LECTURE NOTES FOR MA3H5 MANIFOLDS; WARWICK AUTUMN TERM 2020

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1. The general notion of a manifold

Manifolds show up in many guises in mathematics and the sciences. The following notion helps to introduce some order into the variety of ways that the term "manifold" is used in different contexts.

Definition 1.1. A *pseudogroup of transformations* on a topological space S is a set Γ of transformations satisfying the following axioms:

a) Each $f \in \Gamma$ is a homeomorphism of an open set (called the domain of f) onto another open set (called the range of f) of S;
b) If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;
c) Suppose an open subset U of S is a union $U = \bigcup_i U_i$ of open subsets U_i of S. Then a homeomorphism f of U onto an open subset of S belongs to Γ if the restriction of f to U_i is in Γ for every i;
d) For every open set U of S, the identity transformation of U is in Γ;
e) If $f \in \Gamma$, then also $f^{-1} \in \Gamma$;
f) If $f \in \Gamma$ is a homeomorphism of U onto V and $f' \in \Gamma$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \circ f$ of $f^{-1}(V \cap U')$ onto $f'(V \cap U')$ is in Γ.

Phew! That's a long definition! Luckily, none of its parts is too deep: b) and c) can be paraphrased by saying that belonging to Γ is a local condition, and f) can be expressed more simply by saying that compositions of maps in Γ give an element of Γ whenever defined. Let's look at some examples to make this seem less revolting:

Example 1.2. (1) Let $S = \mathbb{R}^n$ and let $\Gamma_{\mathscr{C}^0}$ be the set of all homeomorphisms $f\colon U \to V$ of an open set U in $\mathbb{R}^n$ onto an open set V in $\mathbb{R}^n$.

(2) Let $S = \mathbb{R}^n$ and let $\Gamma_{\mathscr{C}^r}$ be the set of all homeomorphisms $f\colon U \to V$ of an open set U in $\mathbb{R}^n$ onto an open set V in $\mathbb{R}^n$ such that both f and f^{-1} are differentiable of class $\mathscr{C}^r$, $r = 1, 2, \dots, \infty$, and
(3) Let $S = \mathbb{R}^n$ and let $\Gamma_{\mathscr{C}^r,0}$ be the set of all homeomorphisms $f\colon U \to V$ of an open set U in $\mathbb{R}^n$ onto an open set V in $\mathbb{R}^n$ such that both f and f^{-1} are differentiable of class $\mathscr{C}^r$, $r = 1, 2, \dots, \infty$ and moreover, the Jacobian matrix of f has positive determinant everywhere on U.
(4) Let $S = \mathbb{R}^n$ and let $\Gamma_{\mathscr{C}^\omega}$ be the set of all homeomorphisms $f\colon U \to V$ of an open set U in $\mathbb{R}^n$ onto an open set V in $\mathbb{R}^n$ such that both f and f^{-1} are real analytic.
(5) Let $S = \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$, and let Γ_{hol} be the set of all homeomorphisms $f\colon U \to V$ of an open set U in $\mathbb{C}^n$ onto an open set V in $\mathbb{C}^n$ such that both f and f^{-1} can be developed in a convergent power series in the z_i locally around every point of their domains.

(6) Let $S = \mathbb{R}^n$ and Γ_{aff} be the set of all homeomorphisms $f: U \to V$ of an open set U in $\mathbb{R}^n$ onto an open set V in $\mathbb{R}^n$ that locally around every point of U can be written as
$$f(x) = Ax + b$$
for A some nonsingular $n \times n$-matrix and $b \in \mathbb{R}^n$. If we require in addition A to have positive determinant, we get another pseudogroup Γ^+_{aff}.

The good news is that now we are almost ready to define manifolds already!

Definition 1.3. Let M be a topological space. An *atlas* of M *compatible with a pseudogroup of transformations* Γ is a family of pairs $\{(U_i, \varphi_i)\}$, called *charts*, such that

(i) Each U_i is an open subset of M and $M = \bigcup_i U_i$, i.e. the U_i cover M.
(ii) Each φ_i is a homeomorphism of U_i onto an open set of S.
(iii) Whenever $U_i \cap U_j$ is non-empty, the mapping $\varphi_j \circ \varphi_i^{-1}$ of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$ is an element of Γ.

Sometimes the φ_i are also called local Γ-coordinates, especially when $S = \mathbb{R}^n$, or $\mathbb{C}^n$. For a point $p \in U_i$ one then often writes
$$\varphi_i(p) = (x_1(p), \dots, x_n(p))$$
and calls $x_1(p), \dots, x_n(p)$ the local coordinates of p, and the function $x_1, \dots, x_n$ simply local coordinates. The atlas itself is sometimes also called a Γ-structure on M. Basically, we would like to call such an M a Γ-manifold. However, to be politically correct, we need to choose the atlas maximal in a certain sense:

Definition 1.4. A *complete atlas* on M *compatible with* Γ is an atlas of M compatible with Γ which is not contained in any other atlas of M compatible with Γ.

Note: every atlas of M compatible with Γ is contained in a unique complete atlas of M compatible with Γ. In fact, given an atlas $A = \{(U_i, \varphi_i)\}$ of M compatible with Γ, let $\tilde{A}$ be the family of all pairs (U, φ) such that φ is a homeomorphism of an open set U of M onto an open set of S and that
$$\varphi_i \circ \varphi^{-1}: \varphi(U \cap U_i) \to \varphi_i(U \cap U_i)$$
is an element of Γ whenever $U \cap U_i$ is nonempty. Then $\tilde{A}$ is the complete atlas containing A.

Definition 1.5. A Γ-*manifold* is a Hausdorff topological space M that is second countable, together with a fixed complete atlas compatible with Γ (sometimes also called a Γ-*structure* on M). Depending on the particular Γ, people have come up with a plethora of special names; we keep the notation from Example 1.2.

a) If $\Gamma = \Gamma_{\mathscr{C}^0}$, M is called a *topological manifold.*
b) If $\Gamma = \Gamma_{\mathscr{C}^r}$, $r = 1, 2, \dots,$ M is called a *differentiable manifold of class* $\mathscr{C}^r$. If $\Gamma = \Gamma_{\mathscr{C}^\infty}$, M is called a *smooth manifold* or a *differentiable manifold* (without further qualification).
c) If $\Gamma = \Gamma_{\mathscr{C}^r,0}$, M is called an *orientable differentiable manifold of class* $\mathscr{C}^r$, $r = 1, 2, \dots,$ and if $\Gamma = \Gamma_{\mathscr{C}^\infty,0}$, M is called an *orientable differentiable manifold* or *orientable smooth manifold.*
d) If $\Gamma = \Gamma_{\mathscr{C}^\omega}$, then M is called a *real analytic manifold.*
e) If $\Gamma = \Gamma_{hol}$, M is called a *complex manifold.*

f) If $\Gamma = \Gamma_{aff}$, M is called an *affine manifold*, and if $\Gamma = \Gamma_{aff}^{+}$, M is called an *orientable affine manifold*.

In cases a), b), c), d), f), $S = \mathbb{R}^n$, and this n is called the (real) *dimension* of M; in case e), $S = \mathbb{C}^n$, and this n is called the (complex) dimension of M.

There are even other types in common use: symplectic manifolds etc. To avoid chaos and to focus on a particularly rewarding and useful class we agree:

For us the term *manifold*, without qualification, will always mean smooth manifold.

The additional requirement that M be *Hausdorff* seems reasonable. The requirement that M be *second countable* is another tameness assumption on the topological space M (it should not be too "wild" or "big"). We will never encounter a not second countable space in the sequel and will not dwell too much on this extra assumption.

Note that if $\Gamma' \subset \Gamma$ for pseudogroups Γ', Γ then every atlas of M compatible with Γ' is also an atlas of M compatible with Γ. For example, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by splitting each complex coordinate z_j, $j = 1, \dots, n$ into real and imaginary parts: $z_j = x_j + \sqrt{-1}y_j$, then we see that every complex manifold can also be viewed as a smooth manifold, and every smooth manifold is also a topological manifold.

Definition 1.6. Let M and N be smooth manifolds. A continuous map $f\colon M \to N$ is called a *morphism* of manifolds if for every chart (U_i, φ_i) of M and every chart (V_j, ψ_j) of N with $f(U_i) \subset V_j$, the composition $\psi_j \circ f \circ \varphi_i^{-1}$ is a $\mathscr{C}^\infty$. We also call such an f simply a *smooth map* or a *differentiable map* or even just a *map*. An *isomorphism* f of smooth manifolds is a morphism f as above with an inverse $g\colon N \to M$ which is a morphism. Recall that being an inverse means $g \circ f = \mathrm{id}_M$, $f \circ g = \mathrm{id}_N$. Such an isomorphism is also called a *diffeomorphism*, and the corresponding manifolds are said to be *diffeomorphic*.

Remark 1.7. Of course one can make similarly define morphisms for other types of manifolds: for topological manifolds one would require the $\psi_j \circ f \circ \varphi_i^{-1}$ to be merely continuous, for differentiable manifolds of class $\mathscr{C}^r$ one would demand that $\psi_j \circ f \circ \varphi_i^{-1}$ is differentiable of class $\mathscr{C}^r$, for complex manifolds the $\psi_j \circ f \circ \varphi_i^{-1}$ should be complex analytic, i.e. locally developable into convergent power series in the complex coordinates z_i, and so on.

Exercise 1.8. Show that it is sufficient for a map $f\colon M \to N$ to be a morphism of manifolds if the following holds: for every $x \in M$ there *exists* a chart (U, φ), $x \in U$, and there *exists* a chart (V, ψ), $f(x) \in V$, with $\psi \circ f \circ \varphi^{-1}$ a $\mathscr{C}^\infty$ map.

Cultural digression. For $\Gamma' \subset \Gamma$ pseudogroups, it is one of the most important (and difficult) problems in geometry to determine the different Γ'-structures on a given M, up to isomorphism, that give rise to one and the same Γ-structure on M (up to isomorphism). For example, it is easy to see that the spheres $S^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ are topological manifolds and also have a smooth structure induced from the ambient $\mathbb{R}^{n+1}$ (we'll get to that in the next section). Milnor (late 1950's, early 1960's) (and subsequently) others proved that some spheres can be given distinct differentiable structures that give rise to the same underlying topological manifold (the 7-sphere carries 15 such different differentiable structures; Milnor was awarded the Fields Medal largely for this work).

Kodaira and Spencer around the same time started the investigation of different complex structures on the same underlying differentiable manifold. For example, on

S^2 there exists only one, on S^6 (A. Adler) there exists none, but in general, there can exist a continuous supply of complex structures on a given (even-dimensional) smooth manifold, and these structures themselves form a space with rich geometry.

Benzecri (1959) showed that if a Γ^+_{aff}-structure exists on a compact surface it must be a torus. More recently, symplectic structures have gained greater importance placing them on equal footing with complex structures in the framework of mirror symmetry.

2. Some examples of construction (or description) of manifolds

OK, so the big issue is how do we actually write down a manifold? There are some recurring basic methods, all of which are important.

2.1. **Basic examples.** First, as hopefully no one doubts, $\mathbb{R}^n$ is a manifold. You can just use the one chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ to define an atlas! However, note that there can be other differentiable structures on the topological manifold $\mathbb{R}^n$ than this standard one (exotic ones, e.g. for n=4, found by Michael Freedman in 1982, and studied further by Taubes). We will always consider this standard one (unless explicitly stated otherwise).

Second, if M is a manifold and $U \subset M$ an open subset, then U is a manifold in a natural way: U is given the induced topology, and if $A = \{(U_i, \varphi_i)\}$ is an atlas for M, then $A_U = \{(U \cap U_i, (\varphi_i) \mid_{U \cap U_i})\}$ is one for U.

Definition 2.1. If M is a smooth manifold, $U \subset M$ open, we call a function $f: U \to \mathbb{R}$ *smooth* or *differentiable* if it is a smooth map of manifolds if we endow U and $\mathbb{R}$ with their smooth manifold structures as explained above.

If M is an m-dimensional manifold with atlas $A_M = \{U_i, \varphi_i)\}$, N an n-dimensional manifold with atlas $A_N = \{V_j, \psi_j)\}$, then $M \times N$ is an $m+n$-dimensional manifold in the following way: with the product topology, we endow $M \times N$ with the atlas

$$A = \{(U_i \times V_j, \varphi_i \times \psi_j)\}.$$

So far nothing really to knock your socks off, but there are tons of other more interesting manifolds and constructions, which we will come to in a minute; here is something a little more entertaining to begin with: take $K = \mathbb{R}$ or $K = \mathbb{C}$; we will construct a manifold $\mathbb{P}^n_K$ called (real or complex) projective space over K of (real or complex) dimension n. As a set

$$\mathbb{P}^n_K = (K^{n+1} - \{0\})/ \sim$$

where $\sim$ is the equivalence relation

$$(p_0, \dots, p_n) \sim (q_0, \dots, q_n) \iff \exists c \in K \backslash \{0\} : c(p_0, \dots, p_n) = (q_0, \dots, q_n).$$

We give $\mathbb{P}^n_K$ the quotient topology of the usual Euclidean topology on $K^{n+1} - \{0\}$ (meaning a subset in $\mathbb{P}^n_K$ is declared open iff its preimage in $K^{n+1} - \{0\}$ is open). We denote the equivalence class of $(p_0, \dots, p_n)$ in $\mathbb{P}^n_K$ by $(p_0 : \dots : p_n)$. We define an atlas on $\mathbb{P}^n_K$ in the following way: For $i = 0, \dots n$ we let

$$U_i = \{(p_0 : \dots : p_n) \mid p_i \neq 0\}$$

and

$$\varphi_i : U_i \to K^n, \quad \varphi_i((p_0 : \dots : p_n)) = \left(\frac{p_0}{p_i}, \dots, \widehat{\frac{p_i}{p_i}}, \dots, \frac{p_n}{p_i} \right) = (x_0^{(i)}, \dots, x_{i-1}^{(i)}, x_{i+1}^{(i)}, \dots, x_n^{(i)}).$$

In this case $\varphi_i(U_i) = K^n$ and

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is given by

$$x_\nu^{(j)} = \frac{x_\nu^{(i)}}{x_j^{(i)}},\ \nu \neq i, j$$
$$x_i^{(j)} = (x_j^{(i)})^{-1}.$$

Thus $\mathbb{P}^n_K$ is a smooth manifold (of *real* dimension $2n$ if $K = \mathbb{C}$). By the way, this shows that $\mathbb{P}^n_{\mathbb{C}}$ is also a complex manifold in this way.

2.2. **Submanifolds of known manifolds, such as $\mathbb{R}^n$.** If you are given a manifold M, of dimension d, such as $\mathbb{R}^n$ or $\mathbb{P}^n_K$ for example, you can always make more interesting manifolds by considering appropriate zero sets of smooth functions in M as we will now explain.

Definition 2.2. Let us identify $\mathbb{R}^e$, for $e \leq d$, with its image $\mathbb{R}^e \subset \mathbb{R}^d$ under the canonical inclusion

$$\mathbb{R}^e \to \{0\} \times \mathbb{R}^e \subset \mathbb{R}^{d-e} \times \mathbb{R}^e = \mathbb{R}^d.$$

Suppose M is a manifold of dimension d. Then a subset $Y \subset M$ is called a *submanifold* of dimension e if for every $y \in Y$ there exists a chart (U, φ) on M with $y \in U$ such that

$$\varphi(U \cap Y) = \varphi(U) \cap \mathbb{R}^e.$$

We also say that Y has *codimension* $c = d - e$.

Proposition 2.3. *A submanifold of dimension e of a manifold M is a manifold in its own right in a canonical way. Indeed, for (U, φ) ranging over all charts satisfying the conditions of Definition 2.2, the pairs $(U \cap Y, \varphi\mid_{U\cap Y})$ give a smooth atlas for Y endowed with the subspace topology.*

Proof. If (U, φ) and (V, ψ) are two charts satisfying the condition, we have

$$\varphi((U \cap Y) \cap (V \cap Y)) = \varphi(U \cap V) \cap \mathbb{R}^e$$

is open in $\mathbb{R}^e$ and

$$(\psi\mid_{U\cap V\cap Y}) \circ (\varphi\mid_{U\cap V\cap Y})^{-1} = (\psi \circ \varphi^{-1})\mid_{\varphi(U\cap V)\cap\mathbb{R}^e}$$

which is smooth on $\varphi(U \cap V) \cap \mathbb{R}^e$ if $\psi \circ \varphi^{-1}$ is smooth on $\varphi(U \cap V)$. We endow Y with the topology in which a subset $\Omega \subset Y$ is open iff for each chart (U, φ) satisfying the conditions of Definition 2.2

$$\varphi(\Omega \cap U)$$

is open in $\mathbb{R}^e$; we leave it as an easy exercise to check that this is just the subspace topology. $\square$

There is an easy criterion to check when charts as required in Definition 2.2 exist for a subset Y.

Proposition 2.4. *Let M be a manifold of dimension d and $Y \subset M$ a subset. Suppose that for every $y \in Y$ there exists a chart (U', φ'), $\varphi' = (x_1, \dots, x_d)$, $y \in U'$, and smooth functions $f_1, \dots, f_c$ on U' such that*

$$Y \cap U' = \{p \in U' \mid f_1(p) = \dots = f_c(p) = 0\}$$

and the determinant

$$\det\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1\le i,j\le c}\neq 0$$

for all $p\in U'$; here we view the f_i as functions of the local coordinates $x_1,\dots,x_d$ on U'. Then Y is a submanifold of codimension c; indeed, a chart (U,φ) of the type required in Definition 2.2 can be obtained by using as local coordinates $\varphi=(w_1,\dots,w_c,w_{c+1},\dots,w_d)$ on some open set $U\subset U'$ containing y, where

$$w_i(p)=f_i(p),\ 1\le i\le c,\quad w_{c+1}=x_{c+1},\ w_{c+2}=x_{c+2},\ \dots,\ w_d=x_d.$$

Proof. This is really just an immediate consequence of the Inverse Function Theorem you know from MA259 Multivariable Calculus: indeed, identifying U' with an open subset of $\mathbb{R}^d$ via φ we just have to show that the map $F\colon U'\to\mathbb{R}^d$ (coordinates in the target $\mathbb{R}^d$ are $w_1,\dots,w_d$) given by

$$w_i=f_i(x_1,\dots,x_d),\ 1\le i\le c,\quad w_{c+1}=x_{c+1},\ w_{c+2}=x_{c+2},\ \dots,\ w_d=x_d$$

is a local diffeomorphism around p; and the Inverse Function Theorem says that for this it is sufficient to check that the Jacobian of the mapping F in p has nonzero determinant. But this Jacobian is of the form

$$\begin{pmatrix}\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1\le i,j\le c} & *_{c\times(d-c)}\\ 0_{(d-c)\times c} & \mathrm{Id}_{(d-c)\times(d-c)}\end{pmatrix}$$

(the subscripts indicate the format of the matrices involved, thus $0_{(d-c)\times c}$ is the $(d-c)\times c$-matrix with all entries 0, $\mathrm{Id}_{(d-c)\times(d-c)}$ the $(d-c)\times(d-c)$ identity matrix, and $*_{c\times(d-c)}$ an $c\times(d-c)$-matrix whose entries we do not care about. The determinant of the preceding big matrix is just

$$\det\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1\le i,j\le c}$$

hence nonzero by assumption. □

The careful reader will have noted that I was being sloppy with the notation in the preceding proof inasmuch as I tacitly identified $U'\subset M$ and $\varphi'(U')\subset\mathbb{R}^d$ via φ', and did not distinguish notationally corresponding objects on U' and $\varphi'(U')$, for example f (on U') and $f\circ(\varphi')^{-1}$ on $\varphi'(U')$. In my opinion, such sloppiness, when used with discretion and accompanied by an explanation when first used, does not do any harm and even helps making the main ideas of the proof more transparent. It is also extremely common in the literature on manifolds.

Using Proposition 2.4, we can now give many more examples of manifolds in the form of submanifolds of some known manifolds; for example, the spheres

$$S^n=\left\{x=(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}\mid x_1^2+\dots+x_{n+1}^2-1=0\right\}$$

are submanifolds of $\mathbb{R}^{n+1}$ since the gradient of $x_1^2+\dots+x_{n+1}^2-1$ vanishes at no point of S^n. We will consider those spheres with this "standard" differentiable structures unless explicitly stated otherwise (though there are other exotic differentiable structures on spheres).

For another example, look at the orthogonal group $O(n)$ of $n\times n$ invertible matrices A with $A^t=A^{-1}$. This is a submanifold of $\mathbb{R}^{n^2}$ (of dimension $n(n-1)/2$). Indeed, it is defined by the $n(n+1)/2$ equations inside $\mathbb{R}^{n^2}$ given by

$$A^tA-\mathrm{Id}_{n\times n}=0$$

and it suffices to show that the differential/Jacobian of the map

$$\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$$
$$A \mapsto A^t A - \mathrm{Id}_{n\times n}$$

has rank equal to $n(n+1)/2$ (the dimension of the space of symmetric matrices) at every point of $O(n)$. But this Jacobian, at the point corresponding to a matrix A_0, is easily computed to be (Exercise!)

$$\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$$
$$B \mapsto B^t A_0 + A_0^t B.$$

Since $A_0 \in O(n)$ is invertible we can write $B = A_0 B'$ and get

$$B^t A_0 + A_0^t B = (B')^t + B'.$$

Hence the rank of the Jacobian is $n(n+1)/2$ as desired.

For yet another example, we already know that $S^1 \subset \mathbb{R}^2$ is a submanifold, so taking products

$$\mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$$

is a manifold, called an n-dimensional torus; we will see another method to construct such tori in the next section.

Exercise 2.5. Consider $\mathbb{P}^n_{\mathbb{C}}$ and let $f = f(z_0, \dots, z_n)$ be a nonzero homogeneous polynomial in $z_0, \dots, z_n$ of degree d with complex coefficients; here homogeneous means that very monomial

$$z_0^{\alpha_0} z_2^{\alpha_2} \dots z_n^{\alpha_n}$$

occurring in f has total degree $\alpha_0 + \cdots + \alpha_n$ equal to d. Let $M_f = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n_{\mathbb{C}} \mid f(z_0, \dots, z_n) = 0\}$. Show that if at least one of the partials

$$\frac{\partial f}{\partial z_i},\ i = 0, \dots, n$$

is nonzero at every point of M_f, then M_f is a submanifold of $\mathbb{P}^n_{\mathbb{C}}$.

Such M_f are very varied and interesting. For example, for $n = 2$, it can be shown that M_f is homeomorphic to a compact surface with

$$g = \frac{1}{2} d(d-3) + 1$$

holes (a "g-holed torus"). To show this requires a little more sophisticated techniques, though, than the ones we can get to here.

2.3. **Quotient manifolds.** We get a much larger supply of interesting examples of manifolds by taking quotients of manifolds by suitable group actions.

Definition 2.6. An isomorphism of a smooth manifold M onto itself is also called a *diffeomorphism*. All diffeomorphisms of M form a group $\mathrm{Diff}(M)$ under composition. Let $G \subset \mathrm{Diff}(M)$ be a subgroup.

Definition 2.7. G is called a *properly discontinuous group of diffeomorphisms* of M if for any two compact subsets K_1, K_2 of M, the set

$$\{g \in G \mid g(K_1) \cap K_2 \neq \emptyset\}$$

is finite.

We say that G *has no fixed points* if for all $g \in G$, $g \neq \mathrm{id}$, it holds that g has no fixed points on M.

