomorphism restricts to an isomorphism

\[S_k(n) \xrightarrow{\sim} \bigoplus_{i=1}^{r} S_k(m_i, n).\]

Proof. From definition 2.21 we know this map is well defined, and it is clearly \(\mathbb{C}\)-linear. All we need to get the result is to construct an inverse. We do this by sending the \(r\)-tuple \((f_{x_1}, \ldots, f_{x_r})\) to the function \(f : G_{A_F} \rightarrow \mathbb{C}\) given by \(f(\gamma x_j g) = (f_{x_j} \mid g_{\infty})(i)\) for \(j = 1, \ldots, r\) and all \(\gamma \in G_F, g \in K_0(n) G_{\infty}^+\). To see this is well-defined we can do something similar to what we did in Definition 2.15 and use the fact that the \(f_{x_i}\) are well-defined.
2.6 Hecke operators on Hilbert modular forms

In this section we define the Hecke action on $M_2(n)$. We take a prime ideal $p \in \mathcal{O}_F$, and we let $\varpi_p$ be an adelic generator of $p$. Now, the double coset

$$K_0(n) \left( \begin{smallmatrix} 1 & 0 \\ 0 & \varpi_p \end{smallmatrix} \right) K_0(n) = \bigcup_j w_j K_0(n).$$

With this we define $(T_p f)(x) := \sum_j f(xw_j)$. This gives us a linear map $M_2(n) \to M_2(n)$, and it preserves the space of cusp forms. One thing to note is that it does not preserve the space $M_2(m, n)$ in the decomposition above, these get permuted by the action of the Hecke operator.

When we discussed classical modular forms, we mentioned that on the space of cusp forms we can define the Petersson inner product, which allowed us to then define the space of newforms. Now we seek to do something similar in the case of Hilbert modular forms. First note that by Proposition 2.29 we can give a Fourier expansion for a modular form as follows.

$$f(x) = \sum_{c \in \mathcal{O}_K} a(c) e^{2\pi i c x},$$

where $x$ is as in 2.21. One can check that $a(f, \mathfrak{c})$ is independent of the choice of $\mu$ and of the choice of $\mathfrak{m}_i$ representing the ideal classes of the narrow class group. We call $a(f, \mathfrak{c})$ the Fourier coefficient of the form $f$ associated to the ideal $\mathfrak{c}$. For more details on the above see [8], Section 2.

Definition 2.30. We call a modular form $f$ an eigenform if it is an eigenvector for all the Hecke operators. We say an eigenform is normalized if $a(f, \mathcal{O}_F) = 1$.

Remark 2.31. One can check that $a(f, \mathfrak{c})$ is independent of the choice of $\mu$ and of the choice of $\mathfrak{m}_i$ representing the ideal classes of the narrow class group. We call $a(f, \mathfrak{c})$ the Fourier coefficient of the form $f$ associated to the ideal $\mathfrak{c}$. For more details on the above see [8], Section 2.

Now let $\mathfrak{n}_1 \mid \mathfrak{n}$ and $\mathfrak{n}_2 \mid \mathfrak{n}\mathfrak{m}_1^{-1}$. For each $f \in S_2(\mathfrak{n}_1)$, there is a unique form $f|\mathfrak{n}_2 \in S_2(\mathfrak{n})$, with Fourier coefficients satisfying $a(f|\mathfrak{n}_2, \mathfrak{c}) = a(f, \mathfrak{c}\mathfrak{m}_2^{-1})$ for all integral ideals $\mathfrak{c}$, with the convention $a(-, \mathfrak{c}) = 0$ if $\mathfrak{c}$ is not an integral ideal. With this then it is clear to see that we get a map $S_2(\mathfrak{n}_1) \to S_2(\mathfrak{n})$ that sends $f$ to $f|\mathfrak{n}_2$.

Definition 2.32. The space of old forms $S_{2,\text{old}}(\mathfrak{n})$ is the subspace of $S_2(\mathfrak{n})$ generated by the images of the linear maps $f \mapsto f|\mathfrak{n}_2$, as $\mathfrak{n}_1$ and $\mathfrak{n}_2$ vary (as above).

Remark 2.33. It can be shown that the space of old forms is invariant under the action of the Hecke operators.