Theorem 2.8. *If G is a properly discontinuous group of diffeomorphisms of M and has no fixed points, then the set of G-orbits M/G is a manifold in a natural way, to be described in the course of the proof.*

Proof. We endow M/G with the quotient topology, in which a set is open iff its preimage in M under the natural quotient map $M \to M/G$ is open. We use the notation M^* for M/G, the set of G-orbits in M, and for a point $p \in M$ write p^* for its orbit $G \cdot p$, an element of M^*. We will show that for $q \in M$ there exists a neighbourhood U of q such that if $p_1, p_2 \in U$, $p_1 \neq p_2$, then $p_1^* \neq p_2^*$. In fact, we will show that there is an open subset $U \ni q$ such that $gU \cap U = \emptyset$ for all $g \in G$, $g \neq \mathrm{id}$. Let $U_1 \supset U_2 \supset U_3 \supset \dots$ be a base of relatively compact open neighbourhoods of q (relatively compact means that their closure be compact; such a base of neighbourhoods exists since locally around q, M is homeomorphic to an open in $\mathbb{R}^n$). Then each of the sets

$$F_m = \{g \mid gU_m \cap U_m \neq 0\}, \ m = 1, 2, \dots$$

is finite by the assumption that G is properly discontinuous, and moreover,

$$F_1 \supset F_2 \supset F_3 \supset \dots .$$

Suppose, arguing by contradiction, that each F_m contains an element $g_m \in G$, $g_m \neq \mathrm{id}$. Then since each F_m is finite, there will be a $g \in G$, $g \neq \mathrm{id}$, with $g \in F_m$ for all m, meaning that

$$gU_m \cap U_m \neq \emptyset \quad \forall\, m$$

and at the same time the U_m "converge to q" (more precisely, they form a basis of the neighbourhoods of q), hence we would conclude $g(q) = q$, contradicting the non-existence of fixed points which we have assumed.

This shows that we can cover M with open sets U_j such that whenever $p_1, p_2 \in U_j$, $p_1 \neq p_2$, then also $p_1^* \neq p_2^*$; in other words,

$$U_j \to U_j^* = \{p^* \mid p \in U_j\}, \quad p \mapsto p^*$$

is one-to-one. We now give U_j^* the differentiable structure that U_j has (as an open subset of M). In other words, if (V, ψ) is a chart on U, then we let (V^*, ψ^*) be the chart on U_j^* with $\psi^*(p^*) = \psi(p)$. This defines a differentiable/smooth atlas on M/G and endows M/G with the structure of a smooth manifold. □

Example 2.9. As our first example, consider $M = \mathbb{R}^n$ and choose n vectors $\omega_1, \dots, \omega_n \in \mathbb{R}^n$ that are $\mathbb{R}$-linearly independent. Let G be $\mathbb{Z}^n$, and view it ias a subgroup fo the diffeomorphisms of $\mathbb{R}^n$ by identifying $g = (m_1, \dots, m_n)$, $m_k \in \mathbb{Z}$ with the map

$$g\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x + \sum_{k=1}^{n} m_k \omega_k.$$

It is clear that G is without fixed points and properly discontinuous (for the latter note that a compact subset of $\mathbb{R}^n$ is, in particular, bounded). The quotient manifold

$$\mathbb{T}^n = \mathbb{R}^n / G$$

is an n-dimensional torus (as an exercise, you should show that it is indeed diffeomorphic to a product of n copies of S^1).

Example 2.10. We let $W = \mathbb{C}^n - \{0\}$ (viewed as an open submanifold of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$). We choose complex numbers α_ν, $\nu = 1, \dots, n$, with $0 < |\alpha_\nu| < 1$ for all ν, and let

$$G = \{g^m \mid m \in \mathbb{Z}, \ g(z_1, \dots, z_n) = (\alpha_1 z_1, \dots, \alpha_n z_n)\}.$$

It is easy to see that G is isomorphic to $\mathbb{Z}$ and is without fixed points and properly discontinuous (Exercise: Show this!), and the quotient manifold W/G is called a *Hopf manifold*. It can be shown that it is diffeomorphic to $S^1 \times S^{2n-1}$ (you can think about this as a challenge).

Example 2.11. Let $M = S^3$ and view this as the unit sphere in $\mathbb{R}^4 \simeq \mathbb{C}^2$ (note that if z_1, z_2 are complex coordinates in $\mathbb{C}^2$, this is the subset $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$. Fix coprime integers $p, q > 1$ and consider the group G generated by the diffeomorphism of S^3 given by

$$g(z_1, z_2) = \left(e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2\right).$$

Then G is isomorphic to $\mathbb{Z}/p$ and without fixed points (and properly discontinuous since it is a finite group). The quotient manifold in this case is often denoted by $L(p, q)$ and called a *lens space*.

2.4. **Surgeries.** Another very useful method to produce new manifolds from old ones is to modify a given manifold locally in a certain way we will now describe. This is called *surgery*.

Theorem 2.12. *Let M be a manifold, and let $S \subset M$ be a compact submanifold. Suppose we have an open neighbourhood W of S in M and another pair of manifolds $S^* \subset W^*$ with S^* compact and W^* an open neighbourhood of S^*. Suppose we are given a diffeomorphism*

$$f\colon W^* - S^* \to W - S$$

of $W^* - S^*$ onto $W - S$. Then we can "replace" S by S^* and obtain a new manifold $M^* = (M - W) \cup W^*$. More precisely, we consider

$$M^* = ((M - S) \sqcup W^*) / \sim$$

where $\sim$ is the equivalence relation identifying each point $z^* \in W^* - S^*$ with $z = f(z^*)$. This, with the quotient topology, inherits a natural manifold structure from M (and W^*).

Proof. Exercise for the reader. □

Example 2.13. We consider $M = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. We let $\zeta = (\zeta_0 : \zeta_1)$ be (homogeneous) coordinates on the first copy of $\mathbb{P}^1$, and let $z = \frac{\zeta_1}{\zeta_0}$ be a local coordinate on the open subset $U \subset \mathbb{P}^1_{\mathbb{C}}$, $U \simeq \mathbb{C}$ where $\zeta_0 \neq 0$. Then M contains $W = U \times \mathbb{P}^1$, and W contains $S = \{0\} \times \mathbb{P}^1$. Let $W^* = U^* \times (\mathbb{P}^1)^*$ be a second copy of W (on which everything is denoted by the same letters, but with a superscript $*$), and let $S^* = \{0\} \times (\mathbb{P}^1)^*$. Fix an integer $m > 0$ and define $f\colon W^* - S^* \to W - S$ by

$$(z, \zeta) = f(z^*, \zeta^*) := \left(z^*, \left(\zeta_0^* : \frac{\zeta_1^*}{(z^*)^m}\right)\right).$$

Then f is a diffeomorphism and we can define $M^*_m = (M - S) \cup W^*$ by surgery. This is called a *Hirzebruch manifold*. One can show that all M^*_{2n} are homeomorphic to each other, and so are all M^*_{2n+1}, but the ones for even and odd m are not homeomorphic to each other. This requires more techniques from topology than we have right now, however.

Example 2.14. Let M be a complex manifold with a point $S := p \in M$ and a chart $(W, \varphi = (z_1, z_2))$ with $\varphi\colon W \simeq U$ where $U = \{(z_1, z_2) \in \mathbb{C}^2 \mid ||z_1| < 1, |z_2| < 1\}$ and $\varphi(p) = (0, 0)$. In particular, φ is a smooth chart of course. Define a submanifold W^* of $W \times \mathbb{P}^1_{\mathbb{C}}$ as follows:

$$W^* = \left\{(z_1, z_2; (\zeta_1 : \zeta_2)) \in W \times \mathbb{P}^1_{\mathbb{C}} \mid z_1\zeta_2 - z_2\zeta_1 = 0\right\}$$

It is easy to check this is a submanifold since if $g = z_1\zeta_2 - z_2\zeta_1$ we have that

$$\frac{\partial g}{\partial z_i},\ i = 1, 2$$

are never simultaneously zero on W^*. Let $F\colon W^* \to U$ be the restriction of the projection map. Let $S^* = \{0\} \times \mathbb{P}^1_{\mathbb{C}} \subset W^*$. Clearly, F maps S^* onto $(0, 0)$ and

$$F\colon W^* - S^* \to U - \{0, 0)\}$$

is a diffeomorphism: namely, if $(z_1, z_2; (\zeta_1 : \zeta_2)) \notin S^*$, then at least one of z_1, z_2 is nonzero and hence $(\zeta_1 : \zeta_2)$ is determined by (z_1, z_2). Letting $f = \varphi^{-1} \circ F$, we obtain by surgery a manifold $M^* = (M - p) \cup \mathbb{P}^1_{\mathbb{C}}$. This procedure is very important and called the *blowup* of M at p. Indeed, one imagines sticking a little piece of straw into M at p and blowing a bubble $S^1 \simeq \mathbb{P}^1_{\mathbb{C}}$ replacing p. Note that the points of $\mathbb{P}^1_{\mathbb{C}} = S^*$ correspond to the complex lines in $\mathbb{C}^2$ passing through $(0, 0)$.

Example 2.15. There is a real version of the preceding example, replacing $\mathbb{C}$ by $\mathbb{R}$ everywhere. We leave it to the reader as an exercise to work out the details what changes and what doesn't.

3. Tangent space, vector fields, one-parameter groups of diffeomorphisms, 1-forms, tangent and cotangent bundles, vector bundles

Suppose we are given a smooth manifold M and $p \in M$ a point. We shall now define the notion of a *tangent vector* to M at the point p. By a *(parametrised, smooth) curve* in M we mean a smooth map

$$\gamma\colon (a, b) \to M$$

where $(a, b) \subset \mathbb{R}$ is an interval, $a < b$. We moreover define the $\mathbb{R}$-*algebra of germs of smooth functions defined in a neighbourhood of* p $\mathscr{C}^\infty_{M,p}$ in the following way: the elements of $\mathscr{C}^\infty_{M,p}$ are equivalence classes of pairs (f, U), where $U \ni p$ is an open subset of M and $f\colon U \to \mathbb{R}$ a smooth function (we also write $f \in \mathscr{C}^\infty(U)$), where (f, U) and (g, V) are considered to be equivalent if there is a smaller open neighbourhood W of p, contained in both U and V, on which f and g agree. You see, the name for $\mathscr{C}^\infty_{M,p}$ is more intimidating than the actual thing!

We now consider a curve in M as above with $\gamma(t_0) = p$, $a < t_0 < b$. We define the *tangent vector to the curve* γ *at* p to be the map

$$X (= X_{\gamma,p})\colon \mathscr{C}^\infty_{M,p} \to \mathbb{R}$$

defined by

$$Xf = \frac{df(\gamma(t))}{dt}(t_0).$$

This map has the following properties:

a) $X\colon \mathscr{C}^\infty_{M,p} \to \mathbb{R}$ is $\mathbb{R}$-linear;

b) X satisfies the Leibniz rule

$$X(fg) = (Xf)g(p) + f(p)(Xg), \quad \forall\, f, g \in \mathscr{C}^\infty_{M,p}.$$

The maps satisfying a) and b) form a real vector space (and are called *derivations*). We will show that the set of tangent vectors to curves, i.e., derivations of the form $X_{\gamma,p}$, form a vector subspace of dimension n of the vector space of all derivations, where $n = \dim M$. We then define a *tangent vector* to M at the point p to be simply a tangent vector to some curve at p.

Indeed, let $(U, \varphi = (x_1, \dots, x_n))$ be a chart at p, $p \in U$. We shall show that the set of tangent vectors at p is nothing but the vector space with basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Given any curve $\gamma(t)$ with $p = \gamma(t_0)$ let $x_j = \gamma_j(t)$ its components with respect to the local coordinates. Then

$$\left(\frac{df(\gamma(t))}{dt}\right)_{t_0} = \sum_j \left(\frac{\partial f}{\partial x_j}\right)_p \cdot \left(\frac{d\gamma_j(t)}{dt}\right)_{t_0}$$

which proves that every tangent vector to M at p is a linear combination of

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Conversely, given a linear combination

$$\sum_j \xi_j \left(\frac{\partial}{\partial x_j}\right)_p$$

consider the curve given in local coordinates by

$$x_j = x_j(p) + \xi_j t, \quad j = 1, \dots, n.$$

Then the tangent vector to this curve at $t = 0$ is the given one. We still need to prove the linear independence of the $\left(\frac{\partial}{\partial x_j}\right)_p$: suppose that

$$\sum_j \xi_j \left(\frac{\partial}{\partial x_j}\right)_p = 0.$$

Then

$$0 = \sum_j \xi_j \left(\frac{\partial x_k}{\partial x_j}\right)_p = \xi_k, \quad k = 1, \dots, n.$$

Thus we have shown that the set of tangent vectors to curves, i.e., derivations of the form $X_{\gamma,p}$, form a vector subspace of dimension n of the vector space of all derivations, where $n = \dim M$.

Definition 3.1. We denote the n-dimensional vector space of all tangent vectors to M at p by

$$T_p(M)$$

and call it the *tangent space* to M at p.

Remark 3.2. It is known that for a smooth ($\mathscr{C}^\infty$-)manifold M, $T_p(M)$ coincides with the space of all derivations of $\mathscr{C}^\infty_{M,p}$ (not just a subspace).

Definition 3.3. A *vector field* X on a manifold M is assignment to each point $p \in M$ of a tangent vector X_p to M at p. If f is a smooth function, then Xf is the function on M defined by $(Xf)(p) = X_p f$. The vector field X is called *smooth* if Xf is a smooth function on M for every smooth function f on M. We will denote the smooth vector fields on M by $\mathfrak{X}(M)$.

In the following, we will practically exclusively deal with smooth vector fields and therefore agree that the term "vector field" should always be understood as meaning "smooth vector field" from now on unless explicitly stated otherwise.

If $(U, x_1, \dots, x_n)$ is a chart on M, a vector field X on U can be written as

$$\sum_j \xi_j \frac{\partial}{\partial x_j}$$

with ξ_j smooth functions of the $x_1, \dots, x_n$ on U. If M is covered by charts

$$(U_\lambda, x_1^{(\lambda)}, \dots, x_n^{(\lambda)}),$$

a vector field X on all of M can be thought of in very concrete terms in the following way: the datum of X amounts to giving oneself n smooth functions

$$\xi_1^{(\lambda)}, \dots, \xi_n^{(\lambda)}$$

of the local coordinates $x_1^{(\lambda)}, \dots, x_n^{(\lambda)}$ on U_λ, so that on U_λ

$$X \mid_{U_\lambda} = \sum_j \xi_j^{(\lambda)} \frac{\partial}{\partial x_j^{(\lambda)}};$$

and on $U_\lambda \cap U_\mu$ we want

$$\sum_j \xi_j^{(\lambda)} \frac{\partial}{\partial x_j^{(\lambda)}} = \sum_j \xi_j^{(\mu)} \frac{\partial}{\partial x_j^{(\mu)}}$$

meaning that the system of functions need to satisfy

$$\xi_j^{(\lambda)} = \sum_k \frac{\partial x_j^{(\lambda)}}{\partial x_k^{(\mu)}} \xi_k^{(\mu)}$$

on $U_\lambda \cap U_\mu$. Although computations with these local expressions can get messy rather quickly, this description sometimes allows one to obtain useful information about $\mathfrak{X}(M)$.

The set $\mathfrak{X}(M)$ is obviously an $\mathbb{R}$-vector space and also a module over the algebra $\mathscr{C}^\infty(M)$ of smooth functions on M, but it has another useful structure: if $X, Y \in \mathfrak{X}(M)$ define their bracket by

$$[X, Y]f = X(Yf) - Y(Xf).$$

This is again a vector field; indeed, if in terms of a chart $(U, x_1, \dots, x_n)$

$$X = \sum_j \xi_j \frac{\partial}{\partial x_j}, \quad Y = \sum_j \eta_j \frac{\partial}{\partial x_j}$$

then

$$[X, Y]f = \sum_{j,k} \left(\xi_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial \xi_j}{\partial x_k} \right) \frac{\partial f}{\partial x_j}$$

so that $[X, Y]$ is a vector field whose components on U with respect to the $\frac{\partial}{\partial x_j}$ are

$$\sum_k \left(\xi_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial \xi_j}{\partial x_k} \right).$$

The bracket

$$[\cdot, \cdot] \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is $\mathbb{R}$-bilinear and satisfies

$$[[X, Y]Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \quad \forall\, X, Y, Z \in \mathfrak{X}(M)$$

as a direct calculation shows. The last identity is called the *Jacobi identity*, and a vector space endowed with a bilinear bracket $[\cdot, \cdot]$ satisfying the Jacobi identity is called a *Lie algebra*. You will learn more about these in MA4E0 Lie Groups and MA453 Lie Algebras. Suffice it to mention here that Lie algebras usually arise as tangent spaces to the identity of some Lie group, which is a group endowed with some kind of smooth or similar structure; indeed, $\mathfrak{X}(M)$ should be though tof as the tangent space to the "manifold" $\mathrm{Diff}(M)$, the diffeomorphism group of M. Unfortunately, it is not a manifold in our sense here, but a "Fréchet manifold", locally modelled not on some $\mathbb{R}^d$, but on some special infinite-dimensional topological vector spaces called Fréchet spaces (sorry! The world out there is vast and cruel). We will leave it at that here, but for cultural reasons we remark that "Lie" is pronounced "L-ee", after the Norwegian mathematician Sophus Lie.

If f, g are smooth functions on M and X, Y vector fields, a computation shows that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

(Exercise!)

Definition 3.4. Let

$$f\colon M \to M'$$

be a smooth map of manifolds. The *differential* at $p \in M$ of f is the linear map

$$f_*\colon T_p(M) \to T_{f(p)}(M')$$

defined as follows. For each tangent vector $X_p \in T_p(M)$, choose a curve $\gamma(t)$ in M such that X_p is the tangent vector to $\gamma(t)$ at $p = \gamma(t_0)$. Then $f_*(X_p)$ is the vector tangent to the curve $f(\gamma(t))$ at $f(p) = f(\gamma(t_0))$.

It is easy to check that $f_*(X_p)$ is independent of the choice of $\gamma(t)$; in fact, if $(U, x_1, \dots, x_n)$ is a chart around p, and $(V, y_1, \dots, y_m)$ is a chart around $f(p)$ such that $f(U) \subset V$, it is immediate to see that the differential f_* is the linear map given by the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i=1,\dots,m,\, j=1,\dots,n}$$

if we use the bases

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p; \quad \left(\frac{\partial}{\partial y_1}\right)_{f(p)}, \dots, \left(\frac{\partial}{\partial y_m}\right)_{f(p)}$$

for $T_p(M)$ and $T_{f(p)}(M')$, respectively. You should check the preceding assertions as an exercise and also verify that if g is a smooth function defined in a neighbourhood of $f(p)$, then

$$(f_*(X_p))g = X_p(g \circ f).$$

If we want to specify the point p, we write $(f_*)_p$ instead of f_*. Another common notation used for $(f_*)_p$ is df_p (or Df_p).

The following is a useful consequence of the Inverse Function Theorem that you encounter in MA259 Multivariable Calculus:

Proposition 3.5. *Let $f\colon M \to M'$ be a smooth map and $p \in M$ as before.*

a) If $(f_*)_p$ is injective, there exist local coordinates $x_1, \dots, x_n$ in a neighbourhood U of p and local coordinates $y_1, \dots, y_m$ in a neighbourhood of $f(p)$ such that

$$y_i(f(q)) = x_i(q) \quad \text{for } q \in U \text{ and } i = 1, \dots, n.$$

This can be paraphrased by saying that f locally around p "looks like (=is diffeomorphic to) the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^m$".

In particular, f is a homeomorphism of U onto $f(U)$.

b) *If $(f_*)_p$ is surjective, there exists a local coordinate system $x_1, \dots, x_n$ in a neighbourhood U of p and local coordinates $y_1, \dots, y_m$ in a neighbourhood of $f(p)$ such that*

$$y_i(f(q)) = x_i(q) \quad \text{for } q \in U \text{ and } i = 1, \dots, m.$$

This can be paraphrased by saying that f locally around p "looks like (=is diffeomorphic to) the projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$".

In particular, the mapping $f\colon U \to M'$ is open (=maps open sets to open sets).

c) *If $(f_*)_p$ is bijective, then f defines a homeomorphism of a neighbourhood U of p onto a neighbourhood V of $f(p)$ and its inverse $f^{-1}\colon V \to U$ is also differentiable/smooth.*

Proof. Let us first assume that $(f_*)_p$ is injective and let $y_1, \dots, y_m$ a system of local coordinates around $q = f(p)$ on M'. We claim that it is possible to choose from the set of functions

$$y_1 \circ f, \dots, y_m \circ f$$

a subset of n functions that form a system of local coordinates at p in M. Indeed, in terms of some local coordinates $w_1, \dots, w_n$ at p we can write the component functions $y_i \circ f = f_i(w_1, \dots, w_n)$, and the hypothesis that $(f_*)_p$ is injective means that

$$\left(\frac{\partial f_i}{\partial w_j}\right)(p), \quad i = 1, \dots, m,\ j = 1, \dots, n$$

has rank n; we can thus select n indices $i_1, \dots, i_n$ from the set $\{1, \dots, m\}$ such that the submatrix

$$\left(\frac{\partial f_{i_j}}{\partial w_j}\right)(p),$$

is a square invertible matrix with nonzero determinant; by the Inverse Function Theorem, $y_{i_1} \circ f, \dots, y_{i_n} \circ f$ give a local diffeomorphism, hence can be used as local coordinates $x_1 = y_{i_1} \circ f, \dots, x_n = y_{i_n} \circ f$ of the type required in a). A reordering of the y's (so that $y_{i_1}, \dots, y_{i_n}$ become $y_1, \dots, y_n$) gives local coordinates around q such that all properties in a) are satisfied.

For b), let us now assume that $(f_*)_p$ is surjective and let $y_1, \dots, y_m$ be local coordinates at $q \in M'$. Then we claim that

$$y_1 \circ f, \dots, y_m \circ f$$

are part of a system of local coordinates at p on M. Indeed, with the same notation as before, the rank of

$$\left(\frac{\partial f_i}{\partial w_j}\right)(p), \quad i = 1, \dots, m,\ j = 1, \dots, n$$

is now m. Reordering the columns of this matrix of necessary, we can assume without loss of generality, that the submatrix consisting of the first m columns of this matrix is invertible. Hence again by the Inverse Function Theorem,

$$x_1 = y_1 \circ f, \dots, x_m = y_m \circ f,\ x_{m+1} = w_{m+1}, \dots, x_n = w_n$$

are local coordinates at p in M.

Finally part c) is simply obtained by putting parts a) and b) together. ☐

Definition 3.6. Let $f\colon M \to M'$ be a smooth map of manifolds.

a) If $(f_*)_p$ is injective for every $p \in M$, then f is called an *immersion*. We then say that M is immersed in M' by f, or that M is an immersed submanifold of M'. We call f a *submersion* if $(f_*)_p$ is surjective for every $p \in M$.

b) We call f an *embedding* of M into M' if f is an injective immersion and a homeomorphism onto its image $f(M)$.

To tie this in with the notion of submanifold $Y \subset M$ from Definition 2.2, you should convince yourself that if Y is a submanifold of M, then the inclusion map $Y \subset M$ is an embedding. And, conversely, given an embedding $i\colon N \to M$ of manifolds, the image $Y = i(N) \subset M$ is a submanifold.

There is a slight catch here that you should be aware of: an injective immersion need not necessarily be an embedding. That is, if $f\colon M \to M'$ is an injective immersion, then it does not necessarily follow that f is also a homeomorphism onto its image $f(M)$. The topology of M can in general be finer than the relative topology induced from M'. An example to bear in mind is the composite map

$$\varphi\colon \mathbb{R} \to L \subset \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$$

where $L \subset \mathbb{R}^2$ is a line with irrational slope, and $(m, n) \in \mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(x, y) \mapsto (x + m, y + n)$. The image $\varphi(\mathbb{R})$ is dense in the torus $\mathbb{T}^2$.

However, if M is compact, then every injective immersion $f\colon M \to M'$ is an embedding.

We next aim to carry over some of the theory of ordinary differential equations to manifolds. This will also have some very useful consequences as we shall see. In other words, we will talk about *integral curves* of vector fields, *1-parameter groups of diffeomorphisms* and *local and global flows* generated by vector fields on manifolds next. As an application we will prove Ehresmann's result on local triviality of differentiable fibrations.

Let X be a vector field on a manifold M. A curve $\gamma(t)$ in M is called an *integral curve* of X if, for every parameter value t_0, the vector $X_{\gamma(t_0)}$ is tangent to the curve $\gamma(t)$ at $\gamma(t_0)$. For any point p_0 of M, there is a unique integral curve $x(t)$ of X, defined for $|t| < \epsilon$ for some $\epsilon > 0$, such that $p_0 = x(0)$. Indeed, if $x_1, \ldots, x_n$ are local coordinates in a neighbourhood U of p_0 and

$$X\mid_U = \sum_i \xi_i \frac{\partial}{\partial x_i}$$

then an integral curve of X is a solution of the system of differential equations

$$\frac{dx_i}{dt} = \xi_i(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n.$$

The result then follows from the fundamental existence and uniqueness theorem for solutions of ordinary differential equations (with smooth right hand side), that you get to know in MA254 Theory of ODEs. We can view their solution theory a little bit geometrically in terms of the concept of a 1-parameter group of diffeomorphisms, or flow.

Definition 3.7. A 1-*parameter group of diffeomorphisms* or *(global) flow* on M is a mapping $\Phi\colon \mathbb{R} \times M \to M$, $\Phi(t, p) = \Phi_t(p) \in M$, which satisfies the following conditions:

a) For each $t \in \mathbb{R}$, $\Phi_t\colon M \to M$ is a diffeomorphism of M (=a smooth self-map of M with a smooth inverse);

b) for all $t, s \in \mathbb{R}$ and $p \in M$, $\Phi_{t+s}(p) = \Phi_t(\Phi_s(p))$.

Property b) just says that the map $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \Phi_t$ is a group homomorphism.

Each 1-parameter group induces a vector field X on M as follows: the curve $\gamma_p(t) = \Phi_t(p)$ has tangent vector X_p at $p = \Phi_0(p)$. The image of $\gamma_p(t)$ is the orbit of p. The curve $\gamma_p(t)$ is an integral curve of X starting at p.

Definition 3.8. A *local* 1*-parameter group of local diffeomorphisms* or a *local flow* on M is morally the same as a flow except that the time parameter t is now only in a small interval $(-\epsilon, \epsilon)$ of $0 \in \mathbb{R}$, and p is in an open subset of M. More precisely, let I_ϵ be the open interval $(-\epsilon, \epsilon)$ and U an open set of M. A *local* 1*-parameter group of local diffeomorphisms defined on* $I_\epsilon \times U$ is a smooth map $\Phi\colon I_\epsilon \times U \to M$ such that

a) for each $t \in I_\epsilon$, $\Phi_t = \Phi(-, t) : U \to \Phi_t(U)$ is a diffeomorphism;
b) if $t, s, t+s \in I_\epsilon$ and $p, \Phi_s(p) \in U$, then

$$\Phi_{t+s}(p) = \Phi_t(\Phi_s(p)).$$

Obviously, such a local flow induces a vector field on U in the same manner as before. Actually, conversely, a vector field also generates a local 1-parameter group of local diffeomorphisms inducing the given vector field around every point where it is defined:

Proposition 3.9. *Let X be a vector field on a manifold. For each point $p_0 \in M$, there is an open neighbourhood U of p_0 and $\epsilon > 0$ and a local* 1*-parameter group of local diffeomorphisms $\Phi_t\colon U \to M$, $t \in I_\epsilon$, inducing the given X on U.*

Moreover, if two local 1*-parameter groups of local diffeomorphisms Φ and Ψ defined on $I_\epsilon \times U$ induce the same vector field on U, they coincide.*

Proof. Strictly speaking, this is again just a rephrasement of results from MA254 Theory of ODEs in geometric terms: in fact, let $x_1, \dots, x_n$ be local coordinates in a neighbourhood W of p_0 with $x_1(p_0) = \dots = x_n(p_0) = 0$, and write

$$X \mid_W = \sum_i \xi_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$$

as before. Then consider the system of ordinary differential equations

$$\frac{df_i}{dt} = \xi_i(f_1(t), \dots, f_n(t)), \quad i = 1, \dots, n$$

for unknown functions $f_1, \dots, f_n$. By the fundamental theorem for systems of ordinary differential equations from MA254 Theory of ODEs again, there are unique functions

$$f_1(t; x), \dots, f_n(t; x)$$

defined for all $x = (x_1, \dots, x_n)$ with $|x_i| < \delta_1$ and $|t| < \epsilon_1$, δ_1, ϵ_1 some small positive real numbers, and depending smoothly on t and x, which form a system of solutions of the preceding differential equations for each fixed x and satisfy the initial conditions

$$f_i(0; x) = x_i.$$

We put

$$\Phi(t, x) = (f_1(t; x), \dots, f_n(t; x)) \quad \text{for } |t| < \epsilon_1 \text{ and } x \in U_1 = \{x \mid |x_i| < \delta_1 \,\forall\, i\}.$$

Then if $|t|, |t|$ and $|t+s|$ are all less than ϵ_1 and both x and $\Phi_s(x) = \Phi(s, x)$ are in U_1, then the functions

$$g_i(t) = f_i(t + s; x)$$

are a solution to the initial system of differential equations with initial conditions $g_i(0) = f_i(s; x)$, and thus by the uniqueness of the solutions,

$$g_i(t) = f_i(t; \Phi_s(x)).$$

Thus $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$; moreover, there exist $\delta > 0$ and $\epsilon > 0$ such that putting $U = \{x \mid |x_i| < \delta \,\forall\, i\}$, we have $\Phi_t(U) \subset U_1$ for $|t| < \epsilon$: indeed, this follows simply because Φ_0 is the identity on U_1. Since therefore

$$\Phi_{-t}(\Phi_t(x)) = \Phi_t(\Phi_{-t}(x)) = \Phi_0(x) = x$$

for all $x \in U$ and $|t| < \epsilon$, all Φ_t are diffeomorphisms of U onto $\Phi_t(U)$. Thus Φ is a local 1-parameter group of local diffeomorphisms defined on $I_\epsilon \times U$ inducing X. □

If under the hypotheses of Proposition 3.9, there exists a global 1-parameter group of diffeomorphisms $\Phi\colon \mathbb{R} \times M \to M$ (hence defined for all times t) inducing X, we say X is complete. The following is a fact that adds something genuinely new to facts known from the (local) theory of ordinary differential equations.

Proposition 3.10. *On a compact manifold M, every vector field X is complete.*

Proof. For every point p, let $U(p)$ be a neighbourhood of p and $\epsilon(p) > 0$ such that there exists a local 1-parameter group of local diffeomorphisms Φ defined on $I_{\epsilon(p)} \times U(p)$ inducing X. Since M is compact, finitely many of the $U(p)$, $U(p_1), \dots, U(p_k)$ say, cover M. Putting ϵ the minimum of the $\epsilon(p_i)$, $i = 1, \dots, k$, we see that $\Phi_t(p)$ can be defined on $I_\epsilon \times M$, hence on all of $\mathbb{R} \times M$. Another way to think of this is to say that the integral curve γ of X with $\gamma(0) = p$ exists for all time: after time ϵ, you can continue it for another time interval ϵ, and so on. □

We need one further ingredient before getting to Ehresmann's theorem: *partitions of unity*. These are in general quite useful to patch together locally defined objects into a globally defined object on smooth manifolds. They are particular to the context of smooth manifolds: nothing like this exists for complex manifolds for example. Partitions of unity express in some way the fact that the geometry on a differentiable manifold is rather "soft" and "flabby", whereas for complex manifold it is much "harder"and more "rigid". Here we also use for the first time the assumption that manifolds be second-countable.

Theorem 3.11. *Let M be a manifold, $\{U_\alpha\}_{\alpha\in A}$ a collection of open sets covering M. Then there exists a sequence of smooth functions $\{\theta_i\}_{i\in\mathbb{N}}$ on M, called a* partition of unity *subordinate to the open cover $\{U_\alpha\}$, with the following properties:*

(a) *For all $p \in M$ and i we have $0 \le \theta_i(p) \le 1$.*
(b) *Each $p \in M$ has a neighbourhood on which all but finitely many functions θ_i are identically zero.*
(c) *For each i there is an $\alpha(i)$ such that the support of θ_i is contained in $U_{\alpha(i)}$.*
(d) *For each $p \in M$, $\sum_i \theta_i(p) = 1$ (this sum is in effect finite by (b)).*

Proof. The proof is in a sense not very enlightening and we won't use the techniques introduced in it again later on, but we feel a moral obligation to supply one, so bear with us! The result itself, however, is extremely important and will be used over and over again.

Step 1. We first prove the existence of a compact exhaustion on M, namely, a sequence G_j, $j = 1, 2, \dots$ of open subsets of M with compact closure $\bar{G}_j$, and such that $\bar{G}_j \subset G_{j+1}$,

$$M = \bigcup_{j=1}^{\infty} G_j.$$

Indeed, notice first that since M has a countable basis and is locally Euclidean, we can find a countable basis for the topology of M, V_j, $j = 1, 2, 3, \dots$, such that each V_j has compact closure (start with any countable basis and discard those elements in it that do not have compact closure). Put $G_1 = V_1$. Suppose inductively that

$$G_k = V_1 \cup \dots \cup V_{j_k}.$$

Let j_{k+1} be the smallest positive integer greater than j_k such that

$$\bar{G}_k \subset V_1 \cup \dots \cup V_{j_k} \cup \dots \cup V_{j_{k+1}}$$

and put

$$G_{k+1} = V_1 \cup \dots \cup V_{j_k} \cup \dots \cup V_{j_{k+1}}.$$

This gives a compact exhaustion as required.

Step 2. We prove a local result in $\mathbb{R}^n$, namely: let $C(r) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| < r \,\forall\, i\}$ be the open cube of side length $2r$ and centre at the origin. Then there exists a non-negative smooth ($=\mathscr{C}^\infty$-)function φ on $\mathbb{R}^n$ that is equal to 1 on the closed cube $\overline{C(1)}$ and zero on the complement of the open cube $C(2)$.

It suffices to do this in one dimension, $n = 1$ (in the general case, we take a product of such functions, each one depending only on one coordinate $x_1, \dots, x_n$). So let us construct a non-negative smooth function h on the real line $\mathbb{R}$ which is identically 1 on $[-1, 1]$ and zero outside of $(-2, 2)$. There are different ways of doing this (and the existence of such an h is more than plausible on intuitive grounds of course). One way to write down a formula is as follows: start with

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases}$$

Then put

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

which is non-negative, smooth, and equal to 1 for $t \ge 1$, and 0 for $t \le 0$. Then setting

$$h(t) = g(t+2)g(2-t)$$

gives a function h as desired.

Step 3. Suppose now we are given an open cover $\{U_\alpha\}$ of M. Construct a compact exhaustion $\{G_i\}$ for M as in Step 1 and set $G_0 = \emptyset$. For a point $p \in M$, let i_p be the largest integer such that $p \in M - \overline{G_{i_p}}$. We also pick an index $\alpha_p \in A$ such that $p \in U_{\alpha_p}$. Now let (V, τ) be a chart around p with $\tau(p) = 0$ and such that

$$V \subset U_{\alpha_p} \cap (G_{i_p+2} - \overline{G_{i_p}})$$

and such that $\tau(V)$ contains the closed cube $\overline{C(2)}$. (As always, make a picture or else you will be totally lost now! I am imprisoned in a word processor as they say...) Now define

$$\psi_p = \begin{cases} \varphi \circ \tau & \text{on } V \\ 0 & \text{elsewhere} \end{cases}$$

with φ the smooth function constructed in Step 2. Clearly, ψ_p is a smooth function on M taking the value 1 on some open neighbourhood W_p of p and with compact support contained in $V \subset U_{\alpha_p} \cap (G_{i_p+2} - \overline{G_{i_p}})$.

Now for each $i \geq 1$ let us choose a finite set of points $p \in M$ whose neighbourhoods W_p as above cover $\bar{G}_i - G_{i-1}$ (this is certainly possible since $\bar{G}_i - G_{i-1}$ is compact since it is closed in a compact set). Order all functions ψ_p corresponding to such points p (and $i = 1, 2, \dots$) in a sequence

$$\psi_1,\ \psi_2,\ \psi_3, \dots .$$

Clearly, for every point in M only finitely many of the ψ_j's are nonzero at that point (since the set of ψ's with support contained in each G_i is finite by construction), and thus

$$\psi = \sum_{j=1}^{\infty} \psi_j$$

is a well-defined smooth function on M with $\psi(p) > 0$ everywhere. Then define

$$\theta_i = \frac{\psi_i}{\psi}.$$

Then these θ_i have all the properties stated in (a)-(d) of the Theorem by construction. □

We can finally state and prove Ehresmann's result on fibrations; it is not entirely clear if the attribution is accurate as the argument occurs in several places in some guise and might be older. We say that a map of topological spaces is *proper* if the preimage of every compact set in the target is compact.

Theorem 3.12. *Let $\pi\colon M \to I$ be a proper submersion of manifolds, with $I \subset \mathbb{R}$ an open interval containing $0 \in \mathbb{R}$. Note that all fibres $M_t = \pi^{-1}(t)$ are automatically compact submanifolds of M. Then for any two $t_1, t_2 \in I$, the fibres M_{t_1} and M_{t_2} are diffeomorphic.*

Proof. Since π is a submersion, by Proposition 3.5, b), it is locally around each point $p \in M$ diffeomorphic to the projection $\mathbb{R}^n \to \mathbb{R}$, or better, we can find local coordinates $x_1, \dots, x_n$ in a neighbourhood $U(p)$ of p and a local coordinate t around $\pi(p)$ such that π in these local coordinates is given by $(x_1, \dots, x_n) \mapsto x_1 = t$. We can assume that t is the standard coordinate on $\mathbb{R}$ (we proved in b) of Proposition 3.5 that given any local coordinate y around $\pi(p)$, $y \circ \pi$ can be completed to a system of local coordinates around p). Consider the vector field $\partial/\partial t$ on I. By the preceding argument it is clear that on each $U(p)$ there is a vector field $X_{U(p)}$ with

$$\pi_* X_{U(p)} = \frac{\partial}{\partial t}.$$

We choose a partition of unity $\{\theta_i\}$ subordinate to the cover of M given by the $\{U(p)\}$, $p \in M$. Write the cover $\{U_\alpha\}_{\alpha \in A}$ with A some index set. Then

$$X = \sum_i \theta_i X_{U_{\alpha(i)}}$$

is a global vector field on M with the property $\pi_* X = \partial/\partial t$ by construction.

The idea now is to look at the 1-parameter group of diffeomorphisms/flow generated by the vector field X. Suppose $t_1 < t_2$. For a point $p \in M_{t_1}$ consider a local 1-parameter group of local diffeomorphisms Φ defined on $I_{\epsilon(p)} \times U(p)$. Since

π is a submersion, $\pi(U(p)) = V(p) \subset I$ is open (use Proposition 3.5, b)) and since $\pi_* X = \partial/\partial t$, we have that

$$\Phi_t(p) \in M_{t_1+t}$$

since $\pi \circ \Phi_t(p)$ is an integral curve for the vector field $\partial/\partial t$ on $\mathbb{R}$ with initial value for $t = 0$ equal to t_1. Now since π is proper, $K := \pi^{-1}([t_1, t_2]) \subset M$ is compact. We investigate the "maximum life-time" of the integral curve $\gamma_p(t) = \Phi_t(p)$ starting at $p \in M_{t_1}$. More precisely, for any $q \in M$ we look at the set of all integral curves γ of X in a neighbourhood of q with initial condition $\gamma(0) = q$. If $I_\gamma \subset \mathbb{R}$ is the domain of definition of the map γ, then we can set $J(q) = \bigcup_\gamma I_\gamma$, the union being taken over all such integral curves γ, and obtain an open interval $J(q)$ together with an integral curve $\gamma_q \colon J(q) \to M$ with $\gamma_q(0) = q$ that is the *maximal integral curve* with initial condition q (in the sense that $J(q)$ is maximal among all such integral curves). Let $J(q) = (\omega_-, \omega_+)$, $-\infty \le \omega_- < 0$, $0 < \omega_+ \le +\infty$. Then a well-known fact from the theory of ordinary differential equations, which carries over without change to the context of manifolds, gives that $(t, \gamma_q(t))$ leaves every compact subset S of $\mathbb{R} \times M$ for $t \to \omega_-, t \to \omega_+$: indeed, suppose that this was false. Then we can choose a sequence of times $t_k \to \omega_+$, $k = 1, 2, \dots$ (same argument for ω_-) with $x_k = \gamma_q(t_k)$, $(t_k, x_k) \in S$, and passing to a convergent subsequence if necessary (which we can do since S is compact), we can assume that (t_k, x_k) converges to some $(t_0, x_0) \in S$ (then clearly $t_0 = \omega_+$ and this must be finite). Now picking a chart around (t_0, x_0), we can identify a neighbourhood of (t_0, x_0) in M with an open subset Ω of $\mathbb{R}^n$. With our usual deliberate sloppiness of notation, we write $(t_0, x_0) \in \Omega$. By the Picard-Lindelöf Theorem from MA254 Theory of ODEs it follows that there exist real numbers $r, \epsilon > 0$ such that the initial value problem corresponding to the given vector field on M has a solution defined for all times t with

$$t \in \left[t_0 - \frac{r}{2}, t_0 + \frac{r}{2}\right]$$

provided the initial point is inside

$$G = \left[t_0 - \frac{r}{2}, t_0 + \frac{r}{2}\right] \times \bar{B}\left(x_0, \frac{\epsilon}{2}\right)$$

(where $\bar{B}\left(x_0, \frac{\epsilon}{2}\right)$ is the closed ball around x_0 with radius $\epsilon/2$). But for large k, $(t_k, x_k) \in G$, and is arbitrarily close to (t_0, x_0). This is a contradiction since then $\gamma_q(t)$ cannot be maximal (i.e. the existence interval $J(q)$ is not maximal). Hence $(t, \gamma_q(t))$ has to leave every compact subset.

Returning to our geometric situation, the integral curve $t \mapsto \Phi_t(p)$ cannot be contained in K! In other words, it will exist at least on a small time interval containing $[t_1, t_2]$. The same is true for $\Phi_t(q)$ for any $q \in M_{t'}$ with $t' \in [t_1, t_2]$. Therefore, we can define a diffeomorphism

$$\Phi_{t_2 - t_1} \colon M_{t_1} \to M_{t_2}$$

using the flow of X. This is a diffeomorphism since its inverse is $\Phi_{-(t_2 - t_1)}$. □

Remark 3.13. As a cultural digression and to emphasise how remarkable Theorem 3.12 is, we point out that it is completely false in the context of complex manifolds: the complex structure can vary continuously with parameters in a family of complex manifolds! Riemann called these parameters *moduli*. Theorem 3.12 on the contrary shows that differentiable manifolds do not have moduli: the differentiable structure remains constant in a family $\{M_t\}$ (over any connected base manifold). Intuitively, a complex structure on a given differentiable manifold can be thought of geometrically as a *conformal structure*, i.e. a rule to measure angles in the manifold. Thus intuitively it is clear that there are different complex structures on a two-dimensional torus (tori that are short and fat are conformally different from those that are long and thin, though they are diffeomorphic). The theory of moduli leads to extremely rich and interesting objects in complex geometry, called *moduli spaces*. The reason why we cannot just mimic the argument in the proof of Theorem 3.12 for complex manifolds is the absence of (complex analytic) partitions of unity in that set-up.

We return to develop the general theory of tangent spaces and related notions just a little further for later purposes. If M is a smooth manifold, $p \in M$, and $T_p(M)$ the tangent space to M at p, the dual vector space

$$T_p(M)^* = \mathrm{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$$

is called the *cotangent space* to M at p, or the space of covectors at p (if you have always be on a war-footing with dual vector spaces, this would be the time to shake off this paranoia for good because they are here to stay...).

Definition 3.14. A *(smooth)* 1*-form* ω on M is an assignment of a covector ω_p at each point of $p \in M$ in "a way such that ω_p varies smoothly with p"; the last phrase can be made precise as follows: for each smooth function f on M, we define its *(total) differential* df_p at p as the covector satisfying

$$\langle (df)_p, X\rangle = Xf, \quad \text{for } X \in T_p(M).$$

Here $\langle \cdot, \cdot \rangle$ is just the natural pairing between elements of a vector space and its dual. We point out that df_p coincides with f_* introduced in Definition 3.4 when we view the function as a map of manifolds $f\colon M \to \mathbb{R}$ and use the canonical identification $\mathbb{R} \simeq T_{f(p)}\mathbb{R}$.

Then, if $x_1, \ldots, x_n$ is a local coordinate system around p, then

$$(dx_1)_p, \ldots, (dx_n)_p$$

form a basis for $T_p(M)^*$ which is just the dual basis to the basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_1}\right)_p.$$

This in a neighbourhood of every $p \in M$, every 1-form ω can be uniquely written as

$$\omega = \sum_j f_j dx_j$$

with the f_j functions in a neighbourhood of p. We say that ω is smooth, or that ω_p varies smoothly with p, if all such functions f_j are smooth.

In the following 1-form will always mean smooth 1-form. We will use 1-forms in Section 5 to manufacture more general objects on a manifold, called r*-forms*, $0 \le r \le \dim M$ (a 0-form will just be a smooth function on M). However, for this we will need to introduce some background from multilinear algebra in Section 4 first, notably the exterior powers $\Lambda^r V$ (and exterior algebra) of a vector space V. An r-form on M will then be a smooth assignment of an element of $\Lambda^r T_p(M)^*$ to every $p \in M$. The top-dimensional forms ($r = \dim M$) can be used to build an integration theory on manifolds in Section 5 (since they transform correctly under change of local coordinates according to the well-known change of variables formula for integrals in $\mathbb{R}^d$ involving the Jacobian determinant; functions on M don't transform correctly under change of local coordinates, so you can integrate functions on a general manifold, sorry!)

For the time being, we point out that 1-forms are natural objects occurring in many applications in the sciences as well; for example, the work of a force field on $\mathbb{R}^3$ is a 1-form acting on displacements (which are tangent vectors at points of $\mathbb{R}^3$).

It is a remarkable fact that the collection of all tangent vectors to a manifold M itself forms a manifold (of dimension $2n$ if $\dim M = n$), called the *tangent bundle* $T(M)$, in a natural way, which we now explain. The same is true for covectors: they lead to the so-called *cotangent bundle* $T(M)^*$. As sets, as we have mentioned,

$$T(M) = \bigsqcup_{p \in M} T_p(M), \quad T(M)^* = \bigsqcup_{p \in M} T_p(M)^*$$

(we use the square union symbol to emphasise that these unions are disjoint). We can then define the two projections

$$\pi \colon T(M) \to M, \quad \pi(X_p) = p \text{ for } X_p \in T_p(M),$$
$$\pi^* \colon T(M)^* \to M, \quad \pi(\omega_p) = p \text{ for } \omega_p \in T_p(M)^* \ .$$

Notice that for now these are just maps of sets. M comes with a complete atlas $A = \{(U, \varphi)\}$ of charts (U, φ). Let $\varphi = (x_1, \dots, x_n)$ with the x_i local coordinates on U. Then define

$$\tilde{\varphi} \colon \pi^{-1}(U) \to \mathbb{R}^{2n} \text{ and } \tilde{\varphi}^* \colon (\pi^*)^{-1}(U) \to \mathbb{R}^{2n}$$

by

$$\tilde{\varphi}(v) = (x_1(\pi(v)), \dots, x_n(\pi(v)), dx_1(v), \dots, dx_n(v)) \,,$$
$$\tilde{\varphi}^*(\lambda) = \left(x_1(\pi^*(\lambda)), \dots, x_n(\pi^*(\lambda)), \lambda \left(\frac{\partial}{\partial x_1} \right), \dots, \lambda \left(\frac{\partial}{\partial x_n} \right) \right)$$

for all $v \in \pi^{-1}(U)$ and $\lambda \in (\pi^*)^{-1}(U)$. Both $\tilde{\varphi}$ and $\tilde{\varphi}^*$ are one-to-one maps onto open subsets of $\mathbb{R}^{2n}$. One can construct a topology and differentiable structure on $T(M)$ (and $T(M)^*$) as follows. We do the case of $T(M)$ and leave the unproven assertions as (easy, mechanical, hopefully at worst confusing) exercises:

(1) If (U, φ) and (V, ψ) are charts in A, then $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is a smooth map.
(2) The collection of
$$\left\{ \tilde{\varphi}^{-1}(W) \colon W \text{ open in } \mathbb{R}^n, \, (U, \varphi) \in A \right\}$$
is a basis for a topology on $T(M)$ which makes $T(M)$ into a Hausdorff second-countable topological space.
(3) The set
$$\left\{ (\pi^{-1}(U), \tilde{\varphi}) \colon (U, \varphi) \in A \right\}$$
is a smooth atlas on $T(M)$, and the complete atlas containing this makes $T(M)$ into a differentiable manifold.