Now as in the classical case, we can define an inner product on the space $S_2(\mathfrak{n})$ called the Petersson inner product, and once again we won’t give the details of how it is defined, as it won’t be needed, for details see [8], Section 2. Using this inner product we can then define the space of newforms $S_{2,\text{new}}(\mathfrak{n})$ to be the orthogonal complement of $S_{2,\text{old}}(\mathfrak{n})$ with respect to the Petersson inner product.
3 Automorphic forms on quaternion algebras

We begin by first giving some results about quaternion algebras and then explain how one can define automorphic forms over these quaternion algebras.

3.1 Quaternion algebras

Let $K$ be a field.

**Definition 3.1.** A quaternion algebra over $K$ is a $K$-algebra $D$, such that:

(a) The center of $D$, contains $K$.

(b) The dimension of $D$ as a vector space over $K$ is 4.

(c) $D$ has no non-trivial 2-sided ideals.

The standard example of a quaternion algebra is $M_2(K)$, and over $\mathbb{R}$ we have the Hamilton quaternions $\mathbb{H}$. We say that an algebra over $K$ is split if it is isomorphic to $M_2(K)$. If $L/K$ is a field extension, then $D \otimes_K L$ is a quaternion algebra over $L$, and furthermore if this new quaternion algebra over $L$ is split then we say that $L$ splits $D$.

Now we have the following useful result:

**Proposition 3.2.** If $K$ is a field and $D$ a quaternion algebra over $K$, then $D$ is not split if and only if every non-zero element of $D$ has an inverse.

**Proof.** (Sketch) Assume that $D$ is not a division algebra, then there is some proper left ideal $I$ in $D$, and this is also a $K$-vector space, and so must have dimension 1,2 or 3. One can then show that we cannot have $\dim(I) = 1,3$ as we would then get a $K$-algebra morphism from $D$ to $K$ and its kernel would be a two-sided ideal, which cannot happen. For example if $\dim(I) = 1$ then let $x$ be a generator of the $K$-vector space $I$. Now for $d \in D$, we have $dx$ is some constant multiple of $x$, so define $\rho(x) \in K$ for this scalar. Then $\rho : D \rightarrow K$ is a $K$-algebra morphism and hence surjective, so then the kernel would be a 3-dimensional two-sided ideal which cannot happen.

So we are left with $\dim(I) = 2$ from which we can get an isomorphism of $D$ with $M_2(K)$. For the details of the above and a proof of the other direction see §2.1, p. 76. □

Note that in particular the Hamilton quaternions are not split, since for $a + bi + cj + dk \in \mathbb{H}$, we have that $(a - bi - cj - dk)/a^2 + b^2 + c^2 + d^2$ is an inverse. But if we consider $\mathbb{H}$ as a quaternion algebra over $\mathbb{Q}$ and then base change to $\mathbb{Q}_p$ we get the quaternion algebra $\mathbb{Q}_p \oplus \mathbb{Q}_p i \oplus \mathbb{Q}_p j \oplus \mathbb{Q}_p k$, with $i, j, k$ satisfying the usual relations an $p$ and odd prime, one can show that this is split, in other words it is isomorphic to $M_2(\mathbb{Q}_p)$.

For the cases we are interested one can show what all the quaternion algebras are up to isomorphism.

- If $K$ is algebraically closed, then there is only one quaternion algebra up to isomorphism and it is $M_2(K)$.
- If $K$ is a finite field then there is only one up to isomorphism and it is $M_2(K)$.
- Over $\mathbb{R}$ we have two quaternion algebras $M_2(\mathbb{R})$ and $\mathbb{H}$.
- If $K$ is a finite extension of $\mathbb{Q}_p$, then there are two (up to isomorphism), the split one and one other.
If $F$ a totally real field, we can think of a quaternion algebra over $F$ as an $F$-algebra, generated by elements $x, y$ such that $x^2 = i$, $y^2 = j$ and $xy = -yx$ for some $i, j \in F^\times$, which is usually denoted by $D = \left( \frac{\mathbb{Q}}{\mathbb{Q}} \right)$ (note that this could be split).

See [9] Section 2.1.

If $D$ is a quaternion algebra then we have a conjugation $x \mapsto \bar{x}$ and we can use this to define $nr : D \to F$ by $nr(x) = x\bar{x}$. Explicitly, for $D = \left( \frac{\mathbb{Z} h}{\mathbb{Z}} \right)$ we have that for $\alpha = a + bx + cy + dz$, $nr(\alpha) = a^2 - bc + ad - bd$.