We will sometimes write the elements of $T(M)$ as pairs (p, X_p) and those of $T(M)^*$ as (p, ω_p). If $f \colon M \to N$ is a smooth map of manifolds, then it is easy to check that

$$f_* \colon T(M) \to T(N), \quad f_*((p, X_p)) = (f_*)_p(X_p)$$

is a smooth map of manifolds.

In fact, both $T(M)$ and $T(M)^*$ even have a little more structure to them than just the smooth manifold structures, and this comes out automatically from the above construction: they are *vector bundles* (of rank n) over M. We now define this concept precisely. It is ubiquitous in most parts of geometry and shows up whenever linearising certain structures locally. Intuitively, a rank r vector bundle over M is a smoothly varying family of r-dimensional (real) vector spaces V_p, $p \in M$, on M.

Definition 3.15. A *(smooth, real) vector bundle* of rank r on a smooth manifold M is a smooth manifold E together with a smooth map

$$\pi\colon E \to M$$

onto M such that there exists an open cover $\{U_j\}_{j\in J}$ of M, where U_j are domains of charts on M with local coordinates $x^{(j)} = (x_1^{(j)}, \dots, x_n^{(j)})$, with the properties:

(a) There exists a diffeomorphism f_j between $\pi^{-1}(U_j)$ and $U_j \times \mathbb{R}^r$ such that

$$\begin{tikzcd} \pi^{-1}(U_j) \arrow[r, "f_j"] \arrow[d, "\pi"] & U_j \times \mathbb{R}^r \arrow[d, "\pi_j"] \\ U_j \arrow[r, "\mathrm{id}"] & U_j \end{tikzcd}$$

commutes, where $\pi_j(p, \zeta^{(j)})) = p$. (This morally just means that f_j preserves fibres).

(b) If $(p, \zeta_1^{(j)}, \dots, \zeta_r^{(j)}) \in U_j \times \mathbb{R}^r$ and $(p, \zeta_1^{(k)}, \dots, \zeta_r^{(k)}) \in U_k \times \mathbb{R}^r$, then

$$f_j \circ f_k^{-1}(p, \zeta_1^{(k)}, \dots, \zeta_r^{(k)}) = (p, f_{jk}(p) \cdot (\zeta^{(k)})^t)$$

where

$$(\zeta^{(k)})^t = \begin{pmatrix} \zeta_1^{(k)} \\ \vdots \\ \zeta_r^{(k)} \end{pmatrix}$$

and $f_{jk}\colon U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R})$ are smooth maps (here we view $\mathrm{GL}(r, \mathbb{R})$ as an open submanifold of $\mathbb{R}^{r^2}$).

We call $E_p = \pi^{-1}(p)$ the *fibre* of E over p. By (a) and (b) we can endow E_p with a well-defined structure of $\mathbb{R}$-vector space of dimension r for which the f_j become fibrewise linear isomorphisms.

A *morphism of vector bundles*, sometimes also called a (smooth) map of vector bundles, E and E' of ranks r, r' over M is a smooth map of manifolds $\varphi\colon E \to E'$ that maps the fibre E_p into the fibre E'_p and such that $\varphi\mid_{E_p}$ is a linear map of some constant rank s that is independent of the point $p \in M$. One then defines the notion of *isomorphism* in the usual way as a morphism with a two-sided inverse.

A *(smooth) section* in a vector bundle E over an open subset $U \subset M$ is a smooth map $\sigma\colon U \to E$ with $\pi \circ \sigma = \mathrm{id}_U$.

There are variants of the above notion: for example, to get a smooth, complex vector bundle of rank r, one would replace $\mathbb{R}^r$ by $\mathbb{C}^r$ in the above definition etc.

Note that (smooth) sections in $T(M)$ and $T(M)^*$ over an open subset U are nothing but (smooth) vector fields on U and (smooth) 1-forms on U as defined earlier.

Vector bundles on a manifold M can be (and are often) (re-)constructed starting from an open cover $\{U_j\}_{j\in J}$ as in Definition 3.15 and smooth functions

$$f_{jk}\colon U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R})$$

satisfying

$$(*) \quad f_{ij} \circ f_{jk} = f_{ik} \quad \text{on } U_i \cap U_j \cap U_k \text{ for all } i, j, k.$$

The condition $(*)$ is called the *cocycle condition* and an open cover $\{U_j\}_{j\in J}$ together with a system of functions satisfying the cocyle condition is called a $\mathrm{GL}(r, \mathbb{R})$-*cocycle* on M. Clearly every vector bundle E gives rise to a $\mathrm{GL}(r,\mathbb{R})$-cocycle on M, but starting from just a $\mathrm{GL}(r,\mathbb{R})$-cocycle on M, we can also construct a vector bundle (which gives back E, up to isomorphism, if we take the cocycle coming from E): indeed, take the disjoint union

$$\tilde{F} = \bigsqcup_j U_j \times \mathbb{R}^r$$

and define an equivalence relation on $\tilde{F}$ by

$$(p,\zeta_1^{(j)},\dots,\zeta_r^{(j)}) \in U_j\times\mathbb{R}^r \sim (p,\zeta_1^{(k)},\dots,\zeta_r^{(k)}) \in U_k\times\mathbb{R}^r : \iff (\zeta^{(j)})^t = f_{jk}(p)\cdot(\zeta^{(k)})^t.$$

Then the set of equivalence classes F comes with a natural projection map $\pi\colon F \to M$, as well as with a natural differentiable structure, making F into a vector bundle over M. We leave it to you as an exercise to work out the details here.

This allows us to construct new bundles from old ones. First note that Definition 3.15 supplies us, given a vector bundle, with a $\mathrm{GL}(r,\mathbb{R})$-cocycle on M associated to a specific cover $\{U_j\}$. It is clear that this also gives a $\mathrm{GL}(r,\mathbb{R})$-cocycle for any refinement of the given cover, and the $\mathrm{GL}(r,\mathbb{R})$-cocycle associated to the refined cover determines a vector bundle on M isomorphic to E. Therefore, given two vector bundles E, E' of ranks r and r' on M, we can always assume they are determined by a $\mathrm{GL}(r,\mathbb{R})$-cocycle $\{f_{jk}\}$ and a $\mathrm{GL}(r',\mathbb{R})$-cocycle $\{f'_{jk}\}$ associated to the same cover $\{U_j\}$ of M.

We define the *(Whitney) direct sum* of E, E', denoted by $E\oplus E'$, to be the vector bundle of rank $r+r'$ associated to the $\mathrm{GL}(r+r',\mathbb{R})$-cocycle (with cover $\{U_j\}$) given by $\{g_{jk}\}$

$$g_{jk}(p) = \begin{pmatrix} f_{jk}(p) & 0 \\ 0 & f'_{jk}(p) \end{pmatrix}.$$

The *dual bundle* E^* is defined by the $\mathrm{GL}(r,\mathbb{R})$-cocycle $\{(f_{jk}^{-1})^t\}$. It is easy to verify that $T(M)^*$ is the dual bundle of $T(M)$ in this sense. The fibre E_p^* can be identified with the dual vector space $\mathrm{Hom}(E_p,\mathbb{R})$ for every $p\in M$.

We could also define the *tensor product* $E\otimes E'$ of two vector bundles and the *exterior powers* $\Lambda^r E$ of a vector bundle at this point explicitly via cocycles, but this would be a bit too ad hoc and everything will be much clearer after a short discussion of multilinear algebra.

4. Algebraic interlude: tensor, symmetric and exterior algebras

This Section feels a bit out of place here: it might fit better into a second course on linear algebra, and may get incorporated into such a course after a curriculum reform. We will introduce some notions from (multi-)linear algebra that you cannot expected to be familiar with and that are indispensable to proceed further. We will try to keep everything down to the bare minimum needed for proceeding. Principally, we need the *exterior algebra* of a vector space, and its functorial properties, in subsequent Sections.

We fix a field K, which you can think of as either $\mathbb{R}$ or $\mathbb{C}$ (we certainly do not need anything else in this course later). All vector spaces we will consider will tacitly be assumed to be K-vector spaces. Moreover, we also assume they are finite-dimensional over K unless otherwise stated. Much of what follows holds in much greater generality (for example, many statements still hold if you replace K-vector spaces by modules over a commutative ring). But this is not a course in algebra, and we do not think we should strive for maximum generality here.

First, given two vector spaces U, V we define their *tensor product* $U \otimes V$ (sometimes also denoted by $U \otimes_K V$ if we want to recall the ground field) as follows. Let $F(U, V)$ be the vector space which has the set $U \times V$ as a basis, i.e., the free vector space (over K of course as always) generated by the pairs (u, v) where $u \in U$ and $v \in V$. Let R be the vector subspace of $F(U, V)$ spanned by all elements of the form

$$(u + u', v) - (u, v) - (u', v), \quad (u, v + v') - (u, v) - (u, v'),$$
$$(ru, v) - r(u, v), \quad (u, rv) - r(u, v)$$

where $u, u' \in U$, $v, v' \in V$, $r \in K$.

Definition 4.1. The quotient vector space

$$U \otimes V := F(U, V)/R$$

is called the *tensor product* of U and V. The image of $(u, v) \in F(U, V)$ under the projection $F(U, V) \to U \otimes V$ will be denoted by $u \otimes v$. We define the *canonical bilinear mapping*

$$\beta \colon U \times V \to U \otimes V$$

by $\beta(u, v) = u \otimes v$. Being very precise, one should refer to the pair $(U \otimes V, \beta)$ as the tensor product of U and V, but usually people just use the term for $U \otimes V$ with β tacitly understood.

Sometimes one does not need to know the construction of $U \otimes V$ when working with it, but only has to use the following property it enjoys in proofs.

Proposition 4.2. *Let W be a vector space with a bilinear mapping $\psi \colon U \times V \to W$. We say that (W, ψ) has the* universal factorisation property *for $U \times V$ if for every vector space S and every bilinear mapping $f \colon U \times V \to S$ there exists a unique linear mapping $g \colon W \to S$ such that $f = g \circ \psi$.*

Then the couple $(U \otimes V, \beta)$ has the universal factorisation property for $U \times V$. If a couple (W, ψ) has the universal factorisation property for $U \times V$, then $(U \otimes V, \beta)$ and (W, ψ) are canonically isomorphic in the sense that there exists a unique isomorphism $\sigma \colon U \otimes V \to W$ such that $\psi = \sigma \circ \beta$.

Proof. Suppose we are given any bilinear mapping $f \colon U \times V \to S$. Since $U \times V$ is a basis of $F(U, V)$ we can extend f to a unique linear mapping $f' \colon F(U, V) \to S$. Now f' vanishes on R since f is bilinear so induces a linear mapping $g \colon U \otimes V \to S$ on the quotient. Clearly, $f = g \circ \beta$ by construction. The uniqueness of such a map g follows from the fact that $\beta(U \times V)$ spans $U \otimes V$, so we have no other choice in defining g.

Now if (W, ψ) is a couple having the universal factorisation property for $U \times V$, then by the universal factorisation property of $(U \otimes V, \beta)$ (resp. of (W, ψ)), there exists a unique linear mapping $\sigma \colon U \otimes V \to W$ (resp. $\tau \colon W \to U \otimes V$) such that $\psi = \sigma \circ \beta$ (resp. $\beta = \tau \circ \psi$). Hence

$$\beta = \tau \circ \sigma \circ \beta, \quad \psi = \sigma \circ \tau \circ \psi.$$

Using the uniqueness of the g in the universal factorisation property, we conclude that $\tau \circ \sigma$ and $\sigma \circ \tau$ are the identity on $U \times V$ and W respectively. □

This universal property of the tensor product can be used to prove a great many formal properties of the tensor product in a way that is almost mechanical once one gets practice with it. All these proofs are boring. So we give one, and you can easily work out the rest for some practice with this.

Proposition 4.3. *The tensor product has the following properties.*

a) *There is a unique isomorphism of $U \otimes V$ onto $V \otimes U$ sending $u \otimes v$ to $v \otimes u$ for all $u \in U$, $v \in V$*
b) *There is a unique isomorphism of $K \otimes U$ with U sending $r \otimes u$ to ru for all $r \in K$ and $u \in U$; similarly for $U \otimes K$ and U.*
c) *There is a unique isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$ sending $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$ for all $u \in U$, $v \in V$, $w \in W$.*
d) *Given linear mappings*
$$f_i \colon U_i \to V_i,\ i = 1, 2,$$
there exists a unique linear mapping $f \colon U_1 \otimes U_2 \to V_1 \otimes V_2$ such that
$$f(u_1 \otimes u_2) = f_1(u_1) \otimes f_2(u_2)$$
for all $u_1 \in U_1, u_2 \in U_2$.
e) *There is a unique isomorphism from $(U_1 \oplus U_1) \otimes V$ onto $(U_1 \otimes V) \oplus (U_1 \otimes V)$ sending $(u_1, u_2) \otimes v$ to $(u_1 \otimes v, u_2 \otimes v)$ for all $u_1 \in U_1$, $u_2 \in U_2$, $v \in V$.*
f) *If $u_1, \dots, u_m$ is a basis for U and $v_1, \dots, v_n$ is a basis for V, then $u_i \otimes v_j$, $i = 1, \dots, m$, $j = 1, \dots, n$, is a basis for $U \otimes V$. In particular, $\dim U \otimes V = \dim U \dim V$.*
g) *Let U^* be the dual vector space to U. Then there is a unique isomorphism g from $U \otimes V$ onto $Hom(U^*, V)$ such that*
$$(g(u \otimes v))(u^*) = \langle u, u^* \rangle v \quad \text{for all } u \in U,\ v \in V,\ u^* \in U^*.$$
h) *There is a unique isomorphism h of $U^* \otimes V^*$ onto $(U \otimes V)^*$ such that*
$$(h(u^* \otimes v^*))(u \otimes v) = \langle u, u^* \rangle \langle v, v^* \rangle$$
for all $u \in U$, $u^* \in U^*$, $v \in V$, $v^* \in V^*$.

Proof. We prove a) and g) just to illustrate the method, and leave the rest as easy exercises.

For a) let $f \colon U \times V \to V \otimes U$ be the bilinear mapping with $f(u, v) = v \otimes u$. By the universal property of the tensor product, there is a unique linear mapping
$$g \colon U \otimes V \to V \otimes U$$
such that $g(u \otimes v) = v \otimes u$. Similarly, there is a unique linear mapping $g' \colon V \otimes U \to U \otimes V$ with $g'(v \otimes u) = u \otimes v$. Clearly, $g' \circ g$ and $g \circ g'$ are the identity transformations.

We now prove g) (using f)). Consider the bilinear mapping $f \colon U \times V \to \mathrm{Hom}(U^*, V)$ given by
$$(f(u, v))(u^*) = \langle u, u^* \rangle v$$
and apply the universal property of the tensor product in Proposition 4.2. Thus there exists a unique linear mapping $g \colon U \otimes V \to \mathrm{Hom}(U^*, V)$ such that $(g(u \otimes v))(u^*) = \langle u, u^* \rangle v$. To prove that g is an isomorphism, let $u_1, \dots, u_m$ be a basis for U, $u_1^*, \dots, u_m^* \in U^*$ the dual basis, and $v_1, \dots, v_n$ a basis for V. We show that
$$\{g(u_i \otimes v_j) \colon i = 1, \dots, m,\ j = 1, \dots, n\}$$
is a linearly independent set of vectors. Indeed
$$\sum_{i,j} a_{ij} g(u_i \otimes v_j) = 0, \quad a_{ij} \in K$$

gives

$$0 = \sum_{i,j} a_{ij} g(u_i \otimes v_j)(u_k^*) = \sum_j a_{kj} v_j$$

hence all the a_{ij} vanish. Since $\dim U \otimes V = \dim \mathrm{Hom}(U^*, V)$ by f), g is an isomorphism. $\square$

We now consider a vector space V and put $V^{\otimes r} := V \otimes \cdots \otimes V$ (r-times), and set

$$T^\bullet(V) = \bigoplus_{r \geq 0} V^{\otimes r}.$$

If $e_1, \dots, e_n$ is a basis for V, then

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \colon 1 \leq i_1, \dots, i_r \leq n\}$$

is a basis for $V^{\otimes r}$, applying f) of Proposition 4.3 inductively. $T^\bullet(V)$ has more structure than just the structure of a K-vector space (of infinite dimension in general!): it is a graded $\mathbb{R}$-algebra, associative, but not commutative, if we define the product

$$(v_1 \otimes \cdots \otimes v_r) \cdot (w_1 \otimes \cdots \otimes w_s) = v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s$$

and extend by K-linearity to all of $T^\bullet(V)$. We call $T^\bullet(V)$ the *tensor algebra* of V.

Definition 4.4. Let I be the two-sided ideal of $T^\bullet(V)$ generated by all elements of the form $v \otimes v$ for $v \in V$. The quotient

$$\Lambda^\bullet(V) := T^\bullet(V)/I$$

inherits a natural structure of a graded algebra (since I is a homogeneous ideal), called the *exterior algebra* of V.

Similarly, if J denotes the two-sided ideal of $T^\bullet(V)$ generated by all elements of the form $v \otimes w - w \otimes v$ for $v, w \in V$, then

$$\mathrm{Sym}^\bullet(V) = T^\bullet(V)/J$$

is a graded algebra, called the *symmetric algebra* of V.

We denote the image of $v_1 \otimes \cdots \otimes v_r$ in $\Lambda^\bullet(V)$ by $v_1 \wedge \cdots \wedge v_r$, and the image in $\mathrm{Sym}^\bullet(V)$ by $v_1 \cdot \ldots \cdot v_r$, or simply $v_1 \ldots v_r$. We will also denote the algebra product in $\Lambda^\bullet(V)$ simply by $\wedge$ and call it the wedge product. Similarly, we denote the algebra product in $\mathrm{Sym}^\bullet(V)$ by a dot or simply by concatenation.

The r-th graded component $\Lambda^r(V)$ of $\Lambda^\bullet(V)$ (resp. $\mathrm{Sym}^r(V)$ of $\mathrm{Sym}^\bullet(V)$) is called the r-th exterior power of V (resp. r-th symmetric power of V).

In fact, other types of important algebras can be defined in a similar way as quotients of the tensor algebra $T^\bullet(V)$ by graded ideals, for example, Clifford algebra. But in fact, only the exterior algebra

$$\Lambda^\bullet(V) = \bigoplus_{r=0}^{\infty} \Lambda^r(V)$$

will concern us in applications in later sections. We only mentioned the symmetric algebra because it would have weighed too heavily on our conscience if we hadn't- it is so important in other contexts. In fact, it is a good exercise to convince yourself that $\mathrm{Sym}^\bullet(V)$ is simply isomorphic to a polynomial algebra $K[X_1, \dots, X_n]$ with one variable X_i corresponding to each basis vector e_i of V.

We now turn to the properties of the exterior algebra we will need later. First of all it is clear that for any $v, w \in V$ we have

$$v \wedge v = 0, \quad v \wedge w = -w \wedge v,$$

the first because $v \otimes v$ maps to zero under the quotient map $T^\bullet(V) \to \Lambda^\bullet(V)$, and the second is implied by $(v+w) \wedge (v+w) = 0$. We say the wedge-product is *alternating* or *anti-symmetric*. More generally, this implies that if $\omega \in \Lambda^r(V)$ and $\varphi \in \Lambda^s(V)$, then

$$\omega \wedge \varphi = (-1)^{rs} \varphi \wedge \omega.$$

Proposition 4.5. *The exterior powers and exterior algebra have the following properties.*

a) If $F \colon V \times \dots \times V \to W$ (r copies of V) is a multilinear alternating mapping of vector spaces (which means $F(v_1, \dots, v_r)$ is linear in each argument separately and zero if two of the v_i are equal), then there is a unique linear map

$$\bar{F} \colon \Lambda^r(V) \to W$$

with $\bar{F}(v_1 \wedge \dots \wedge v_r) = F(v_1, \dots, v_r)$.

b) If $\varphi \colon V \to W$ is a linear mapping, there is a unique linear mapping

$$\Lambda^r(\varphi) \colon \Lambda^r(V) \to \Lambda^r(W)$$

with the property

$$\Lambda^r(\varphi)(v_1 \wedge \dots \wedge v_r) = \varphi(v_1) \wedge \dots \wedge \varphi(v_r).$$

c) If $e_1, \dots, e_n$ is a basis of V, then

$$\{e_{i_1} \wedge \dots \wedge e_{i_r} \colon 1 \le i_1 < \dots < i_r \le n\}$$

is a basis of $\Lambda^r(V)$. Consequently,

$$\dim \Lambda^r(V) = \binom{n}{r}$$

and $\Lambda^i(V) = 0$ for $i > n$. Moreover, note that $\dim \Lambda^n(V) = 1$.

d) If $f \colon V \to V$ is an endomorphism, $\dim V = n$, then the induced map

$$\Lambda^n(f) \colon \Lambda^n(V) \to \Lambda^n(V)$$

is multiplication by $\det(f)$.

e) For an n-dimensional vector space V we have a natural non-degenerate bilinear pairing

$$\Lambda^r(V^*) \times \Lambda^r(V) \to K$$

mapping

$$(v_1^* \wedge \dots \wedge v_r^*, w_1 \wedge \dots \wedge w_r) \mapsto \det(v_i^*(w_j))_{1 \le i,j \le r}$$

which induces an isomorphism

$$\Lambda^r(V^*) \simeq (\Lambda^r(V))^*.$$

Proof. For a) notice that repeated application of the universal property of the tensor product furnishes us with a linear map

$$\tilde{F} \colon V^{\otimes r} \to W$$

with $\tilde{F}(v_1 \otimes \dots \otimes v_r) = F(v_1, \dots, v_r)$; this factors over

$$\Lambda^r(V) = V^{\otimes r} / \left(V^{\otimes r} \cap I \right)$$

since the ideal I is generated by elements $v \otimes v$ that get mapped to zero since F is alternating.

To prove b) notice that inductive application of Proposition 4.3, d), gives an induced mapping

$$\otimes^r \varphi \colon V^{\otimes r} \to W^{\otimes r}$$

and this maps $V^{\otimes r} \cap I$ into the corresponding graded piece of the ideal we divide out by to get $\Lambda^r W$, so descends to give $\Lambda^r(\varphi)$ as desired.

For c) we first show that $\Lambda^n(V) \simeq K$ via the map induced by the determinant. Indeed, since the elements $v_1 \wedge \cdots \wedge v_r$ generate $\Lambda^r(V)$, it is clear that, if $e_1, \ldots, e_n$ is a basis of V, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} \colon 1 \le i_1 < \cdots < i_r \le n\}$$

is at least a generating set for $\Lambda^r(V)$. In particular, $\Lambda^n(V)$ is at most one-dimensional, and exactly one-dimensional, generated by $e_1 \wedge \cdots \wedge e_n$, if we can show it is nonzero. But by a), the determinant gives a map $\overline{\det} \colon \Lambda^n(V) \to K$ sending $e_1 \wedge \cdots \wedge e_n$ to 1.

Now suppose there was a linear dependence relation between the $e_{i_1} \wedge \cdots \wedge e_{i_r}$:

$$\sum_I a_I e_I = 0$$

where we use multi-index notation $I = (i_1, \ldots, i_r)$, $1 \le i_1 < \cdots < i_r \le n$, $a_I \in K$, $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}$. For a certain multi-index $J = (j_1, \ldots, j_r)$, let $\bar{J}$ be the complimentary indices to J in $\{1, \ldots, n\}$, increasingly ordered. Then

$$(\sum_I a_I e_I) \wedge e_{\bar{J}} = \pm a_J e_1 \wedge \cdots \wedge e_n = 0.$$

Hence all coefficients a_J are zero, proving c).

The endomorphism $f \colon V \to V$ gives a commutative diagram

$$\begin{array}{ccc} \Lambda^n(V) & \xrightarrow{\Lambda^n(f)} & \Lambda^n(V) \\ \downarrow{\scriptstyle \overline{\det}} & & \downarrow{\scriptstyle \overline{\det}} \\ K & \xrightarrow{\text{mult}(c)} & K \end{array}$$

where the lower horizontal arrow is multiplication by some constant c. We want to show that $c = \det(f)$ and for this it suffices to consider what happens to $\overline{\det}(e_1 \wedge \cdots \wedge e_n) = 1$: this gets mapped to the determinant of the matrix with columns $(f(e_1), \ldots, f(e_n))$, which is $\det(f)$. This proves d).

For e) first note that

$$\det((v_i^*(w_j))_{1 \le i,j \le r}$$

is alternating in both the $v_1^*, \ldots, v_n^*$ and the $w_1, \ldots, w_n$, and multilinear in these sets of variables; hence by an application of a), we get a well-defined map

$$\beta \colon \Lambda^r(V^*) \times \Lambda^r(V) \to K$$

of the type in e). All that remains to prove is that this pairing is nondegenerate, i.e. that for any nonzero $\omega \in \Lambda^r(V)$ there is a $\psi \in \Lambda^r(V^*)$ with $\beta(\psi, \omega) \neq 0$, and vice versa, for any nonzero $\psi' \in \Lambda^r(V^*)$ there is an $\omega' \in \Lambda^r(V)$ with $\beta(\psi', \omega') \neq 0$. We prove the first assertion since the second is then proven completely analogously. If $e_1, \ldots, e_n$ is a basis of V, write in multi-index notation

$$\omega = \sum_I a_I e_I.$$

Since $\omega \neq 0$, there is an $a_J \neq 0$. Then let $\psi = e_J^* = e_{j_1}^* \wedge \dots \wedge e_{j_r}^*$ where $e_1^*, \dots, e_n^*$ is the dual basis to $e_1, \dots, e_n$. We have $\beta(\psi, \omega) = a_J \neq 0$ then. □

5. Differential r-forms, integration of forms, exterior derivative, manifolds with boundary, Stokes' theorem

Let M be a smooth manifold as usual, $p \in M$, $T_p(M)$ the tangent space to M at p.

Definition 5.1. An *r-form* ω on M (or, more precisely, a smooth r-form, or differential form of degree r) is an assignment of an element in $\Lambda^r T_p(M)^*$ to each point $p \in M$ such that, if we express ω in terms of local coordinates $x_1, \dots, x_1$ around p:

$$\omega = \sum_{1 \le i_1 < \dots < i_r \le n} f_{i_1 \dots i_r}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

then the component functions $f_{i_1 \dots i_r}$ are smooth.

We write $\mathscr{A}^r(M)$ for the $\mathbb{R}$-vector space of all r-forms on M (this is also a module over the smooth functions $\mathscr{C}^\infty(M)$ on M).

So why are differential forms cool and why should you even bother about them? There are at least 3 reasons, but more could be given.

(1) Differential forms show up naturally in numerous places in the sciences when modelling real-life problems, e.g. in physics.
(2) They are the appropriate objects to build an integration theory on manifolds and vastly generalise familiar theorems from vector calculus (Gauss, Green, ...) and put all of these into one conceptual framework.
(3) They allow us to measure what type of "holes" a space has, via the so-called *de Rham cohomology*; in this way, we can attach quantities to manifolds that are invariant under diffeomorphism and that, in particular, allow us to decide in many cases that two given manifolds are not diffeomorphic.

We will get to (2) and (3) later in this Section and the next; for (1), just think of the work of a field along a path or the flux of a fluid through a surface. We will see more concrete examples later.

Suppose that $f\colon M \to N$ is a smooth map of manifolds and $\omega \in \mathscr{A}^r(N)$, then we can define the *pullback* $f^*\omega \in \mathscr{A}^r(M)$ of ω by f via

$$(f^*\omega)_p = \Lambda^r((f_*)_p)^*(\omega_{f(p)}), \quad p \in M$$

where $\Lambda^r((f_*)_p)^*$ is the linear map

$$\Lambda^r((f_*)_p)^*\colon \Lambda^r T_{f(p)}(N)^* \to \Lambda^r T_p(M)^*$$

obtained as the r-th exterior power of the map

$$(f_*)_p^*\colon T_{f(p)}(N)^* \to T_p(M)^*$$

which in turn is just the dual map of the differential

$$(f_*)_p\colon T_p(M) \to T_{f(p)}(N).$$

We will make the pullback of forms more explicit in a second, but first note that obviously

$$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2, \quad \text{for all } \omega_1, \omega_2 \in \mathscr{A}^r(N),$$

$$f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta), \quad \text{for all } \omega \in \mathscr{A}^r(N),\ \theta \in \mathscr{A}^s(N),$$

$$(f \circ h)^*\omega = h^* f^*\omega, \quad \text{where } h\colon M' \to M,\ f\colon M \to N \text{ are smooth maps, } \omega \in \mathscr{A}^r(N).$$

Let us work out what f^* does in local coordinates. Given a smooth map $f\colon M \to N$ of manifolds, $\dim M = k$, $\dim N = l$, $p \in$, we can choose a chart $(V, y_1, \ldots, y_l)$ around $f(p)$ and a chart $(U, x_1, \ldots, x_k)$ around p with $f(U) \subset V$; in terms of these coordinates we can write $f = (f_1, \ldots, f_l)$ with the f_i smooth functions of $x_1, \ldots, x_k$ on U. Since $(f_*)_q$, for $q \in U$, is represented in the local coordinates by the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(q)\right)_{1\le i\le l, 1\le j\le k}$$

and $((f_*)_q)^*$ is given by the transpose matrix, we get

$$f^* dy_i = \sum_{j=1}^{k} \frac{\partial f_i}{\partial x_j} dx_j = df_i.$$

Thus for an r-form ω on V, which we can write as

$$\omega = \sum_I a_I dy_I$$

(using multi-index notation $I = (i_1, \ldots, i_r)$, $1 \le i_1 < \cdots < i_r \le l$, $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_r}$, the a_I are smooth functions on V), we obtain

$$f^*\omega = \sum_I (f^* a_I) df_I$$

where $f^* a_I = a_I \circ f$ is the pullback of the function a_I and $df_I = df_{i_1} \wedge \cdots \wedge df_{i_r}$.

Now suppose that $f\colon U \to V$ is a diffeomorphism, $k = l =: n$, and $\omega = dy_1 \wedge \cdots \wedge dy_n$. Then we get

$$f^*(dy_1 \wedge \cdots \wedge dy_n) = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le k} dx_1 \wedge \cdots \wedge dx_n.$$

More specifically suppose now that $(\Omega, x_1, \ldots, x_n)$ and $(\Omega', y_1, \ldots, y_n)$ are two domains of charts on the same manifold M, $U = V = \Omega \cap \Omega'$, and f the coordinate change map expressing the y-coordinates in terms of the x-coordinates. If $a(y_1, \ldots, y_n)$ is any function, we thus have the relation on $\Omega \cap \Omega'$

$$a(y_1, \ldots, y_n) dy_1 \wedge \cdots \wedge dy_n = a(y_1(x), \ldots, y_n(x)) \det\left(\frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n.$$

This is temptingly reminiscent of the change of variables formula in a multiple integral in $\mathbb{R}^n$:

$$\int a(y_1, \ldots, y_n) dy_1 \ldots dy_n = \int a(y_1(x), \ldots, y_n(x)) \left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right| dx_1 \ldots dx_n.$$

(Here a is any "nice" integrable function, for example a smooth function with compact support). The only difference is the presence of the absolute value around $\det(\partial y_i/\partial x_j)$. This suggest that we should be able to develop a consistent integration theory on manifolds, using partitions of unity to piece together local contributions to an integral over a manifold, provided we can make consistent sign choices for $\det(\partial y_i/\partial x_j)$ to make the local contributions independent of the choice of local coordinates. This requires the introduction of the concept of an *orientation.*

Definition 5.2. An n-dimensional manifold M is called *orientable* if there exists an everywhere non-vanishing n-form ω on it (meaning $\omega \in \mathscr{A}^n(M)$ and $\omega_p \neq 0$ in $\Lambda^n T_p(M)^*$ for all $p \in M$). Given an orientable manifold M, an *orientation* of M is an equivalence class of non-vanishing n-forms on M where $\omega \sim \omega'$ if $\omega = f\omega'$

with f an everywhere positive smooth function on M. An *oriented* manifold is an orientable manifold together with a fixed choice of orientation.

It is thus clear that a connected orientable manifold M has precisely two orientations: the equivalence class of ω and of $-\omega$.

The notion is consistent with the nomenclature introduced in Definition 1.5, c):

Proposition 5.3. *A manifold M is orientable if and only if it has a covering by coordinate charts such that*

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$$

on each intersection.

Proof. Suppose first that M is orientable, so we are given a nowhere vanishing n-form ω on M. In a coordinate chart

$$\omega = f(x_1, \dots, x_n)dx_1 \wedge \dots \wedge dx_n$$

and possibly replacing x_1 by $-x_1$, we can assume that M is covered by such coordinate charts where the function corresponding to f above is positive in each chart. Then on the overlap of two charts we have

$$\begin{aligned} g(y_1, \dots, y_n)dy_1 \wedge \dots \wedge dy_n &= g(y_1(x), \dots, y_n(x)) \det\left(\frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n \\ &= f(x_1, \dots, x_n)dx_1 \wedge \dots \wedge dx_n \end{aligned}$$

so if $f > 0$ and $g > 0$, we must have $\det(\partial y_i/\partial x_j) > 0$ on the intersection.

Conversely, suppose we are given such a coordinate covering. Choose a partition of unity θ_i subordinate to the given cover $\{U_\alpha\}$. Put

$$\omega = \sum_i \theta_i dy_1^{\alpha(i)} \wedge \dots \wedge dy_n^{\alpha(i)}$$

which is a global smooth n-form on M. Then on U_β with coordinates $y_1^\beta, \dots, y_n^\beta$ we have

$$\omega\mid_{U_\beta} = \sum_i \theta_i \det\left(\frac{\partial y_k^{\alpha(i)}}{\partial y_l^\beta}\right) dy_1^\beta \wedge \dots \wedge dy_n^\beta$$

and this is nonzero because $\theta_i \geq 0$ and $\det\left(\frac{\partial y_k^{\alpha(i)}}{\partial y_l^\beta}\right) > 0$ by hypothesis. $\square$

Suppose now that M is an orientable manifold and that we have moreover chosen an orientation. We can then define

$$\int_M \omega$$

for any n-form ω with compact support on M, in the following way: choose a covering $\{U_\alpha\}$ satisfying the condition in Proposition 5.3 (transition functions have positive Jacobian on intersections of charts) and coordinates y_i^α on U_α such that $dy_1^\alpha \wedge \dots \wedge dy_n^\alpha$ is equivalent to the given orientation form. Then on U_α we can write

$$\omega\mid_{U_\alpha} = f(y_1^\alpha, \dots, y_n^\alpha)dy_1^\alpha \wedge \dots \wedge dy_n^\alpha.$$

Choosing a partition of unity $\{\theta_i\}$ subordinate to $\{U_\alpha\}$, we have

$$\theta_i\omega\mid_{U_{\alpha(i)}} = g_i(y_1^\alpha, \dots, y_n^\alpha)dy_1^\alpha \wedge \dots \wedge dy_n^\alpha$$

with g_i a function which is smooth and has compact support in (an open subset of) the $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$. Then define

$$\int_M \omega := \sum_i \int_M \theta_i \omega := \sum_i \int_{\mathbb{R}^n} g_i(x_1, \dots, x_n) dx_1 dx_2 \dots dx_n.$$

The local finiteness of the family $\{\theta_i\}$, property b) of Theorem 3.11, implies that only finitely many θ_i are not identically zero on the compact support of ω, so the above sum is finite. We leave it to you as an exercise to show this definition is independent of the partition of unity.

We still owe you to exhibit some examples of orientable manifolds and orientations. This is rather easy. Clearly $\mathbb{R}^n$ is orientable; if $x_1, \dots, x_n$ are the standard global coordinates, then the two orientations correspond to $\pm dx_1 \wedge \dots \wedge dx_n$.

Suppose that $M \subset \mathbb{R}^n$ is a subset given by $f(x) = c$ where $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has surjective derivative at all points of M, hence is a submersion in a neighbourhood of M. Such an M is a submanifold either by Proposition 2.4 or Proposition 3.5, b). If $f = (f_1, \dots, f_m)$ the tangent space $T_a(M)$ can be identified with the subspace of $T_a(\mathbb{R}^n)$ of tangent vectors annihilated by the 1-forms $df_1, \dots, df_m$. Dually, $T_a(M)^*$ is the quotient of $T_a(\mathbb{R}^n)^*$ by the subspace U spanned by $df_1, \dots, df_m$. Thus

$$\Lambda^m U \otimes \Lambda^{n-m} T_a(M)^* \simeq \Lambda^m U \otimes \Lambda^{n-m}(T_a(\mathbb{R}^n)^*/U) \simeq \Lambda^n(T_a(\mathbb{R}^n)^*)$$

(prove this as an exercise! It's just linear algebra). Therefore, since $df_1 \wedge \dots \wedge df_m$ gives a non-vanishing section in the $\Lambda^m U$ on M, and $dx_1 \wedge \dots \wedge dx_n$ gives a non-vanishing section in $T(\mathbb{R}^n)^*$, the isomorphism furnishes us with a non-zero section in $\Lambda^{n-m} T(M)^*$.

In particular, it follows that all spheres S^n are orientable, but obviously we get tons of other examples as well. However, there are also compact manifolds that are not orientable (we will give an example in a second). This gives an interesting conclusion: it can be proved that any compact manifold M of dimension d can be embedded into some $\mathbb{R}^N$ for N sufficiently large (we will not do this in this course, though); thus it follows that not every such M can be defined by $N - d$ global functions on $\mathbb{R}^N$ with independent derivatives on M because such an M would necessarily have to be orientable.

Here is a non-orientable (compact) example: consider the real projective space $\mathbb{P}^n_{\mathbb{R}}$ with the smooth quotient map

$$p\colon S^n \to \mathbb{P}^n_{\mathbb{R}}$$

mapping a unit vector in $\mathbb{R}^{n+1}$ to the one-dimensional subspace it spans (this shows that $\mathbb{P}^n_{\mathbb{R}}$ is compact since S^n is). We have the diffeomorphism

$$\sigma\colon S^n \to S^n, \quad \sigma(x) = -x.$$

Moreover, if $x_1, \dots, x_{n+1}$ are global coordinates on $\mathbb{R}^{n+1}$ and $x_1 \neq 0$ at a point of S^n, we can use $x = (x_2, \dots, x_{n+1})$ as local coordinates on S^n at that point and the ratios

$$(x_2/x_1, \dots, x_{n+1}/x_1)$$

as local coordinates around the image point in $\mathbb{P}^n_{\mathbb{R}}$. With $||\cdot||$ the Euclidean norm on $\mathbb{R}^{n+1}$ one thus has in these coordinates

$$p(x) = \frac{1}{\sqrt{1 - ||x||^2}} x$$

which is smooth with smooth inverse

$$q(y) = \frac{1}{\sqrt{1+||y||^2}} y;$$

therefore, we can also use $x_2, \dots, x_{n+1}$ as local coordinates on $\mathbb{P}^n_{\mathbb{R}}$. We can define a nowhere vanishing n-form ω on S^n by using local coordinates $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}$ around points where $x_i \neq 0$ and putting

$$\omega = (-1)^i \frac{1}{x_i} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$$

in the chart of S^n where $x_i \neq 0$. Note that

$$\sum_{i=1}^{n+1} x_i dx_i = 0$$

on S^n, so when $x_j \neq 0$,

$$dx_j = -\frac{1}{x_j}(x_1 dx_1 + \cdots + x_{j-1}dx_{j-1} + x_{j+1}dx_{j+1} + \cdots + x_{n+1}dx_{n+1})$$

and substituting this into the above formula for ω on $x_i \neq 0$ gives the analogous formula for ω on the open set $x_j \neq 0$, so the formula defines a global ω on S^n.

If $\mathbb{P}^n_{\mathbb{R}}$ is orientable, there exists a nonzero n-form θ on $\mathbb{P}^n_{\mathbb{R}}$, and then $p^*\theta$ would be a nonzero n-form on S^n, whence $p^*\theta = f\omega$ for a nowhere vanishing smooth function f on S^n. On the other hand

$$\sigma^*\omega = (-1)^i \frac{1}{-x_i} d(-x_1) \wedge \cdots \wedge d(-x_{i-1}) \wedge d(-x_{i+1}) \wedge \cdots \wedge d(-x_{n+1}) = (-1)^{n-1}\omega.$$

Thus, since $p \circ \sigma = p$, we get

$$f\omega = p^*\theta = \sigma^* p^* \theta = (f \circ \sigma)(-1)^{n-1}\omega.$$

If n is even, we conclude $f \circ \sigma = -f$, and if $f(a) > 0$ at some point a of S^n, $f(-a) < 0$. Since S^n is connected, f must vanish somewhere, contradiction. Thus $\mathbb{P}^{2m}_{\mathbb{R}}$ is not orientable.

On the other hand, if n is odd, we have $\sigma^*\omega = \omega$, and this σ-invariant top form descends to the quotient $S^n/\sigma = \mathbb{P}^n_{\mathbb{R}}$ to give a nowhere vanishing n-form on $\mathbb{P}^n_{\mathbb{R}}$. Hence $\mathbb{P}^{2m+1}_{\mathbb{R}}$ is orientable.

When a manifold arises as a quotient, such as the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, there is frequently a better method to prove it is orientable: in the case of the torus, for example, the n-form $dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{R}^n$ is invariant under all translations in $\mathbb{Z}^n$, hence descends to a nowhere vanishing n-form on $\mathbb{T}^n$.

It is instructive to notice that you have already integrated over manifolds before although you didn't notice it back then.

Example 5.4. Suppose that

$$\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

is a compactly supported smooth 1-form on $\mathbb{R}^3$. Let $\gamma\colon I \to \mathbb{R}^3$ be a diffeomorphism of some open interval $I \subset \mathbb{R}$ onto $C = \gamma(I)$; then C is a 1-dimensional manifold, and

$$\int_C \omega = \int_I \gamma^*\omega.$$

Concretely, if

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

then
$$\gamma^* dx_i = d\gamma_i = \frac{d\gamma_i}{dt},$$
whence
$$\int_C \omega = \sum_{i=1}^{3} \int_I f_i(\gamma(t)) \frac{d\gamma_i}{dt} dt.$$
If we define F to be the vector field (f_1, f_2, f_3) you will recognise this as the *line integral* of F over C (sometimes denoted by
$$\oint \vec{F}\, d\gamma$$
if your lecturer is into calligraphy.

Next consider a compactly supported 2-form on $\mathbb{R}^3$
$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2.$$
Suppose we want to integrate ω over a surface S in $\mathbb{R}^3$, which we assume to be given as (locally) the graph of a smooth function $G\colon \mathbb{R}^2 \to \mathbb{R}$, $x_3 = G(x_1, x_2)$. Then $h\colon \mathbb{R}^2 \to S$
$$h(x_1, x_2) = (x_1, x_2, G(x_1, x_2))$$
is a diffeomorphism, and
$$h^* dx_1 \wedge dx_2 = dx_1 \wedge dx_2,$$
$$h^* dx_2 \wedge dx_3 = dx_2 \wedge dG = dx_2 \wedge \left(\frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial x_2} dx_2 \right)$$
$$= -\frac{\partial G}{\partial x_1} dx_1 \wedge dx_2;$$
and similarly
$$h^* dx_3 \wedge dx_1 = -\frac{\partial G}{\partial x_2} dx_1 \wedge dx_2.$$
Therefore
$$\int_S \omega = \int_{\mathbb{R}^2} (n_1 f_1 + n_2 f_2 + n_3 f_3) dx_1\, dx_2,$$
where
$$\vec{n} = (n_1, n_2, n_3) = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1 \right).$$
At any point of the surface $x = (x_1, x_2, G(x_1, x_2))$ the vector $\vec{n}(x)$ is normal to the surface S meaning
$$\vec{n}(x) \perp T_x(S)$$
when we view $T_x(S)$ as a subspace of $\mathbb{R}^3$ in the natural way (check this!) You will be familiar with this formula from vector calculus: writing
$$\vec{u} = \frac{\vec{n}}{|\vec{n}|}$$
for the unit normal vector, and letting $\vec{F} = (f_1, f_2, f_3)$, and defining a smooth 2-form
$$dA = |\vec{n}| dx_1 \wedge dx_2$$
we get
$$\int_S \omega = \int_{\mathbb{R}^2} (\vec{F} \cdot \vec{u})\, dA.$$

Here dA is usually called the *area form* of S (if you don't recall from vector calculus, try to figure out why!), and the integral on the right hand-side the *flux* of the vector field $\vec{F}$ through the surface S.

We have already seen that for a 0-form on a manifold M, $f \in \mathscr{A}^0(M)$, which is nothing but a smooth function on M, we can define a 1-form df, the differential of f. This operation d can be extended in a canonical way to all r-forms and is then called *exterior differentiation*. We will see that it provides a key link between the geometry of a manifold and analysis on the manifold. The operation d is defined in terms of the manifold structure alone without any additional choices.

Proposition 5.5. *Let M be a smooth manifold, $\mathscr{A}(M) = \bigoplus_r \mathscr{A}^r(M)$. There is a natural $\mathbb{R}$-linear map $d\colon \mathscr{A}(M) \to \mathscr{A}(M)$, called* exterior differentiation *or the* exterior derivative*, that has the following properties:*

(i) *$d(\mathscr{A}^r(M)) \subset \mathscr{A}^{r+1}(M)$;*
(ii) *For every function $f \in \mathscr{A}^0(M)$, df is the differential defined earlier.*
(iii) *For every $\omega \in \mathscr{A}^r(M)$ and $\pi \in \mathscr{A}^s(M)$, we have*

$$d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^r \omega \wedge d\pi.$$

(iv) *$d^2 = d \circ d = 0$.*

Moreover, if $x_1, \dots, x_n$ are local coordinates on some open subset U of M and we write on U

$$\omega = \sum_{i_1 < \dots < i_r} f_{i_1 \dots i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

then

$$d\omega = \sum_{i_1 < \dots < i_r} df_{i_1 \dots i_r} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r}.$$

Proof. We will define a map d having all the properties claimed by breaking up ω as a sum of terms with support in some local coordinate system (using a partition of unity), then define a d operator locally using a coordinate system, then showing that the definition is independent of the choice of coordinate system. So we write

$$\omega = \sum_{i_1 < \dots < i_r} f_{i_1 \dots i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

and *define*

$$d\omega = \sum_{i_1 < \dots < i_r} df_{i_1 \dots i_r} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r}.$$

When $r = 0$, this is just the differential, so (ii) will be satisfied, and (i) is by construction. Now

$$d\omega = \sum_{j,\, i_1 < \dots < i_r} \frac{\partial f_{i_1 \dots i_r}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r}.$$

whence

$$d^2\omega = \sum_{j,k,\, i_1 < \dots < i_r} \frac{\partial^2 f_{i_1 \dots i_r}}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r}.$$

Now

$$\frac{\partial^2 f_{i_1 \dots i_r}}{\partial x_k \partial x_j}$$

is symmetric in k, j, but gets multiplied by $dx_k \wedge dx_j$, which is skew-symmetric, so $d^2\omega = 0$, in accordance with (iv).