Now let $K$ be a number field and let $D$ be a quaternion algebra over $K$, then if we complete $K$ with respect to a place $v$, to get $K_v$. Let $D_v = D \otimes_K K_v$. Now this new quaternion algebra can be split, in which case we say that $D$ is split at $v$, otherwise we say $D$ is ramified at $v$. It turns out that the set of places $S(D)$ at which $D_v$ is ramified can determine $D$ (up to isomorphism).

**Theorem 3.3.** Let $D$, $S(D)$ and $K$ be as above, then:

1. The set $S(D)$ is finite and has an even number of elements, none of which is complex.
2. For any set $S$ of places of $K$ not containing the complex places and having an even number of element, then there is exactly one quaternion algebra (up to isomorphism) $D/K$ with $S(D) = S$.

**Proof.** The proof of 1, basically comes down to using the product formula for Hilbert symbols, but for 2 we require a little more work. Both these results can be deduced from [10] XIII.3, Theorem 2 and XIII.6, Theorem 4. \hfill \square

We say a quaternion algebra $D/\mathbb{Q}$ is definite if $\infty \in S(D)$, and in general for a quaternion algebra $D/K$, we say it is totally definite if it ramifies at all the real places of $K$.

**Definition 3.4.** The discriminant $\text{disc}(D)$ of a quaternion algebra is the product of the finite primes in $S(D)$.

**Definition 3.5.** An order in a quaternion algebra $D/K$ is a subring $\mathcal{O} \subset D$ such that as abelian groups $\mathcal{O} \cong \mathcal{O}_K^4$ and $\mathcal{O} \otimes K = D$. We call an order maximal if it is not properly contained in any other order. Also we call an order an Eichler order if it can be written as the intersection of two maximal orders.

Furthermore is $\mathfrak{n}$ is an ideal of $\mathcal{O}_K$ coprime to $\text{disc}(D)$ and we let $\mathcal{O}_{K, \mathfrak{n}}$ be the completion of $\mathcal{O}_K$, then for a maximal order $\mathcal{O}$ we have an isomorphism $f : \mathcal{O} \to \mathcal{O} \otimes_{\mathcal{O}_K} \mathcal{O}_{K, \mathfrak{n}} \cong M_2(\mathcal{O}_{K, \mathfrak{n}})$. Define an Eichler order of level $\mathfrak{n}$ as $\mathcal{O}(\mathfrak{n}) = \{ x \in \mathcal{O} \mid f(x) \text{ is upper triangular } \bmod \mathfrak{n} \}$.

**Remark 3.6.** It is a theorem that maximal orders exist.

### 3.2 Automorphic forms for a quaternion algebra over $\mathbb{Q}$

In this section I am following [12]. In order to define these automorphic forms over a quaternion algebra $D/\mathbb{Q}$ we first need some preliminaries.

In this section we will work with a definite quaternion algebra $D/\mathbb{Q}$ for simplicity and fix a maximal order $\mathcal{O}$. Let $D_p = D \otimes \mathbb{Q}_p$ and $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$, then $\mathcal{O}_p$ will be an order of $D_p$.

Similarly we set $D_\mathfrak{a} = D \otimes \mathfrak{a}$, and $D_f = D \otimes_{\mathbb{Q}} \mathfrak{a}_f$. Note that $D_f$ will inherit a topology from $\mathfrak{a}_f$, also we can embed $D$ diagonally into $D_f$. Now we will mainly work with $D_f^\times$, which has the subspace topology when embedded in $D_f \times D_f$ (this has product topology) as $x \mapsto (x, x^{-1})$. Also we write $D_\infty = D \otimes_{\mathbb{Q}} \mathbb{R}$.

**Theorem 3.7.** If $D$ is a definite quaternion algebra over $\mathbb{Q}$ and $K \subset D_\mathfrak{a}^\times$ is an open compact subgroup, then the double coset space $D^\times \backslash D_\mathfrak{a}^\times / KD_\infty^\times$ is finite.

\footnote{Here by completion we mean as in [11], Chapter 10.}

\footnote{The case for indefinite quaternion algebras will be a special case of what we define in the next section.}
Proof. (Sketch) The work here is in proving that $D^\times \setminus D^\times \mathfrak{A}$ is compact, once we have this we use the fact that taking a quotient of a topological group by an open subgroup gives a discrete group, to deduce the double quotient is a discrete and compact topological space, hence it is finite. For details see [1] Theorem 2.8, p. 85.

Next we need some examples of $K$ as above. Pick some integer $M$ prime to the $\text{disc}(D)$, and for a prime $p$ not dividing $\text{disc}(D)$, we let $e = v_p(M)$ (number of times $p$ divides $M$). If we define $K_{0,p}$ (respectively $K_{1,p}$) to be the elements in $O_p^\times = \text{GL}_2(Z_p)$ which are congruent to $\begin{pmatrix} a & * \\ 0 & a \end{pmatrix}$ (respectively $\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}$) mod $p^e$ and $K_p = O_p^\times$ if $p \nmid M$. Define

$$U_0(M) = \prod_{p|M} K_p \times \prod_{p|M} K_{0,p}$$

and similarly $U_1(M)$. Then both $U_0(M)$ and $U_1(M)$ are open compact subgroups of $D^\times \mathcal{L}$. See [13] for more detail on how to compute the coset representatives explicitly.

Now for an integer $m \geq 0$, we let $P_m$ be the subspace of $\mathbb{C}[x,y]$ of homogeneous polynomials of degree $m$. We define a right action of $\text{GL}_2(\mathbb{C})$ on $P_m$ as follows: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C})$, we let $\overline{\gamma}$ denote its adjoint. Now for $f \in P_m$, we let $(f \cdot \overline{\gamma})(x,y) = f(dx-cy,-bx+ay)$. Now let $L_k$ be $P_{k-2}$ viewed as a $\text{GL}_2(\mathbb{C})$-module under the right action defined above.

Next we fix an isomorphism of $D \otimes \mathbb{C} \cong M_2(\mathbb{C})$, which gives us a way of viewing elements of $D^\times$ as elements of $\text{GL}_2(\mathbb{C})$. Hence by the above we have that $D^\times$ acts on $L_k$.

Definition 3.8. Let $D$ be as above, we define the space of quaternionic automorphic forms of level $U_1(M)$ and weight $k$ for $D$ as

$$S_k^D(U_1(M)) = \left\{ f : D^\times \longrightarrow L_k \mid f(dgu) = f(g)u_\infty, \text{ for } d \in D^\times, g \in D^\times_f, u \in U_1(M) \times D^\times_\infty \right\}.$$ 

Here $u_\infty$ is the component of $u$ in $D^\times_\infty$.

Contrary to the definition of an automorphic form over a number field, in this case we do not need a holomorphy or growth condition as in this case we have a one point topological space at infinity.

Note that since $D^\times = \bigcup_{i=1}^n D^\times d_i U_1(M) D^\times_\infty$, then we see that $f \in S_k^D(U_1(M))$ is determined by its values on \{ $f(d_1), \ldots, f(d_n)$ \}.

Furthermore, we can decompose the space $S_k^D(U_1(M))$ as follows: Let $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a character, we consider it a character of $U_0(M) \times D^\times_\infty$ via the map $U_0(M) \longrightarrow (\mathbb{Z}/M\mathbb{Z})^\times$ defined by sending a matrix to its lower right entry modulo $M$ and acting trivially on $D^\times_\infty$.

$$S_k^D(U_1(M)) = \bigoplus_{\chi} S_k^D(U_1(M)) (\chi),$$

where the sum is over all the characters $\chi$ as above, and

$$S_k^D(U_1(M)) (\chi) = \left\{ f : D^\times_f \longrightarrow L_k \mid f(dgu) = \chi(u)f(g)u_\infty, \text{ for } d \in D^\times, g \in D^\times_f, u \in U_1(M) \times D^\times_\infty \right\}.$$ 

3.3 Hecke operators

As before we can define Hecke operators on these quaternionic modular forms, but in this case the definition is much more similar to definition for standard modular forms. Let $\eta \in D^\times_\mathfrak{A}$, and for $f : D^\times \longrightarrow A$ define $(f \cdot k\eta)(g) = f(g \eta)\eta_\infty^{-1}$. Now for $U = U_0(M)$ or $U_1(M)$, we can write the double coset $U \eta U = \bigcup_i U \eta_i$, where $\eta_i$ are coset representatives. We can now define the Hecke operator $[U \eta U]$ as

$$[U \eta U] f = \sum_i f|_{k \eta_i}.$$
Now, if we are looking for the analogues of $T_l$ for $l$ not dividing $\text{disc}(D)$. We define them in this setting as $T_l = [U \eta U]$, where $\eta = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$, and $\omega_l \in \mathbb{A}_F$ is the finite adele which is $l$ at the place $l$ and the identity at all other places. We say a quaternionic modular form (as defined above) is an eigenform, if it is an eigenvector for all the Hecke operators.