We check (iii) for decomposable forms

$$\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_r} =: f dx_I, \quad \pi = g dx_{j_1} \wedge \cdots \wedge dx_{j_s} =: g dx_J$$

and conclude using $\mathbb{R}$-linearity. Now

$$\begin{aligned} d(\omega \wedge \pi) &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= (f dg + g df) \wedge dx_I \wedge dx_J \\ &= (-1)^r f\, dx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge g dx_J \\ &= (-1)^r \omega \wedge d\pi + d\omega \wedge \pi. \end{aligned}$$

So we have constructed an operator d in terms of a chosen coordinate system, which satisfies the conditions of the Proposition. Now let us write in terms of another coordinate system $y_1, \dots, y_n$

$$\omega = \sum_{i_1 < \cdots < i_r} \tilde{f}_{i_1 \dots i_r} dy_{i_1} \wedge \cdots \wedge dy_{i_r}.$$

We then define in the same way

$$d'\omega = \sum_{i_1 < \cdots < i_r} d\tilde{f}_{i_1 \dots i_r} \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_r}.$$

We now show that then $d = d'$ (on the intersection), using the properties (i)-(iv). Now (ii) and (iii) give

$$d\omega = d\Big(\sum_{i_1 < \cdots < i_r} \tilde{f}_{i_1 \dots i_r} dy_{i_1} \wedge \cdots \wedge dy_{i_r}\Big)$$

$$= \sum_{i_1 < \cdots < i_r} d\tilde{f}_{i_1 \dots i_r} \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_r} + \sum_{i_1 < \cdots < i_r} \tilde{f}_{i_1 \dots i_r} d(dy_{i_1} \wedge \cdots \wedge dy_{i_r})$$

and from (iii)

$$d(dy_{i_1} \wedge \cdots \wedge dy_{i_r}) = d(dy_{i_1}) \wedge \cdots \wedge dy_{i_r} - dy_{i_1} \wedge d(dy_{i_2} \wedge \cdots \wedge dy_{i_r}).$$

By (iv) the first summand is zero and continuing in the same way, the whole expression is seen to be zero. Thus

$$d\omega = \sum_{i_1 < \cdots < i_r} d\tilde{f}_{i_1 \dots i_r} \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_r} = d'\omega$$

is independent of the coordinate system chosen and hence globally well-defined. □

The operation d commutes with pullback:

Proposition 5.6. *If $f\colon M \to N$ is a smooth map of manifolds, and $\omega \in \mathscr{A}^r(N)$, then*

$$d(f^*\omega) = f^*(d\omega).$$

Proof. We know that $f_*\colon T_p(M) \to T_{f(p)}(N)$ satisfies $f_*(X_p)(\varphi) = X_p(\varphi \circ f)$ (where φ is a smooth function defined locally around $f(p)$). Therefore the dual map

$$(f_*)^*\colon T_{f(p)}(N)^* \to T_p(M)^*$$

satisfies

$$(f_*)^*\left((d\varphi)_{f(p)}\right) = d(\varphi \circ f)_p$$

(you should recall the definition of the differential of a function $d\psi_p$: $\langle (d\psi)_p, X_p \rangle = X_p(\psi)$). Thus

$$f^*(d\varphi) = d(\varphi \circ f) = d(f^*\varphi)$$

so our claim holds for 0-forms (functions) on N. Now in general in terms of local coordinates $y_1, \dots, y_r$ on N

$$\omega = \sum_{i_1 < \dots < i_r} g_{i_1 \dots i_r} dy_{i_1} \wedge \dots \wedge dy_{i_r}$$

whence

$$f^*\omega = \sum_{i_1 < \dots < i_r} f^*(g_{i_1 \dots i_r}) f^* dy_{i_1} \wedge \dots \wedge f^* dy_{i_r}$$

and therefore

$$\begin{aligned} d(f^*\omega) &= \sum_{i_1 < \dots < i_r} d(f^*(g_{i_1 \dots i_r})) \wedge f^* dy_{i_1} \wedge \dots \wedge f^* dy_{i_r} \\ &= \sum_{i_1 < \dots < i_r} f^* d(g_{i_1 \dots i_r}) \wedge f^* dy_{i_1} \wedge \dots \wedge f^* dy_{i_r} \\ &= f^*(d\omega). \end{aligned}$$

□

Let us work out what d does on $\mathbb{R}^3$; we will re-encounter some old friends (and if not friends, at least acquaintances) in this way.

Example 5.7. If f is a function on $\mathbb{R}^3$ (with coordinates x_1, x_2, x_3), then

$$df = g_1 dx_1 + g_2 dx_2 + g_3 dx_3$$

where

$$(g_1, g_2, g_3) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) = \mathrm{grad}\,(f)$$

the *gradient* vector field of f.

If

$$\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

is a 1-form, then

$$\begin{aligned} d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2 \end{aligned}$$

where

$$g_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3},\ g_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1},\ g_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

Thus if F and G are the vector fields (f_1, f_2, f_3) and (g_1, g_2, g_3), then $G = \mathrm{curl}\, F$.

For a 2-form

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

we get

$$\begin{aligned} d\omega &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 \\ &= (\mathrm{div}\,(F)) dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Of course d of any 3-form is zero, and all r-forms, $r > 3$ are zero on $\mathbb{R}^3$.

From $d^2 = 0$ we thus get

$$\mathrm{curl}\,(\mathrm{grad}\, f) = 0, \quad \mathrm{div}(\mathrm{curl}\, F) = 0$$

which is well-known to you (but $d^2 = 0$ is certainly easier to remember than those two formulas!)

It turns out that the connection between what you already know from vector calculus and differential forms can be carried much further: the classical theorems of Stokes and Green and Gauss in vector calculus are special cases of a single result in the theory of differential forms, which is also called "Stokes' theorem" (in the present form it was formulated in 1945 by Élie Cartan following earlier work by Volterra, Goursat and Poincaré on generalisations of the classical integral theorems of vector calculus. Apparently this is an example of Arnold's principle (named after the famous Russian mathematician V.I. Arnold): If a famous result or notion bears a personal name, then this name is not the name of the discoverer).

We first prove a simple version of the result and the full version afterwards.

Theorem 5.8. *Let M be an oriented n-dimensional manifold (i.e., M is orientable and we have chosen one of the two possible orientations). If $\omega \in \mathscr{A}^{n-1}(M)$ is a differential form with compact support, then*

$$\int_M d\omega = 0.$$

Proof. Choose a coordinate covering of M (with positive Jacobian determinant of the transition functions on intersections) and a partition of unity $\{\theta_i\}$ subordinate to this covering to write

$$\omega = \sum_i \theta_i \omega.$$

Then in a chart we can write

$$\theta_i \omega = f_1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n - f_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \dots$$

whence

$$d(\theta_i \omega) = \left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} \right) dx_1 \wedge \cdots \wedge dx_n.$$

From the definition of the integral over M, we simply need to sum all contributions

$$\int_{\mathbb{R}^n} \left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} \right) dx_1\, dx_2 \dots dx_n.$$

Now consider

$$\int_{\mathbb{R}^n} \frac{\partial f_1}{\partial x_1} dx_1\, dx_2 \dots dx_n.$$

Using Fubini's theorem to evaluate this as a repeated integral we get that this equals

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \frac{\partial f_1}{\partial x_1} dx_1 \Big) dx_2 dx_3 \dots dx_n.$$

But since f_1 has compact support, it vanishes for $|x_1| \geq N$, N large, so

$$\int_{\mathbb{R}} \frac{\partial f_1}{\partial x_1} dx_1 = [f_1]_{-N}^{N} = 0.$$

Continuing in the same way, we see that the entire integral is thus zero. □

As you will recall from vector calculus, Green's theorem relates an area integral (in the plane) to a line integral (and so does the classical Stokes' theorem for surfaces in 3-space), and Gauss's theorem relates a surface integral to a volume integral. To extend such theorems to the manifold context, we thus have to enlarge the class of objects under study to include spaces like a closed ball in $\mathbb{R}^{n+1}$ (with boundary the sphere S^n), and the cylinder $S^1 \times [0,1]$ in $\mathbb{R}^3$ which has boundary two copies of the circle. We will call these *manifolds with boundary* and we need to integrate over them.

Definition 5.9. Define the upper half-space H^n in $\mathbb{R}^n$ as a set by

$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

and endow it with the induced topology from $\mathbb{R}^n$. For $U \subset H^n$ open, we call $f\colon U \to \mathbb{R}^n$ smooth if it is locally around every point of its domain the restriction of a smooth function defined on some open set of $\mathbb{R}^n$. A local diffeomorphism on H^n is a smooth map $f\colon U \to V$ with $U, V \subset H^n$ open, that has a smooth inverse. The local diffeomorphisms form a pseudogroup $\Gamma_{H^n, diff}$ on H^n.

A *manifold with boundary* M (of dimension n) is nothing but a $\Gamma_{H^n, diff}$-manifold in the sense of Definition 1.5. More concretely, for the convenience of the reader, we recall that such a structure is determined by an atlas $\{(U_\alpha, \varphi_\alpha\}$ on M that has the following properties:

a) $M = \bigcup_\alpha U_\alpha$;

b) $\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subset H^n$ is a homeomorphisms between U_α and an open subset $\varphi_\alpha(U_\alpha)$ of H^n.

c) $\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map between open subsets of H^n for all α, β (in the sense defined above: it is the restriction of a $\mathscr{C}^\infty$-map from a neighbourhood in $\mathbb{R}^n$ of $\varphi_\alpha(U_\alpha \cap U_\beta) \subset H^n \subset \mathbb{R}^n$ to $\mathbb{R}^n$).

We define the *boundary* of M by

$$\partial\, M = \{p \in M \colon \varphi_\alpha(p) \in \{(x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid x_n = 0\} \text{ for some chart } (U_\alpha, \varphi_\alpha)\}\,.$$

It is immediate to check that this is independent of the chart, and that $\{(U_\alpha\mid_{\partial M}, \varphi_\alpha\mid_{\partial M})\}$ defines a smooth atlas on ∂M making it into a smooth manifold of dimension $n-1$ (not a manifold with boundary! we mean an actual manifold).

The following are some obvious examples of manifolds with boundary.

(1) H^n itself, with boundary $\partial H^n = \{x_n = 0\}$.
(2) The closed unit ball $\{x \in \mathbb{R}^n \mid ||x|| = 1\}$ with boundary S^{n-1}.
(3) The cylinder $[0,1] \times S^1$ with boundary two copies of S^1.

We can now define differential forms on manifolds with boundary in a straightforward way. Namely, in some local chart, they are just restrictions of smooth forms on some open subset of $\mathbb{R}^n$ to H^n. A differential form on M is then simply given by a collection of differential forms in the local charts of some atlas, transforming correctly under the transitions functions from one chart to the other. Moreover, a form on M restricts to a form on the boundary ∂M. We also have the notions of an orientable manifold with boundary (one which admits a nowhere vanishing global differential form of top degree), and an oriented manifold with boundary in complete analogy to the situation for manifolds.

[If a student feels uncomfortable with the few words we said here, they should work out more details for themselves, but we also want to warn them that nothing but boredom awaits them in that enterprise.]

Proposition 5.10. *If a manifold with boundary M is orientable, then ∂M is orientable.*

Proof. If $\dim M = 1$, $\partial M = 0$ and consists of points, thus the assertion is trivial. So let $\dim M \geq 2$ in what follows. Then the same argument as in the proof of Proposition 5.3 shows that we can choose a local coordinate systems $x_1, \dots, x_n$ on M such that ∂M is defined by $x_n = 0$ locally and on intersections with other charts $\det(\partial y_i / \partial x_j) > 0$. Where two local charts, with coordinates $x_1, \dots, x_n$ and

$y_1, \dots, y_n$ overlap

$$y_i(x_1, \dots, x_n), \quad y_n(x_1, \dots, x_{n-1}, 0) = 0,$$

so the Jacobian matrix at a point of the boundary has the form

$$\begin{pmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \dots & \partial y_1/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \partial y_n/\partial x_n \end{pmatrix}.$$

Now $\varphi_\beta \varphi_\alpha^{-1}$ maps $x_n > 0$ to $y_n > 0$, thus if $x_n = 0$, then $y_n = 0$, and if $x_n > 0$, $y_n > 0$. Therefore

$$\frac{\partial y_n}{\partial x_n} |_{x_n=0} > 0$$

and since the determinant of the preceding matrix is positive, the determinant of the upper left $(n-1) \times (n-1)$-matrix is positive on $x_n = 0$. Thus the $(U_\alpha |_{\partial M}, \varphi_\alpha |_{\partial M})$ give an atlas for ∂M whose transition functions have everywhere positive Jacobian, so ∂M is orientable. □

Once we have chosen a specific orientation for M, we will always choose a certain orientation of ∂M (one of the two possible ones) that is compatible with the orientation of M in a way we now make precise. For surfaces in $\mathbb{R}^3$, for example, one can think of the convention we are about to make as the choice of an outward or inward normal.

Definition 5.11. Suppose M is an orientable manifold with boundary for which we have chosen a specific orientation, represented by a nowhere vanishing n-form ω. If in local coordinates around a boundary point

$$\omega \simeq \epsilon \, dx_1 \wedge \dots \wedge dx_n$$

where $\epsilon = \pm 1$ and with $x_n \geq 0$ defining M locally, then we define the *induced orientation* ω' of ∂M locally by

$$(-1)^n \epsilon \, dx_1 \wedge \dots \wedge dx_{n-1}.$$

Example 5.12. It is worthwhile to think about what this means, especially the funny sign $(-1)^n$; for example, if we take the closed interval $M = [0, 1] \subset \mathbb{R}$ with coordinate x, we see that $\omega = dx$ is a nowhere vanishing 1-form on $[0, 1]$, giving it an orientation. At 0, the local coordinate $x_1 = x$ is satisfied that M is locally given by $x_1 \geq 0$ and $\omega \simeq dx_1$, so the point 0 gets the induced orientation -1 (since $n = 1$). Note that a choice of orientation for a point is simply the choice of $+1$ or -1 in $\mathbb{R}$.

At the point 1, instead we can use as local coordinate $y_1 = 1 - x$, whence locally around 1, M is given by $y_1 \geq 0$, and now $\omega \simeq -dy_1$ whence the point 0 gets the induced orientation $+1$. Note that this already indicates that our choice of orientation of the boundary is well-suited for generalising the fundamental theorem of differential and integral calculus: indeed,

$$\int_0^1 f'(x)dx = f(1) - f(0)$$

and this can be interpreted as saying that integrating the differential of a function over M is the same as "integrating the function over the boundary", which in this case is just evaluating the function at the points of the boundary ∂M, with the boundary orientation taken into account: $f(1)$ enters with a plus-sign, and $f(0)$ with a minus-sign.

More generally, if we consider $H^n \subset \mathbb{R}^n$ with the standard basis $e_1, \dots, e_n$ in $\mathbb{R}^n$ and dual basis $dx_1, \dots, dx_n$, we can give H^n an orientation by taking the nowhere vanishing n-form $\omega = dx_1 \wedge \dots \wedge dx_n$. We call an ordered basis $(v_1, \dots, v_n)$ in $\mathbb{R}^n$ positively oriented (with respect to ω) if

$$\omega(v_1 \wedge \dots \wedge v_n) > 0.$$

The ω' equivalent to ω are precisely those having positive values on a positively oriented basis, and so the choice of a positively oriented basis in $\mathbb{R}^n$ is equivalent to the choice of an orientation. The induced orientation on $\partial H^n = \mathbb{R}^{n-1} = \langle e_1, \dots, e_{n-1} \rangle$ in this case is given by $\omega_{\partial H^n} = (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$: this means that a basis $(w_1, \dots, w_{n-1})$ of $\partial H^n = \mathbb{R}^{n-1}$ is positively oriented with respect to $\omega_{\partial H^n}$ if and only if

$$(n,\ w_1, \dots, w_{n-1})$$

is a positively oriented basis with respect to ω in $\mathbb{R}^n$, where n is the *outward normal* $n = -e_n$ to H^n at $0 \in \mathbb{R}^n$.

We can now prove the full version of Stokes' theorem; in part its power lies in the fact that it establishes a connection between the analytical operations $\int$ and d and the geometric operation ∂ of forming the (oriented) boundary of a manifold with boundary.

Theorem 5.13 (Stokes' Theorem, which, as you will recall, is not entirely due to Stokes...)**.** *Let M be an n-dimensional oriented manifold with boundary ∂M carrying the induced orientation. Let $\omega \in \mathscr{A}^{n-1}(M)$ be an $(n-1)$-form of compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega \mid_{\partial M} .$$

Proof. We need to start with a word about how the integral $\int_M d\omega$ over the oriented manifold M with boundary is defined; we completely mimic the procedure for manifolds without boundary: if φ is the orientation form on M, as in the proof of Proposition 5.3, we can write

$$\varphi = \varphi(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n$$

in local charts, and *provided φ is positive in every chart* get a cover by charts of this type such that the Jacobian of the transition functions is positive on overlaps. We can then give a well-posed definition of the integral just as before: choose a partition $\{\theta_i\}$ subordinate to such a cover and repeat the construction described after the proof of Proposition 5.3. However, if you have been super vigilant, there is one small catch here: to make φ positive in each chart, we may have to replace x_1 by the local coordinate $-x_1$, and if $n = 1$, we cannot do that at a boundary point because it would mess up the requirement that M be locally defined by $x_n \geq 0$ at a boundary point! Thus if the dimension of M is 1 (and only then) are we seemingly in trouble and cannot proceed in complete analogy with what we did before. It is easy to sort out this small inconvenience and we leave it to the student to do this as a nice and easy exercise to test his understanding of the material (maybe the easiest fix is to allow local models for manifolds with boundary M that are either open sets in the upper half space $H^n = H^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ *or* the lower half-space $H^n_- = \{x \in \mathbb{R}^n \mid x_n \leq 0\}$; then one can always cover M by such charts with positive Jacobian on overlaps and $dx_1 \wedge \dots \wedge dx_n$ equivalent to the chosen orientation of M in each chart). In any case, for $n = 1$, Theorem 5.13 is nothing but the one-dimensional calculus assertion

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

so we proceed undeterred and give a proof now that works for all $n \geq 2$ without the need for modification.

As in the proof of Theorem 5.8, we write again $\omega = \sum_i \theta_i \omega$ whence

$$\int_M d\omega = \sum_i \int_M d(\theta_i \omega)$$

and write

$$\theta_i\omega = f_1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n - f_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^{n-1} f_n dx_1 \wedge \cdots \wedge dx_{n-1}.$$

There are now two types of open sets among the coordinate charts in use here: those that do not intersect ∂M, and those which do intersect it. For those which do not intersect it, the contribution to the integral is zero by what was shown in the proof of Theorem 5.8. For those which do, we have

$$\begin{aligned}
\int_M d(\theta_i\omega) &= \int_{x_n \geq 0} \left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} \right) dx_1\, dx_2 \ldots dx_n \\
&= \int_{\mathbb{R}^{n-1}} [f_n]_0^\infty dx_1 \ldots dx_{n-1} \\
&= -\int_{\mathbb{R}^{n-1}} f_n(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1} \\
&= \int_{\partial M} \theta_i \omega \mid_{\partial M}
\end{aligned}$$

where the last equality follows because

$$\theta_i \omega \mid_{\partial M} = (-1)^{n-1} f_n dx_1 \wedge \cdots \wedge dx_{n-1}$$

and we use the induced orientation

$$(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}.$$

□

If our endeavour here is likened to a mountain hike, you should not view Stokes' Theorem as a peak or pinnacle that affords a beautiful view of the scenery and everything that surrounds it, but rather as a solid pair of mountain boots. In other words, we are after geometric applications. Here is a first one, we will give many more in the next Section.

Theorem 5.14 (Brouwer's fixed point theorem)**.** *Let B be the closed unit ball in $\mathbb{R}^n$ and $f \colon B \to B$ a smooth map from B to itself. Then f has a fixed point.*

Proof. Arguing by contradiction, suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B$. Extend the segment $\overline{f(x)x}$ to a half ray starting from $f(x)$ and intersecting the boundary S^{n-1} of B in $g(x)$. Then

$$g \colon B \to \partial B$$

is a smooth map with the property that $g(x) = x$ for $x \in \partial B$. Now let ω be a nowhere vanishing $(n-1)$-form on $\partial B = S^{n-1}$ (we have seen above that such an ω exists), normalised so that

$$\int_{\partial B} \omega = 1.$$

Then

$$1 = \int_{\partial B} \omega = \int_{\partial B} g^* \omega$$

since g is the identity on ∂B. By Stokes' Theorem

$$\int_{\partial B} g^* \omega = \int_B d(g^* \omega) = \int_B g^*(d\omega) = 0$$

using Proposition 5.6 and the fact that $d\omega = 0$ as ω is a form in dimension $n-1$ on S^{n-1}. This is a contradiction (1=0) and hence f must have a fixed point. $\square$

6. Applications of Stokes' theorem, de Rham cohomology

From the point of view of geometric imagination, it is intuitively clear that a sphere, the surface of a solid ball, has a hole, the hollow in its interior; and likewise the surface of a rubber tyre (or doughnut), a torus, has another kind of hole, or even two holes, the space where the air of the tyre is, and the space you can put your arm through if you remove the bike spokes. These are certainly different, though, from the type of holes in a plane with two punctures. On the other hand, $\mathbb{R}^n$ itself apparently does not have any holes. This can be made precise and quantified by certain objects, called *(de Rham) cohomology groups*. In combination with Stokes' Theorem these will lead us to a lot of interesting geometric insights about different types of manifolds.

Definition 6.1. Let M be a manifold. An r-form $\alpha \in \mathscr{A}^r(M)$ is said to be *closed* if $d\alpha = 0$, and *exact* if there exists an $(r-1)$-form β such that $d\beta = \alpha$. We denote the $\mathbb{R}$-vector subspace of $\mathscr{A}^r(M)$ consisting of the closed forms by

$$F^r(M)$$

(the "F" is for "fermée", paying homage to our good friends and neighbours the French), and the $\mathbb{R}$-vector subspace of $\mathscr{A}^r(M)$ consisting of the exact forms by

$$B^r(M)$$

(since elements in the image of d are also called boundaries and d the boundary operator, the "B" stands for "bords", paying homage to you can guess who). The quotient vector space

$$H^r(M) = F^r(M)/B^r(M)$$

is called the r-th *de Rham cohomology group* of M; note that since $d^2 = 0$, $B^r(M)$ is a subspace of $F^r(M)$.

Actually $H^r(M)$ is a real vector space, not just an additive group, but people usually refer to it as a de Rham cohomology *group* since there are a lot of other instances in mathematics where similar cohomology groups make their appearance and in these other situations they are frequently really just abelian groups without a vector space structure.

It is easy to figure out what $H^0(M)$ signifies: indeed, this is the space of smooth functions f on M satisfying $df = 0$; on each connected component M_i of M (an open submanifold), such an f must be constant because a function on an open ball in $\mathbb{R}^n$ with vanishing gradient is constant, hence the subset of points $p \in M_i$ where $f(p) = c$ is both open and closed, whence equal to M_i. Hence if M is connected, $H^0(M) = \mathbb{R}$, and if M is a manifold with k connected components, $H^0(M) = \mathbb{R}^k$ (of course there could also be infinitely many connected components, and then $H^0(M)$ is the free $\mathbb{R}$-vector space on the components of M).

The next result summarises some basic formal and functorial properties of the de Rham groups that are indispensable to get the theory off the ground.

Theorem 6.2. *Let M be a manifold of dimension n. Then*

(i) *For $r > n$ we have $H^r(M) = 0$.*

(ii) *The totality*

$$H^*(M) = \bigoplus_r H^r(M)$$

is an $\mathbb{R}$-algebra with product between $a = [\alpha] \in H^r(M)$ and $b = [\beta] \in H^s(M)$ (where we denote by $[\alpha]$ is the equivalence class of the closed r-form α etc.):

$$ab = [\alpha \wedge \beta].$$

The product on $H^*(M)$ thus defined is not commutative, but what is sometimes called graded commutative *meaning*

$$ab = (-1)^{rs} ba.$$

(iii) *If $f\colon M \to N$ is a smooth map of manifolds, pull-back of differential forms induces an $\mathbb{R}$-linear map*

$$f^*\colon H^*(N) \to H^*(M)$$

compatible with the product (in the sense that $f^*(cd) = f^*(c)f^*(d)$; moreover, this operation is functorial in the sense that id_M^* is the identity on $H^*(M)$ and for two smooth maps

$$M \xrightarrow{f} N \xrightarrow{g} P$$

we have $f^* \circ g^* = (g \circ f)^*$. In particular, if M and N are diffeomorphic via f, $f^*\colon H^r(N) \to H^r(M)$ is an isomorphism for all r.

Proof. Property (i) follows since there are no nonzero r-forms on a manifold M for $r > n = \dim M$.

For (ii), everything follows immediately from properties of the exterior product of forms, but we need to check that

$$ab = [\alpha \wedge \beta]$$

is well-defined and really does define a cohomology class; in other words, what we have to check is that $\alpha \wedge \beta$ is closed and that its cohomology class does not change if we change α or β by an exact form. For the first part, if α and β are closed

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta = 0$$

so $\alpha \wedge \beta$ is closed as well. For the second part, suppose we put $\alpha' = \alpha + d\gamma$. Then

$$\alpha' \wedge \beta = (\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta)$$

since $d\beta = 0$, hence the cohomology class of $\alpha' \wedge \beta$ is the same as that of $\alpha \wedge \beta$.

Finally, for (iii), one only needs to check that pullback of differential forms really descends to cohomology groups; we need that the pull-back of a closed form α is closed. This follows from

$$d(f^*\alpha) = f^*(d\alpha) = 0.$$

And we need that if we change α by an exact form, replacing it by $\alpha' = \alpha + d\gamma$, the pullback $f^*\alpha'$ still represents the same cohomology class as $f^*\alpha$: again, this follows from the compatibility of f^* and d

$$f^*(\alpha + d\gamma) = f^*\alpha + d(f^*\gamma).$$

Since $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$, the operation f^* respects the product we defined. □

We now come to a result that has more geometric content and says that the induced maps f_t^* on cohomology do not vary if $f_t\colon M \to N$ is a family of smooth maps, smoothly varying with a parameter t. This is sometimes referred to as homotopy invariance of de Rham cohomology. Precisely, it means the following.

Theorem 6.3. *Let $f\colon M \times [0,1] \to N$ be a smooth map (by which of course we mean it is the restriction of some smooth map $M \times (-\epsilon, 1+\epsilon) \to N$ for some small $\epsilon > 0$). Put $f_t(x) = f(x,t)$. The for the induced maps $f_t^*\colon H^r(N) \to H^r(M)$ we have*

$$f_0^* = f_1^*.$$

Proof. We start with a class $a \in H^r(N)$ and represent it by a closed r-form α; let us consider $f^*\alpha$, an r-form on $M \times [0,1]$. We can write uniquely

$$f^*\alpha = \beta + dt \wedge \gamma$$

with β an r-form on M, depending smoothly on t, and γ an $(r-1)$-form on M, depending smoothly on t. Here β is nothing but $f_t^*\alpha$. Denote by d_M the exterior derivative on M. Then from the preceding displayed formula we get

$$0 = d_M\beta + dt \wedge \frac{\partial\beta}{\partial t} - dt \wedge d_M\gamma$$

because α is closed. Therefore we must have

$$\frac{\partial\beta}{\partial t} = d_M\gamma.$$

Since also

$$\frac{\partial}{\partial t} f_t^*\alpha = \frac{\partial\beta}{\partial t}$$

integration with respect to t yields

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 \frac{\partial}{\partial t} f_t^*\alpha = d_M \int_0^1 \gamma\, dt.$$

Thus the cohomology class of $f_1^*\alpha$ is the same as that of $f_0^*\alpha$ since their difference is exact on M. □

As an application we immediately get

Corollary 6.4. *For $M = \mathbb{R}^n$ we have*

$$H^0(\mathbb{R}^n) = \mathbb{R}, \quad H^r(\mathbb{R}^n) = 0 \textit{ for } r > 0.$$

Proof. Since $\mathbb{R}^n$ is connected, $H^0(\mathbb{R}^n) = \mathbb{R}$. Now consider the map

$$f\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

given by $f(x,t) = tx$. Then f_1 is the identity, and f_0 is the constant map to the origin $0 \in \mathbb{R}^n$. In particular, any r-form for $r > 0$ pulls back to 0 via f_0 because its derivative vanishes identically. Therefore Theorem 6.3 implies $H^r(\mathbb{R}^n) = 0$ for $r > 0$. □

Call a region Ω in $\mathbb{R}^n$ *star-shaped* (with respect to a point $p \in \Omega$) if for each $x \in \Omega$ the line segment $\overline{px}$ joining p and x lies entirely in Ω. Then one can use the same argument as in the proof of Corollary 6.4 to show

$$H^0(\Omega) = \mathbb{R}, \quad H^r(\Omega) = 0 \text{ for } r > 0$$

writing down a smooth family $f_t\colon \Omega \to \Omega$ with f_0 mapping everything to p and f_1 the identity (write down the details as an exercise if you have trouble seeing it). This result is usually called the *Poincaré lemma*.

Moreover, the same type of argument applied to the family of maps

$$f_t\colon M \times \mathbb{R}^n \to M \times \mathbb{R}^n$$

with $f_t(a,x) = (a,tx)$ shows that for all manifolds M

$$H^r(M \times \mathbb{R}^n) \simeq H^r(M) \text{ for all } r.$$

Computing de Rham cohomology groups is usually not very easy, but we are now already in a position to calculate them for some "easy" manifolds, for example, spheres.

Theorem 6.5. *Let S^n be the n-sphere in $\mathbb{R}^{n+1}$. Then*

$$H^0(S^n) = H^n(S^n) = \mathbb{R} \text{ and } H^r(S^n) = 0 \text{ otherwise.}$$

Proof. We start with the case $n = 1$ and then make an induction.

Thus consider $S^1 \simeq \mathbb{R}^1/\mathbb{Z}$ first. The 1-form dx on $\mathbb{R}$ is translation-invariant, hence descends to a nowhere vanishing (closed) 1-form, also denoted dx, on $\mathbb{R}/\mathbb{Z}$. This dx is not the differential of a function f on S^1 because such a function would have to have a minimum on the compact S^1 and at such a point we would have $df = 0$, but dx is nowhere vanishing. Hence

$$H^1(\mathbb{R}/\mathbb{Z}) \neq 0.$$

Now suppose $g(x)dx$ is any 1-form on S^1 (automatically closed); here g can be viewed as a periodic function on $\mathbb{R}$: $g(x+1) = g(x)$. If we want to write $\alpha = df$ on S^1 we must have $f'(x) = g(x)$ on $\mathbb{R}$ whence we can put

$$f(x) = \int_0^x g(s)\,ds.$$

But we need f periodic $f(x+1) = f(x)$ for it to descend to S^1 whence we must have

$$\int_0^1 g(s)\,ds = 0.$$

Thus if $\alpha = g(x)\,dx$ with g some smooth periodic function on $\mathbb{R}$ (not necessarily integral 0 over $[0,1]$) we can write

$$\alpha = g(x)\,dx = \left(\int_0^1 g(s)\,ds\right)\,dx + df$$

because $\tilde{g} = g - \int_0^1 g(s)\,ds$ has integral zero over $[0,1]$, and we can put $f = \int_0^x \tilde{g}(s)\,ds$. Thus any 1-form on S^1 is of the form

$$c\,dx + df$$

with $c \in \mathbb{R}$ some constant and f a function on S^1, whence $H^1(S^1) = \mathbb{R}$.

Now we assume by induction that the statement of the Theorem holds for all spheres S^j for j up to dimension $n-1$ and we prove it for S^n. As S^n is connected, $H^0(S^n) = \mathbb{R}$. We now prove first that $H^p(S^n) = 0$ for all $1 < p < n$. For this let α be a closed p-form on S^n representing some cohomology class in $H^p(S^n)$. The type of argument we will employ now has been formalised in the form of the *Mayer-Vietoris exact sequence* in topology, but we give a proof without the technical baggage. Let U and V be the complements in S^n of small closed balls in $\mathbb{R}^{n+1}$ around the North and South pole of S^n. Then U and V are diffeomorphic to open balls in $\mathbb{R}^n$ via stereographic projection, which are star-shaped with respect to the center, hence they have trivial higher de Rham cohomology. Thus the restriction of α to U and V is exact:

$$\alpha = du \text{ on } U \text{ and } \alpha = dv \text{ on } V.$$

On $U \cap V$, $u - v$ is closed since $d(u-v) = \alpha - \alpha = 0$ on the intersection. Since

$$U \cap V \simeq S^{n-1} \times \mathbb{R}$$

and $H^{p-1}(S^{n-1} \times \mathbb{R}) = H^{p-1}(S^{n-1}) = 0$ by induction, $u - v$ is exact on $U \cap V$:

$$u - v = dw \text{ on } U \cap V.$$

Now $U \cap V$ is also diffeomorphic to $S^{n-1} \times (-2,2)$ and we can choose a smooth function φ that is equal to 1 on $S^{n-1} \times (-1,1)$ and has support in $S^{n-1} \times (-2,2)$. Shrinking U and V slightly to get $U' \subset U$ and $V' \subset V$ such that $U' \cap V' = S^{n-1} \times (-1,1)$, we can extend φw by 0 outside of its support to get a globally defined $(p-2)$-form on S^n. Moreover we have the $(p-1)$-form u on U' and the $(p-1)$-form

$$v + d(\varphi w)$$

on V' that satisfy $u = v + dw = v + d(\varphi w)$ on $U' \cap V'$. Thus we can patch these together to get a $(p-1)$-form β on all of S^n with $\alpha = d\beta$ globally (because this holds on the open cover U', V' be construction). Thus the cohomology class of α is trivial and we have shown

$$H^p(S^n) = 0 \text{ for } 1 < p < n.$$

Now what about $p = 1$? Well, we can start the argument above as before and get a function $u - v$ on $U \cap V$ with $d(u-v) = 0$. Hence $u - v$ is a constant c since $U \cap V$ is connected for $n > 1$, which we can assume since the case $n = 1$ was the induction base. Then $d(v + c) = \alpha$ on V as well, and $v + c$ and u agree on $U \cap V$, hence patch together to give a globally defined function f on S^n with $df = \alpha$. Hence

$$H^1(S^n) = 0; \text{for } n > 1$$

as well.

For $p = n$, the argument gives us an $(n-1)$-form $u - v$ on $U \cap V$ defining a class in

$$H^{n-1}(S^{n-1} \times \mathbb{R}) \simeq H^{n-1}(S^{n-1}) = \mathbb{R}.$$

Taking an $(n-1)$-form ω on S^{n-1} with a nontrivial cohomology class (which we know exists by induction), we pull it back to

$$S^{n-1} \times (-2,2) \simeq U \cap V$$

via the projection of the left-hand side on the first factor. Since $H^{n-1}(S^{n-1}) = \mathbb{R}$ is known by induction, we can conclude that $H^{n-1}(S^{n-1} \times (-2,2))$ is then generated by (the pullback of) $[\omega]$ and thus

$$u - v = \lambda\,\omega + dw$$

for some $\lambda \in \mathbb{R}$ and a $(n-2)$-form w on $U \cap V$. If $\lambda = 0$, the argument above gives that the cohomology class of the n-form α is trivial, and we can conclude that $H^n(S^n)$ is at most one-dimensional: indeed, every α determines some $\lambda = \lambda(\alpha)$ such that we have an equation $u - v = \lambda\,\omega + dw$ on $U \cap V$ with some $(n-2)$-form w, and $(n-1)$-forms u, v on U and V with $\alpha = du$ on U and $\alpha = dv$ on V; if we choose different u' and v' with $\alpha = du'$ and $\alpha = dv'$ on U and V, then $d(u' - u) = 0$ (and $d(v' - v) = 0$) so we have changed u and v by some closed forms on U and V which are exact since $H^{n-1}(U) = 0$ and $H^{n-1}(V) = 0$ since they are star-shaped and $n > 1$. So then again

$$u' - v' = \lambda\omega + dw'$$

on $U \cap V$ for some different w'. Thus λ does not depend on the choice of u and v. Moreover,

$$\lambda\omega + dw = \lambda'\omega + dw'$$

can only hold if $\lambda = \lambda'$ since ω gives a nontrivial class on $U \cap V$. Also, $\lambda = \lambda(\alpha)$ is obviously $\mathbb{R}$-linear in α. For an exact form α, λ is zero, and if λ is zero, α must be exact. Thus we get an injection $H^n(S^n) \to \mathbb{R}$ associating to an α the number $\lambda(\alpha)$, which shows that $H^n(S^n)$ is at most one-dimensional.

Thus it only remains to show that $H^n(S^n)$ is exactly one-dimensional: this follows because integration $\int$ over S^n gives a well-defined linear form (by Stokes' Theorem)

$$\int_{S^n} : H^n(S^n) \to \mathbb{R}$$

and we know that, by the definition of the integral, integrating a nowhere vanishing form giving an orientation, say an everywhere positive such form, gives a nonzero (positive) result. □

As an anecdote we mention that Leopold Vietoris was an Austrian topologist who lived from 1891 to 2002 when he died aged 110; he was also an enthusiastic alpinist, but the story goes that his doctor told him to stop skiing once he got past 100. He also wrote a mathematical paper at age 103. This may prove that the theory of manifolds and the subject of topology (or mathematics as a whole) is conducive to long life!

Here is another very entertaining consequence of the previous calculation.

Theorem 6.6. *For $p, q \geq 1$, the sphere S^{p+q} is not diffeomorphic to a product $M \times N$ of manifolds M of dimension p and N of dimension q.*

Proof. First remark that if X is any non-orientable manifold, a product $X \times Y$, where Y is another manifold, can never be orientable: for if it were, $X \times Y$ contains an open submanifold diffeomorphic to $X \times \mathbb{R}^n$, which would also be orientable whence X itself would be orientable, contradiction. Therefore, in order to prove the Theorem, it suffices to derive a contradiction from the assumption that S^{p+q} is diffeomorphic to $M \times N$ with M and N orientable; we can also assume we have chosen some orientations for M and N and the product orientation for S^{p+q}.

Since $p, q \geq 1$, we know from Theorem 6.5 that $H^p(S^{p+q}) = H^p(M \times N) = 0$. Let

$$\pi \colon M \times N \to M$$

be the projection and $\omega \in \mathscr{A}^p(M)$ inducing the orientation of M. Clearly, for fixed $n \in N$ the map $\pi \colon M \times \{n\} \to M$ is an orientation-preserving diffeomorphism, so

$$0 < \int_M \omega = \int_{M \times \{n\}} \pi^* \omega \mid_{M \times \{n\}} .$$

Now since $d\omega = 0$ (ω is of top degree $\dim M = p$ on M), we have $d(\pi^*\omega) = \pi^*(d\omega) = 0$. Therefore, $H^p(M \times N) = 0$ implies the existence of a $(p-1)$-form α on $M \times N$ with $\pi^*\omega = d\alpha$. But then

$$\int_{M \times \{n\}} \pi^* \omega \mid_{M \times \{n\}} = \int_{M \times \{n\}} d\alpha \mid_{M \times \{n\}} = \int_{M \times \{n\}} d(\alpha \mid_{M \times \{n\}}) = 0$$

by Theorem 5.8, contradiction. □

Theorem 6.7. *Every (smooth) vector field on an even-dimensional sphere S^{2m} vanishes at some point of the sphere.*

Proof. We argue by contradiction and assume there was such a nowhere vanishing vector field. As S^{2m} is a submanifold of $\mathbb{R}^{2m+1}$ we can think of such a vector field as a smooth map

$$v \colon S^{2m} \to \mathbb{R}^{2m+1}$$

with $\langle x, v(x) \rangle = 0$ for all $x \in S^n$; let us also normalise such a v so that $||v(x)|| = 1$ for all x. Define

$$f_t \colon S^{2m} \to \mathbb{R}^{2m+1}, \quad f_t(x) = (\cos t)x + (\sin t)v(x).$$

Since $x, v(x)$ are unit length vectors and everywhere perpendicular, $f_t(x)$ is a vector of unit length everywhere as well. Thus f_t maps S^{2m} to itself. Now

$$f_0(x) = x, \quad f_\pi = -x$$

and using our standard orientation form on S^{2m} defined for $x_i \neq 0$ by

$$\omega = (-1)^i \frac{1}{x_i} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{2m+1}$$

we find

$$f_0^*\omega = \omega, \quad f_\pi^*\omega = -\omega.$$

But by Theorem 6.3, the two maps f_0^*, $f_1^* \colon H^{2m}(S^{2m}) \to H^{2m}(S^{2m})$ are equal; this is a contradiction since $[\omega]$ gives a nontrivial cohomology class which cannot be equal to its negative. Hence the vector field v has to have a zero. □

Sometimes this result is called the "Hairy Ball Theorem" since it says that one cannot comb a hairy ball such that each hair lies flat.

We will now calculate the de Rham cohomology groups of projective spaces and tori, which will acquaint us with further computational techniques for these groups.

Theorem 6.8. *The de Rham cohomology groups of real projective spaces*

$$H^r(\mathbb{P}^n_{\mathbb{R}})$$

are zero except for

$$H^0(\mathbb{P}^n_{\mathbb{R}}) \quad \textit{and} \quad H^{2m+1}(\mathbb{P}^{2m+1}_{\mathbb{R}})$$

which are isomorphic to $\mathbb{R}$.

Proof. We use Theorem 6.5 and the description of $\mathbb{P}^n_{\mathbb{R}}$ as the quotient

$$p\colon S^n \to \mathbb{P}^n_{\mathbb{R}} = S^n/\sigma$$

for the involution $\sigma\colon S^n \to S^n$ sending x to $-x$. First, since $\mathbb{P}^n_{\mathbb{R}}$ is the image of a connected manifold S^n, it is connected and $H^0(\mathbb{P}^n_{\mathbb{R}}) = \mathbb{R}$. Now for $0 < r < n$ we have $H^r(S^n) = 0$. Take a closed r-form $\alpha \in F^r(\mathbb{P}^n_{\mathbb{R}}) \subset \mathscr{A}^r(\mathbb{P}^n_{\mathbb{R}})$. Then since $d \circ p^* = p^* \circ d$, the pullback $\beta = p^*\alpha$ is closed whence there exists a $\gamma \in \mathscr{A}^{r-1}(S^n)$ with

$$\beta = d\gamma.$$

Now $\sigma^* \circ d = d \circ \sigma^*$, so $d(\sigma^*\gamma) == \sigma^*(d\gamma) = \sigma^*\beta = \beta$ whence

$$\beta = d\left(\frac{\gamma + \sigma^*\gamma}{2}\right).$$

But the form

$$\delta = \frac{\gamma + \sigma^*\gamma}{2}$$

is σ-invariant, hence descends to a form $\overline{\delta} \in \mathscr{A}^{r-1}(\mathbb{P}^n_{\mathbb{R}})$ with $p^*\overline{\delta} = \delta$ and $d\overline{\delta} = \alpha$ by construction. Thus for $0 < r < n$ we have $H^r(\mathbb{P}^n_{\mathbb{R}}) = 0$.

It remains the case $r = n$. In that case $\beta = p^*\alpha$ defines a class in $H^n(S^n) = \mathbb{R}$. Let ω be the standard n-form inducing the orientation of S^n that we used before. Then

$$\beta = k\omega + d\gamma$$

for some $k \in \mathbb{R}$ and $\gamma \in \mathscr{A}^{n-1}(S^n)$. We know $\sigma^*\beta = \beta$ and $\sigma^*\omega = (-1)^{n+1}\omega$, so

$$\beta = (-1)^{n+1}k\omega + d(\sigma^*\gamma)$$

and we can write

$$\beta = k\frac{1 + (-1)^{n+1})}{2}\omega + d\delta$$

with

$$\delta = \frac{\gamma + \sigma^*\gamma}{2}.$$

Then δ is invariant under σ. If n is even, $\beta = d\delta$, and δ descends to a form $\overline{\delta} \in \mathscr{A}^{n-1}(\mathbb{P}^n_{\mathbb{R}})$ with $\alpha = d\overline{\delta}$ by construction. Thus $H^n(\mathbb{P}^n_{\mathbb{R}}) = 0$ in this case.

If n is odd, $\beta = k\omega + d\delta$ with $\sigma^*\delta = \delta$. But also

$$\sigma^*\omega = \omega.$$

Thus there exist $\bar{\sigma}$ and $\bar{\delta}$ on $\mathbb{P}^n_{\mathbb{R}}$ with

$$\alpha = k\bar{\omega} + d\bar{\delta}.$$

Thus $\dim H^n(\mathbb{P}^n_{\mathbb{R}}) \leq 1$ in this case (every class is a multiple of $[\bar{\omega}]$), but since $\bar{\omega}$ is a volume form inducing an orientation in this case,

$$H^n(\mathbb{P}^n_{\mathbb{R}}) = \mathbb{R}.$$

□

The case of the complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$ is rather different: as a real manifold, $\mathbb{P}^n_{\mathbb{C}}$ has dimension $2n$ and it can be shown that $H^r(\mathbb{P}^n_{\mathbb{C}})$ is equal to $\mathbb{R}$ precisely if $0 \leq r \leq 2n$ and r is even, and otherwise these groups are zero. In fact, we mention here as a cultural remark that this and many other calulations become much simpler once one recognises that for most "nice" spaces M, such as manifolds, the groups $H^r(M)$ cannot only be calculated with the aid of differential forms, but via a number of different other methods! There is morally just one "cohomology" of the space M, and depending on how one initially defines it (e.g. via differential forms, or so-called singular co-chains, or...) one speaks of de Rham cohomology, singular cohomology, Alexander-Spanier cohomology, Čech cohomology, ... and the good news is they all essentially give the same answer! This can be proven using sheaf theory and sheaf cohomology. There is one further remark we would like to make since we will see some "shadow" of this phenomenon in the next Section: the cohomology $H^r(M)$ actually "comes from" an object over $\mathbb{Z}$ (i.e., with an integral structure), in the sense that one can define natural abelian groups (=$\mathbb{Z}$-modules) $H^r(M, \mathbb{Z})$ with $H^r(M) = H^r(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. One can use other "coefficients" than $\mathbb{Z}$ as well, but $\mathbb{Z}$ is in some sense a universal choice.

We finally compute the de Rham cohomology of tori.

Theorem 6.9. *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (where $(m_1, \dots, m_n) \in \mathbb{Z}^n$ acts on $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by $x \mapsto (x_1 + m_1, \dots, x_n + m_n)$). The translation-invariant 1-forms $dx_1, \dots, dx_n$ on $\mathbb{R}^n$ descend to 1-forms on $\mathbb{T}^n$ which we denote by the same letters. Then*

$$\dim H^r(\mathbb{T}^n) = \binom{n}{r}$$

and $H^r(\mathbb{T}^n)$ has a basis consisting of the classes of the r-form

$$dx_{i_1} \wedge \dots \wedge dx_{i_r}, \quad 1 \le i_1 < \dots < i_r \le n.$$

Proof. Let $\alpha \in F^r(\mathbb{T}^n)$ be a closed r-form. We will show the following:

(1) There are n natural commuting one-parameter groups of diffeomorphisms $G_t^{(i)}$, $i = 1, \dots, n$, acting on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: indeed,

$$G_t^{(i)} \colon [(x_1, \dots, x_n)] \mapsto [(x_1, \dots, x_i + t, \dots, x_n)],$$

so the action of $\prod_i G_t^{(i)}$ is just the action of $\mathbb{T}^n$ on itself by translations. We will define an *average* $\bar{\alpha}$ of α under all of these translations, and show that this average $\bar{\alpha}$ is closed, defines the same cohomology class as α, and is invariant under all translations.

(2) We show that r-forms invariant under all translations are combinations of the $dx_{i_1} \wedge \dots \wedge dx_{i_r}$ with constant coefficients. We denote these invariant forms by $\mathrm{Inv}^r(\mathbb{T}^n)$. We have $\mathrm{Inv}^r(\mathbb{T}^n) \subset F^r(\mathbb{T}^n)$.

(3) We know at that point that $\mathrm{Inv}^r(\mathbb{T}^n)$ and $B^r(\mathbb{T}^n)$ span $F^r(\mathbb{T}^n)$, so we show

$$\mathrm{Inv}^r(\mathbb{T}^n) \cap B^r(\mathbb{T}^n) = 0$$

to conclude that $\mathrm{Inv}^r(\mathbb{T}^n)$ maps isomorphically onto $H^r(\mathbb{T}^n)$ under the projection $F^r(\mathbb{T}^n) \to H^r(\mathbb{T}^n)$.