### 3.4 Automorphic forms for a quaternion algebra over a totally real field

We now seek to generalize the above for the case of Hilbert modular forms, here I am following [14]. Which means we are working with totally real fields $F$ of degree $n > 1$ over $\mathbb{Q}$.

Let $D$ be a quaternion algebra over $F$, and let $J_D^s$ be the subset of $J_F$ of places where $D$ splits, and $S(D) = J_F - J_D^s$ (complement). With this we have that

$$D^\times = (D \otimes \mathbb{Q} \mathbb{R})^\times \cong \text{GL}_2(\mathbb{R})^\times \times (\mathbb{H}^\times)^r,$$

with $\mathbb{H}$ the usual Hamilton quaternions and $s = |J_F^s|$ and $r = |S(D)|$. From this it is clear that $D^\times$ acts on $\mathcal{H}_\pm^s$ on the right, where $\mathcal{H}_\pm = \mathbb{C} - \mathbb{R}$, and the stabilizer of $(i, \ldots, i) \in \mathcal{H}^s$ is clearly $K_\infty = (\mathbb{R}^\times \text{SO}_2(\mathbb{R}))^s \times (\mathbb{H}^\times)^r$. As before we define $D_F = D \otimes \mathbb{A}_F$. For $n$ an ideal of $\mathcal{O}_F$ coprime to $\text{disc}(D)$, we define an Eichler order $\mathcal{O} = \mathcal{O}(n)$ of level $n$, and let $\mathcal{U} = \mathcal{O}(n) \otimes \mathbb{Z}$.

For a weight vector (with entries all having the same parity) $k = (k_v), v \in \mathbb{Z}^F$ we let (as before) $k_0 = \max_v k_v$, $t_v = (k_0 - k_v)/2$, $\sigma = \frac{k_0 + 1}{2} - 1$ and $m_v = k_v - 2$. Now recall form the previous section we had an action of $\text{GL}_2(\mathbb{C})$ on $P_m$, we now twist this action by the character $\chi_t(\gamma) = \det(\gamma)^t$ for $\gamma \in \text{GL}_2(\mathbb{C})$, and $t$ an integer. We denote this new module by $P_m(t)$. Now define

$$L_k = \bigotimes_{v \in S(D)} P_{m_v}(t_v),$$

with the convention that $L_k = \mathbb{C}$ if $J_F^s = J_F$. For each place $v \in S(D)$ we choose a splitting $D \hookrightarrow D \otimes \mathbb{C} \cong M_2(\mathbb{C})$. For any $d \in D$ we let $d_v$ denote the image of $d$ under the embedding corresponding to the place $v \in S(D)$. With this we can make $L_k$ a $D^\times$ module by sending $d \mapsto (d_v)_{v \in S(D)} \in \text{GL}_2(\mathbb{C})^r$ (note that this is just a twisted ‘higher dimensional’ analogue of what we did in the previous section). For $P \in L_k$ and $d \in D^\times$ we denote by $P^d$ the action of $d$ on $P$ defined above.

Now we define an action of $D^\times$ on the space of functions $f : \mathcal{H}_\pm^s \times D_F^s / \mathcal{U}^\times \longrightarrow L_k$ by

$$(f|_k d)(z, u) = \left( \prod_{v \in J_F^s} \det(d_v)^{\sigma_v} \right) f(dz, du)^d,$$

where we choose $g \in D_\infty^s$ such that $g(i, \ldots, i) = z$ and we let $z_v = g_v(i, \ldots, i)$. Notice here that $F^\times \subset D^\times$ acts by $(f|_k d)(z, u) = N_{F/\mathbb{Q}}(f)(z, du)$, for $d \in F^\times$, where $N_{F/\mathbb{Q}}$ is the usual norm.

Notice that on the places where the quaternion algebra splits we have the usual transformation factor as in the case of classical Hilbert modular forms, and it is on the places where the quaternion algebra is ramified that we need to introduce a new action.

We can now make the following definition:

**Definition 3.9.** A quaternionic modular form of weight $k$ and level $n$ for $D$ is a function $f : \mathcal{H}_\pm^s \times D_F^s / \mathcal{U}^\times \longrightarrow L_k$ that is holomorphic in the first variable and locally constant in the second, such that $f|_k d = f$ for all $d \in D^\times$. We denote the space of such functions by $\mathcal{M}_k^D(n)$.

Notice that here we need a holomorphic condition on the first variable which we did not have in the previous section, since for definite quaternion algebras over $\mathbb{Q}$ (or totally definite over $F$), we have $s = 0$ so the first factor
is trivial. Also with a little work it can be shown that if $D = M_2(\mathbb{Q})$ then we recover the classical definition of modular forms, and for $D = M_2(F)$ we recover the classical definition of Hilbert modular forms over $F$.

Next we define the subset of $\mathcal{M}_D^D(n)$ of cusp forms (which in many cases is the whole space). If we have $D = M_2(F)$ then we are in the case of classical Hilbert modular forms, so we define a cusp form $f \in S^D(n) \subseteq \mathcal{M}_D^D(n)$, to be a quaternionic form such that $f(z, U^x) \to 0$ whenever $z$ tends to a cusp in $\mathbb{P}^1(F)$. If $D$ is indefinite then it can be shown that there are no cusps, so $S^D(n) = \mathcal{M}_D^D(n)$. In the case that $D$ is totally definite then for $k \neq (2, \ldots, 2)$ we have $S^D(n) = \mathcal{M}_D^D(n)$, but for $k = (2, \ldots, 2)$ we have an orthogonal decomposition $\mathcal{M}_D^D(n) = S^D(n) \oplus E^D(n)$ where $E^D(n)$ is the subspace of $\mathcal{M}_D^D(n)$ of forms that factor through $nr: D_f \to \mathbb{A}^n_F$.

In order to show the connection between what we have just defined and what we did in the previous section for definite quaternion algebras over $\mathbb{Q}$, we now look at the case where $D/F$ is a totally definite quaternion algebra (note that for this to work we must have either $[F: \mathbb{Q}]$ being even or we can introduce a finite place in $S(D)$). In this case one can simplify the above definitions of quaternionic modular forms and one gets that $f \in \mathcal{M}_D^D(n)$ is simply a map $f: D_f^x/U^x \to L_\mathbb{A}$ such that $f|_d = f$ for all $d \in D^x$ and once we note that $D^x/D_f^x/U^x$ is again a finite set (same proof as before works here), then we see that this is just a ‘higher dimensional’ analogue of the results from the previous section.

3.5 Hecke operators on quaternionic modular forms over $F$

Lastly we seek to define the Hecke operators on $S^D_L(n)$. These are pairwise commuting diagonalizable operators $T_i$ in the complex vector space $S^D_L(n)$, indexed by the non-zero ideals $c \in \mathcal{O}_F$. For $f \in S^D_L(n)$ and $d \in D_f^x$, we define the Hecke operator $T_d$ as follows: we first write

$$U^x dU^x = \bigcup_i U^x d_i,$$

and we let

$$(T_d f)(z, u) := \sum_i f(z, ud_i^{-1})$$

Now for a prime ideal $p$ coprime to $n \cdot disc(D)$, we denote by $T_p$ the Hecke operator $T_\pi$ where $\pi \in D_f^x$ is such that $\pi_v = 1$ (the component at $v$) for $v \neq p$ and $\pi_p = \begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,p} \cong M_2(\mathcal{O}_{F,p})$,

where $\pi_p$ is the uniformizer at $p$ in $\mathcal{O}_{F,p}$.

As usual we say a quaternionic modular form is an eigenform if it is an eigenvector for all the Hecke operators. Furthermore we say that a cusp form $f$ is a newform if it is an eigenform and it does not belong to $\mathcal{M}_D^D(m)$ for $m \mid n$.

4 Classical Jacquet–Langlands Correspondence

Now that we have set up the theory of automorphic forms over $GL_2$ and over quaternion algebras, we can finally talk about how they are linked. In general, this idea of relating spaces of classical modular forms and quaternionic modular forms, is very useful, since what one does is reduce the problem to computing a finite amount of data on the quaternionic side, and then we can deduce results in the classical setting. For this relation between the classical and quaternionic side we have the following Theorem, which we do not prove as its proof involves the theory of automorphic representations, which we won’t discuss here.
Theorem 4.1. (Eichler, Jacquet–Langlands, Shimizu). Let $D$ be a quaternion algebra over a totally real field $F$, with $\text{disc}(D) = \mathcal{D}$ and $n$ a ideal coprime to $\mathcal{D}$. Then we have an isomorphism of $S^D_k(\mathcal{N}) \cong S^D_k - \text{new}(\mathcal{D} \mathcal{N})$, where $\mathcal{D} - \text{new}$ denotes the forms that are new at all the primes dividing $\mathcal{D}$.

Unpacking all of the above definition, for the case where $D$ is a definite quaternion algebra over $\mathbb{Q}$, then we get:

Corollary 4.2. Let $k \geq 3$ be an integer and $M$ a positive integer prime to $d = \text{disc}(D)$. Then there is an isomorphism

$$S^D_k(U_1(M)) \cong S^D_k - \text{new}(\Gamma_1(M) \cap \Gamma_0(d)),$$

that is compatible with the action of the Hecke operators as defined above. In the case $k = 2$ then $S^D_2 - \text{new}(\Gamma_1(M) \cap \Gamma_0(d))$ is isomorphic to the quotient of $S^D_2(U_1(M))$ by the subspace of forms which factor through the norm map, and once again this isomorphism is compatible with the action of the Hecke operators.

One important consequence of this result is that we now have a good way of computing the spaces $S^D_k - \text{new}(\Gamma_1(M) \cap \Gamma_0(d))$, since we only need to compute $S^D_k(U_1(M))$, and the elements here are determined by the images of the double coset representatives of $D^\times \backslash D_f^\times / U_1(M)$, and for this there are explicit algorithms.

The Theorem above is not stated in the same way as the original Jacquet-Langlands theorem, which is stated in the language of automorphic representations, but it can be shown that this is just a concrete realization of Theorem 16.1 of [1].

5 The $p$-adic Jacquet–Langlands correspondence.

In this last section we will very briefly look at how the Jacquet–Langlands correspondence extends to $p$-adic overconvergent modular forms. On the $p$-adic side, the theory requires a lot more work in order to give a general definition of a $p$-adic or overconvergent modular form. So we will only give a hint as to what the objects involved are, by using the less general definition of a $p$-adic modular form given by Serre. On the quaternionic side things are much easier, and we will define the space of overconvergent automorphic forms on a quaternion algebra.

For this we will first need some more definitions.

Definition 5.1. Let $f$ be a modular form of level one, and such that the $q$-expansion $f(q) = \sum a_n q^n$ has $a_n \in \mathbb{Q}$ for all $n$. Then we define a valuation on $f$ by setting $v_p(f) := \inf_n \text{val}_p(a_n)$, where $\text{val}_p(a_n)$ is the usual $p$-adic valuation.

Now following Serre we make the following definition.

Definition 5.2. A $p$-adic modular form is a formal power series $f = \sum a_n q^n$, where $a_n \in \mathbb{Q}_p$ and there exists a sequence $(f_i)$ of modular forms of level 1 and weight $k_i$, such that

$$\lim_{i \to \infty} v_p(f - f_i) = \infty,$$

i.e. $f$ and $f_i$ get closer $p$-adically as $i$ increases.

Examples 5.3. 1. Using the well-known relation $E_{p^r(p-1)} \equiv 1 \mod p^{r+1}(p-1)$ for $r \geq 1$ for Eisenstein series, we see that we can think of these Eisenstein series as $p$-adic approximations to the constant form 1 of weight 0. We say that $\{E_{p^r(p-1)}(z) \mid r \geq 1\}$ form a $p$-adic family.

2. The quasi-modular form $E_2$, which is well-known not to be a modular form, can be seen to be a $p$-adic modular form by defining $E_2^*(z) = E_2(z) - 2E_2(2z)$ and then noting that

$$E_2 = E_2^* + 2E_2^* + 4E_2^* + \ldots,$$
which shows that $E_2$ is in fact a 2-adic modular form.

The definition given above was first put forward by Serre, and it gives some important motivation for the study of such objects. There is a more comprehensive definition of $p$-adic modular forms given by Katz, but this is harder to define, so we will not go into it here, since we only want to give a flavor of the $p$-adic side of things. Using the definition given by Katz it is possible to define an analogue of the $U_p$ operator from the classical setting. This is where the idea for overconvergent modular forms comes in, since it turns out that on the space of all $p$-adic modular forms, the $U_p$ operator isn’t compact. To fix this we restrict to a subset where it is a compact operator, and this subset is what we call the overconvergent modular forms.

On the quaternionic side we have a simpler way of defining overconvergent automorphic forms as follows.

For any $\alpha \geq 1$, define $M_\alpha$ to be the the monoid consisting of matrices $(a \ b)\ c\ d)$ over $\mathbb{Z}_p$ with non-zero determinant such that $p^\alpha \ | \ c$ and $p \nmid d$.

**Definition 5.4.** For $U$ an open compact subgroup of $D_f^\chi$ and $\alpha \geq 1$, we say that $U$ has wild level $\geq p^\alpha$ if the projection of $U$ onto $GL_2(\mathbb{Q}_p)$ is in $M_\alpha$.

For example if $p^\alpha \ | \ M$, then $U_0(M), U_1(M)$ as above have wild level $\geq p^\alpha$.

Now take $E$ a complete subfield of $\mathbb{C}_p$ and define $A_{k,1}$ to be the ring of power series $\sum_{n=0}^{\infty} a_nz^n$ such that $a_n \to 0$ as $n \to \infty$. We give $A_{k,1}$ an action of $M_\alpha$ as follows:

Let $\gamma = (a \ b)\ c\ d \in U_\alpha$ and let $h \in A_{k,1}$. Then define

$$(h | k \ \gamma)(z) = (cz + d)^{k-2} h \left( \frac{az + b}{cz + d} \right)$$

Now fix a prime $p$ and let $D/\mathbb{Q}$ be a definite quaternion algebra, with discriminant prime to $p$. Furthermore fix an order $\mathcal{O}$ of $D$ and an isomorphism $\mathcal{O} \otimes B \cong M_2(B)$, with $B = \lim\downarrow (\mathbb{Z}/M\mathbb{Z})$ where the limit is taken over all integers $M$ prime to $disc(D)$ (this is an isomorphism since we are taking the limit over all the primes where $D$ is split and $B \cong \prod_{l \mid disc(D)} \mathbb{Z}_p$). This isomorphism induces isomorphisms $\mathcal{O} \otimes \mathbb{Z}_l \cong M_2(\mathbb{Z}_l)$ and $\mathcal{O} \otimes \mathbb{Q}_l \cong M_2(\mathbb{Q}_l)$, for primes $l \mid disc(D)$. Now thinking of $D_f$ as the restricted topological product over all primes $l$ of $D \otimes \mathbb{Q}_l$, then for $g \in D_f$ we let $g_p$ denote the component at $p$, viewed as an element of $M_2(\mathbb{Q}_p)$ by the isomorphism above.

**Definition 5.5.** If $D/\mathbb{Q}$ is a quaternion algebra, and $M > 0$ is an integer coprime to $disc(D)$, we define the space of overconvergent automorphic forms of level $U_1(M)$ and weight $k$ for $D$ as

$$S^D_k(U_1(M)) = \{ f : D_f^\chi \to A_{k,1} | f(dg) = f(g)|_ku_p, \text{ for all } d \in D^\times, u \in U_1(M) \}$$

Now, what the $p$-adic Jacquet–Langlands correspondence does is it relates this space of overconvergent automorphic forms for $D$, with a space of overconvergent modular forms (which we have not defined). The correspondence first proven by Gaëtan Chenevier in [15]. The main idea to prove this correspondence is to use the classical case and then make use of a density argument combined with rigid GAGA, to get an isomorphism between the relevant spaces. The importance of such a result is that on the quaternionic side, we can compute things explicitly, and then get results for overconvergent $p$-adic modular forms.

Now, one can also do something similar and define $p$-adic Hilbert modular forms, by generalizing Katz’s construction to totally real fields, and then one can define overconvergent Hilbert modular forms. But in this case there isn’t (yet) an analogue of the above results, this is what I shall be working on during my PhD.
References