Now let us implement this program in more detail. For (1), let us start looking at $G_t := G_t^{(1)}$. We then define

$$\mu_1(\alpha) = \int_0^1 G_\theta^* \alpha \, d\theta.$$

Since G_t^* is linear and commutes with the integral sign, we get that $\mu_1(\alpha)$ is invariant under G_t, i.e. $G_t^*(\mu_1(\alpha)) = \mu_1(\alpha)$ for every t. Also $\mu_1(\alpha) \in F^r(\mathbb{T}^n)$ since

$$d(\mu_1(\alpha)) = d\left(\int_0^1 G_\theta^* \alpha \, d\theta\right) = \int_0^1 d(G_\theta^* \alpha) \, d\theta = \int_0^1 G_\theta^* d\alpha \, d\theta = 0.$$

Moreover, $\mu_1(\alpha)$ and α define the same class in $H^r(\mathbb{T}^n)$ since, following the method of proof for Theorem 6.3, we can write

$$G_t^*\alpha - G_0^*\alpha = G_t^*\alpha - \alpha = d\beta_t$$

with β some $(r-1)$-form depending smoothly on t. Hence

$$\mu_1(\alpha) = \int_0^1 G_\theta^*\alpha \, d\theta = \int_0^1 (\alpha + d\beta_\theta) \, d\theta = \alpha + \left(d \int_0^1 \beta_\theta \, d\theta \right)$$

and thus $\mu_1(\alpha)$ and α define the same class.

We can then employ these argument inductively to conclude that

$$\bar{\alpha} = \mu_n(\mu_{n-1}(\dots(\mu_2(\mu_1(\alpha)))\dots)),$$

where μ_i denotes the analogous average with respect to $G_t^{(i)}$, is invariant under all translations, closed and of the same class as α.

Now for (2), we suppose we are given an r-form γ that is invariant under all translations. We can write, using multi-index notation again,

$$\gamma = \sum_I a_I dx_I$$

(thus $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_r}$), with the a_I functions that, when we view them as functions on $\mathbb{R}^n$, are periodic for the lattice $\mathbb{Z}^n$. But since the dx_I are invariant under all translations, for γ to be invariant under all translations, it must be true that the a_I are invariant under all translations, i.e. they must be constants. In particular, such a γ is closed.

Now Step (3) can be achieved as follows: assume $\alpha \in \mathrm{Inv}^r(\mathbb{T}^n)$ and also $\alpha = d\beta$ for some β, so $\alpha \in B^r(\mathbb{T}^n)$. We want to show that $\alpha = 0$. Let $\overline{(-)}$ be the averaging operator introduced above. Then

$$\bar{\alpha} = \overline{d\beta} = d\bar{\beta}.$$

But $\bar{\beta}$ is invariant under all translations, hence $\bar{\beta}$ is closed. Hence $d\bar{\beta} = 0$, and thus $\bar{\alpha} = \alpha = 0$. $\square$

The idea used in the proof above can be used to compute the de Rham cohomology of several other manifolds acted on by some groups that are so-called compact Lie groups.

7. Degree theory and applications: fundamental theorem of algebra, linking numbers, indices of singularities of vector fields, the argument principle

In this Section we will show that the n-th de Rham group of an oriented, compact, connected n-dimensional manifold is canonically isomorphic to $\mathbb{R}$. This will allow us to associate to any smooth map $f\colon M \to N$ of oriented, compact, connected manifolds of the same dimension n a real number, called the degree of f, and denoted by $\deg(f)$, defined as the number such that

$$\int_M f^*\omega = \deg(f) \int_N \omega$$

for any n-form ω on N. The fecundity of the concept of degree lies in the surprising fact that we can arrive at the quantity $\deg(f)$ in an entirely different, geometric way: indeed, it is equal to the number of preimages, counted with appropriate signs, of a regular value of f; more precisely, an $a \in N$ is called a *regular value* for a smooth map $f\colon M \to N$ if the differential $(f_*)_p = df_p\colon T_p(M) \to T_a(N)$ is surjective for each $p \in M$ with $f(p) = a$ (if $\dim M = \dim N$ this means df_p is an isomorphism). Sard's Theorem tells us that, in a sense to be made precise, "most" points $a \in N$ are regular values (for example, imagine taking a random point, then this will be a regular value with probability 1). Then

$$\deg(f) = \sum_{p\in f^{-1}(a)} \operatorname{sgn}\left(\det(df_p)\right).$$

In particular, this shows that $\deg(f)$ is always an integer. We will give several applications of this afterwards.

We start with a crucial auxiliary result.

Lemma 7.1. *Let $Q^n = (-1,1)^n$ be the open unit cube in $\mathbb{R}^n$ and let $\omega \in \mathscr{A}^n(\mathbb{R}^n)$ with support in Q^n and*

$$\int_{Q^n} \omega = 0.$$

Then there exists a $\beta \in \mathscr{A}^{n-1}(\mathbb{R}^n)$ with support in Q^n and such that $\omega = d\beta$.

Remark that since Q^n is star-shaped, Poincaré's Lemma tells us that there exists a β with $\omega = d\beta$ (since $d\omega = 0$), but we do not know that such a β will have support in Q^n. In fact, just in case you are wondering why the Lemma is not just Poincaré's Lemma "$+\epsilon$", and why we have to work quite a bit to get it at this point, note that we do not have $\operatorname{supp}\beta \subset Q^n$ if $\int_{Q^n}\omega \neq 0$ (which explains the extra assumption): indeed, by Stokes' Theorem

$$\int_{\partial\overline{Q^n}} \beta = \int_{\overline{Q^n}} d\beta = \int_{\overline{Q^n}} \omega = \int_{Q^n} \omega$$

(this is actually not quite the generality we proved Stokes' Theorem in because $\partial\overline{Q^n}$ has "corners", but one can rather easily generalise it to this case), but of $\operatorname{supp}\beta \subset Q^n$, the first integral would be zero.

Proof. We will prove the statement by induction on the dimension n, but we will prove a slightly stronger assertion (which often simplifies proofs by induction because the you can assume more in the inductive step): namely we will prove the statement with ω and β depending smoothly on some additional parameter $\lambda \in \mathbb{R}^m$, and will also show that if ω vanishes identically for some value of λ, the same holds for β.

We start with $n = 1$. We can then write $\omega = f(x,\lambda)dx$. Setting

$$\beta(x,\lambda) = \int_{-1}^{x} f(t,\lambda)dt$$

defines a function with $d\beta = \omega$; there is a $\delta > 0$ such that f vanishes for $x > 1-\delta$ and $x < -1+\delta$ since $\operatorname{supp}\omega \subset (-1,1)$ and thus

$$\int_{-1}^{x} f(t,\lambda)dt = \int_{-1}^{1} f(t,\lambda)dt = 0$$

(by the hypothesis $\int_{Q^n}\omega = 0$) for $x > 1-\delta$, and similarly for $x < -1+\delta$. So β has support in $(-1,1)$ as well. Also if $f(x,\lambda) = 0$ for all x for some λ, then $\beta(x,\lambda)$ vanishes for all x, too, by its definition.

Now for the induction step, let us assume that the stronger statement we want to prove is true for all dimensions less than n, and we will prove it for n. Write

$$\omega = f(x_1,\dots,x_n,\lambda)dx_1 \wedge \dots \wedge dx_n.$$

To make the induction, let us write temporarily $x_n = t$ and consider

$$f(x_1, \dots, x_{n-1}, t, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1}$$

as a form on $\mathbb{R}^{n-1}$ depending smoothly on t, λ. If σ is a smooth function with support in Q^{n-1} such that

$$\int_{Q^{n-1}} \sigma dx_1 \wedge \cdots \wedge dx_{n-1} = 1$$

then for

$$g(t, \lambda) := \int_{Q^{n-1}} f(x_1, \dots, x_{n-1}, t, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1}$$

we have that the form

$$f(x_1, \dots, x_{n-1}, t, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} - g(t, \lambda) \sigma dx_1 \wedge \cdots \wedge dx_{n-1}$$

has support in Q^{n-1} and integral zero over Q^{n-1}, so by induction there exists a form $\gamma = \gamma(t, \lambda)$ with support in Q^{n-1} such that the previous form in the displayed formula is equal to $d\gamma$. Reinserting $x_n = t$ we compute

$$d(\gamma \wedge dx_n) = f(x_1, \dots, x_{n-1}, x_n, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n - g(x_n, \lambda) \sigma dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n.$$

If we define

$$\xi(x_1, \dots, x_n, \lambda) = (-1)^{n-1} \left(\int_{-1}^{x_n} g(t, \lambda) dt \right) \sigma dx_1 \wedge \cdots \wedge dx_{n-1}$$

we find

$$d\xi = g(x_n, \lambda) \sigma dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n$$

whence

$$f(x_1, \dots, x_{n-1}, x_n, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n = d(\gamma \wedge dx_n + \xi) =: d\beta.$$

Clearly

$$\beta = \gamma \wedge dx_n + \xi$$

also has support in $Q^{n-1} \times \mathbb{R}$ by induction (and the way it is constructed), so we only need to check what happens in the x_n-direction. Now we know that

$$f(x_1, \dots, x_{n-1}, t, \lambda)$$

vanishes for $t < 1 - \delta$, $t > -1 + \delta$ some small δ (since ω was assumed to have support in Q^n). Therefore, by our stronger inductive assumption (including the dependency on parameters), we get that the same is true for γ. Therefore all we have to check is what happens with ξ. We consider the case $x_n = t > 1 - \delta$, the case $x_n < -1 + \delta$ being similar. Then

$$\begin{aligned}
\int_{-1}^{t} g(s, \lambda) ds &= \int_{-1}^{t} \left(\int_{Q^{n-1}} f(x_1, \dots, x_{n-1}, s, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} \right) ds \\
&= \int_{-1}^{1} \left(\int_{Q^{n-1}} f(x_1, \dots, x_{n-1}, s, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} \right) ds \\
&= \int_{Q^n} f(x_1, \dots, x_{n-1}, x_n, \lambda) dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n \\
&= 0
\end{aligned}$$

by assumption. Thus also ξ has support in Q^n. To finish the inductive proof we only need to observe that if $f(x, \lambda)$ is identically zero for some λ, the same holds for β. This holds by construction if you run through the definitions above once more. $\square$

We are now in a position to show

Theorem 7.2. *If M is a compact, orientable, connected n-dimensional manifold, and also suppose we have chosen a fixed orientation; then integration*

$$\int_M : H^n(M) \to \mathbb{R}$$

gives a canonical isomorphism between $H^n(M)$ and $\mathbb{R}$.

Proof. We can cover M by finitely many charts $U_1, \dots, U_N$ such that each U_j is diffeomorphic to Q^n (here we use the compactness of M). Choose an n-form α_0 with support contained in U_1 and such that $\int_M \alpha_0 = 1$. Clearly, by Stokes' Theorem, the cohomology class $[\alpha_0]$ is nontrivial, and we want to show that every other class in $H^n(M)$ is a multiple of this one.

Now take an arbitrary n-form α on M. Take a partition of unity (θ_i) subordinate to the given cover; we can modify the (θ_i) given to us by Theorem 3.11 so that they are indexed by the same finite index set J as the cover $\{U_j\}$: indeed, put η_j equal to the sum of all θ_i with $j(i) = j$ and zero if no θ_i has $j(i) = j$. Then

$$\alpha = \sum_j \eta_j \alpha$$

and this is a finite sum. It suffices to show that each $\eta_j \alpha$ defines a cohomology class proportional to $[\alpha_0]$, so without loss of generality we can suppose that α has support in one of the coordinate neighbourhoods U_m, say. Since M is connected, we can connect any $p \in U_1$ and $q \in U_m$ by a path, and possibly by reordering the U_i and allowing some duplications, we can assume that the path is covered by $U_1, \dots, U_m$ such that $U_i \cap U_{i+1} \neq \emptyset$. Now choose an n-form α_i with support in $U_i \cap U_{i+1}$, $1 \le i \le m-1$, and integral 1. Then on U_1

$$\int (\alpha_0 - \alpha_1) = 0$$

so by Lemma 7.1, there is a form β_1 with support in U_1 with $\alpha_0 - \alpha_1 = d\beta_1$. Notice that we precisely need the assertion in Lemma 7.1 that β_1 has support in Q^n to be able to interpret β_1 as a form on all of M.

Similarly, we find forms $\beta_2, \dots, \beta_{m-1}$ satisfying the equations

$$\begin{aligned} \alpha_0 - \alpha_1 &= d\beta_1, \\ \alpha_1 - \alpha_2 &= d\beta_2, \\ &\dots \\ \alpha_{m-2} - \alpha_{m-1} &= d\beta_{m-1}. \end{aligned}$$

Therefore $\alpha_0 - \alpha_{m-1} = d(\sum_{i=1}^{m-1} \beta_i)$. If $\int \alpha = c \in \mathbb{R}$, then $\int (\alpha - c\alpha_{m-1}) = 0$ and we can apply Lemma 7.1 to write

$$\alpha - c\alpha_{m-1} = d\beta$$

and thus

$$\alpha = c\alpha_{m-1} + d\beta = c\alpha_0 + d(\beta - c\sum_{i=1}^{m-1} \beta_i).$$

□

Definition 7.3. Let $f\colon M \to N$ be a smooth map of manifolds. A point $p \in M$ is called a *regular point* for f if the differential

$$(f_*)_p \colon T_p(M) \to T_{f(p)}(N)$$

is surjective. If the differential fails to be surjective, p is called a *critical point* for f. A point $q \in N$ is called a *critical value* for f if it is the image under f of some critical point $p \in M$ for f. If $q \in N$ is not a critical value, it is called a *regular value* for f.

Notice some potential subtleties here: we do *not* define a regular value as the image under f of a regular point; for example, if a point q is not in the image of f at all, it is a regular value by definition. A point $q \in N$ is a regular value if *all* points in $f^{-1}(q)$ are regular points (and this preimage may be empty whence the requirement is vacuously satisfied), and a point $q \in N$ is a critical value if *some* point in $f^{-1}(q)$ is a critical point.

Theorem 7.4. *Let $f\colon M \to N$ be a smooth map between compact, oriented, connected manifolds of the same dimension n. Then there exists an integer* $\deg(f)$ *called the* degree of f *such that if $\omega \in \mathscr{A}^n(N)$, then*

$$\int_M f^*\omega = \deg(f) \int_N \omega$$

and if $q \in N$ is a regular value for f, then

$$\deg(f) = \sum_{p \in f^{-1}(q)} \operatorname{sgn}(\det(f_*)_p),$$

where $\operatorname{sgn}(\det(f_*)_p$ *is ± 1 and defined as follows: the fact that M and N are assumed to be oriented means that we have chosen orientations for M and N, i.e. equivalence classes of nowhere vanishing global n-forms ω_M and ω_N, which we can also assume to be normalised so that $\int_M \omega_M = \int_N \omega_N = 1$; the differential*

$$(f_*)_p\colon T_p(M) \to T_{f(p)}(N)$$

induces a map

$$\Lambda^n((f_*)_p)^*\colon \Lambda^n T_{f(p)}(N)^* \to \Lambda^n T_p(M)^*$$

whence we have $(\Lambda^n((f_*)_p)^*)(\omega_{N,f(p)}) = \lambda\omega_{M,p}$ for some $\lambda \in \mathbb{R}$. Then $\operatorname{sgn}(\det(f_*)_p)$ *is the sign of λ. More concretely, if we choose local coordinates $(U, x_1, \dots, x_n)$ around p and $(V, y_1, \dots, y_n)$ around $q = f(p)$ such that $(\omega_M)\mid_U$ is equivalent to $dx_1 \wedge \dots \wedge dx_n$ and $(\omega_N)\mid_V$ is equivalent to $dy_1 \wedge \dots \wedge dy_n$, and we write $f = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$ in these local coordinates, then* $\operatorname{sgn}(\det(f_*)_p)$ *is nothing but the sign of the determinant of the Jacobian matrix:*

$$\det\left(\frac{\partial f_i}{\partial x_j}\right).$$

Note that if f fails to be surjective, the theorem says that $\deg(f) = 0$.

Proof. If ω is any n-form on N, its cohomology class is a multiple of the cohomology class of ω_N, $[\omega] = c[\omega_N]$, $c \in \mathbb{R}$, by Theorem 7.2. Therefore (using Stokes' Theorem)

$$\int_N \omega = c.$$

On the other hand $f^*[\omega_N] = k[\omega_M]$ for some number $k \in \mathbb{R}$. So

$$\int_M f^*\omega = ck = k \int_N \omega.$$

We want to show that the number $k = \deg(f)$ is an integer and can be described geometrically as explained in the statement of the Theorem. If $q \in N$ is a regular value, then $(f_*)_p$ is an isomorphism at any point $p \in f^{-1}(q)$, hence by Proposition

2.4, $f^{-1}(q)$, if nonempty, is a submanifold of codimension n, thus dimension 0 of M. Since M is compact, $f^{-1}(q)$ is also compact (since it is closed in a compact space), hence a finite set of points. Proposition 3.5, c) implies that for a sufficiently small open neighbourhood U of q, $f^{-1}(U)$ decomposes as a disjoint union of finitely many opens $U_1, \dots, U_m$, each of which maps diffeomorphically onto U under f.

If σ is an n-form with support in U and $\int_N \sigma = 1$, then $\int_M f^*\sigma = k$. But we can compute this number in a different way now: we just have to sum the

$$\int_{U_i} (f\mid_{U_i})^*\sigma, \quad i = 1, \dots, m.$$

But the change of variables formula for multiple integrals/coordinate invariance of integrals of differential forms then implies

$$\int_{U_i} (f\mid_{U_i})^*\sigma = \mathrm{sgn}(\det(f_*)_{p_i}) \int_U \sigma = \mathrm{sgn}(\det(f_*)_{p_i})$$

if $p_i \in U_i$ is such that $f(p_i) = q$. □

Before giving applications of Theorem 7.4, we feel it is our moral duty to show you that there is always an ample supply of regular values. This is the following (special case of) *Sard's Theorem* (which we will never use in the sequel, however).

Theorem 7.5. *Let M and N be manifolds of the same dimension n. Let $f\colon M \to N$ be a smooth map. Then the set of critical values for f has measure zero in N.*

We will omit the proof because the techniques are of a different flavour than the ones we have used so far and we feel that it would take us too far afield here. If you are interested, proofs can be found in lots of texts on manifolds, for example in M. Berger, B. Gostiaux, Differential Geometry: Manifolds, Curves and Surfaces, Springer (1988), section 4.3.

We add a few comments, however, to explain Sard's result a little better. If M and N have different dimensions, $\dim M = m$, $\dim N = n$ say, it can still be proven that the critical values of f have measure zero in N, but the proof for $m > n$ is more difficult.

Secondly, we have to say a few words about what we mean when we say that a subset $S \subset N$ of a manifold N has "measure zero". We want to give a pedestrian definition of this concept without delving too deeply into measure theory. First, we agree that an n-cube $C \subset \mathbb{R}^n$ of egde length λ, by which we mean a product

$$C = I_1 \times \cdots \times I_n$$

of closed intervals $I_j = [a_j, a_j + \lambda]$ of length λ, has measure $\mu(C) = \mu_n(C) = \lambda^n$. We say that a subset $X \subset \mathbb{R}^n$ has measure zero if for every $\epsilon > 0$ it can be covered by a countable family of n-cubes the sum of whose measures is less than ϵ. Thus a countable union of sets of measure zero in $\mathbb{R}^n$ has measure zero. If $X \subset N$ is a subset of some manifold N, we say that X has measure zero if for every chart (U, φ) the set $\varphi(X \cap U) \subset \mathbb{R}^n$ has measure zero. In fact, it turns out that it suffices to check this for the charts of some atlas.

We turn to some applications of degree theory. First, consider the sphere S^2. We may view this as $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with coordinate z on $\mathbb{C}$ and homogeneous coordinates $(Z_0 : Z_1)$ on $\mathbb{P}^1_{\mathbb{C}}$ so that $z = Z_1/Z_0$ on the open subset of $\mathbb{P}^1_{\mathbb{C}}$ where $Z_0 \neq 0$. Let

$$f(z) = z^k + a_1 z^{k-1} + \cdots + a_k$$

be a monic polynomial with complex coefficients a_i. Then

$$F(Z_0 : Z_1) = (Z_0^k : Z_1^k + a_1 Z_1^{k-1} Z_0 + \cdots + a_k Z_0^k)$$

defines a smooth map from $\mathbb{P}^1_{\mathbb{C}}$ to itself. Now consider the smooth family of maps

$$F_t \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$$

with

$$F_t(Z_0 : Z_1) = \left(Z_0^k : Z_1^k + t(a_1 Z_1^{k-1} Z_0 + \cdots + a_k Z_0^k)\right).$$

Then by Theorem 6.3, the action of F_t^* on cohomology is independent of t, so

$$\deg(F) = \deg(F_0).$$

Here in terms of the coordinate z, $F_0(z) = z^k$. We calculate the degree by choosing a 2-form on $\mathbb{C} = \{Z_0 \neq 0\} \subset \mathbb{P}^1_{\mathbb{C}}$ with compact support in $\mathbb{C}$: indeed, in terms of $z = x + iy = re^{i\theta} = r\cos\theta + ri\sin\theta$ we will choose

$$\omega = f(r) dx \wedge dy = f(r) r dr \wedge d\theta$$

with f of compact support normalised such that

$$\int_{\mathbb{R}^2} \omega = 1.$$

Thus

$$\deg(F_0) = \int_{\mathbb{R}^2} (F_0)^*(\omega) = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} f(r^k) r^k d(r^k) \wedge d(k\theta) = k \int_{\mathbb{R}^2} \omega = k.$$

In particular, if $k > 0$, each F_t is surjective. Therefore

$$f(z) = z^k + a_1 z^{k-1} + \cdots + a_k = 0$$

has a solution, which is the Fundamental Theorem of Algebra.

As a second application we consider *linking numbers of curves in* $\mathbb{R}^3$. Here we will understand by a *curve* in $\mathbb{R}^3$ a smooth embedding

$$f \colon S^1 \to \mathbb{R}^3.$$

Compare Definition 3.6. A *pair of curves* $\{f, g\}$ will consist of two curves f, g such that $f(S^1) \cap g(S^1) = \emptyset$.

Definition 7.6. We call pairs of curves $\{f, g\}$ and $\{f', g'\}$ *isotopic* if there are smooth maps

$$F \colon [0,1] \times S^1 \to \mathbb{R}^3, \quad G \colon [0,1] \times S^1 \to \mathbb{R}^3$$

such that $F_t(x) := F(t, x)$ and $G_t(x) := G(t, x)$ are such that

$$\{F_t, G_t\}$$

is a pair of curves for every $t \in [0,1]$ and $F_0 = f, F_1 = f'$, $G_0 = g, G_1 = g'$. We will call $\{F, G\}$ an *isotopy* between $\{f, g\}$ and $\{f', g'\}$.

The reader should at this point sketch some examples of pairs of curves: they can be very complicated "linked" copies of S^1! Moreover, it would be a good idea to think about what the notion of isotopy means from a visual informal point of view.

We now wish to define an invariant under isotopy of a pair of curves $\{f, g\}$.

Definition 7.7. Let $\{f, g\}$ be a pair of curves. The *linking number* of $\{f, g\}$, denoted by $\mathrm{link}(f, g)$, is the degree of the smooth map $\mu\colon S^1 \times S^1 \to S^2$ given by

$$\mu(t, s) = \frac{f(t) - g(s)}{||f(t) - g(s)||}$$

where $|| \cdot ||$ is the Euclidean norm.

We now wish to relate this number $\mathrm{link}(f, g)$ to the geometric idea that the curves can be "linked" or "unlinked".

We will denote by $\{f_0, g_0\}$ the pair of curves in $\mathbb{R}^3$ consisting of the standard parametrisations of the circles

$$f_0\colon S^1 \to \left\{(x, y, z) \in \mathbb{R}^3\colon z = 0 \text{ and } (x-2)^2 + y^2 = 1\right\},$$
$$g_0\colon S^1 \to \left\{(x, y, z) \in \mathbb{R}^3\colon z = 0 \text{ and } (x+2)^2 + y^2 = 1\right\}.$$

Note that the images of f_0 and g_0 are just translations of the standard copy of S^1 of radius 1 around the origin in the (x, y)-plane, and these translations give what we call "standard parametrisations".

Definition 7.8. We call a pair of curves $\{f, g\}$ *unlinked* if it is isotopic to the pair $\{f_0, g_0\}$. Otherwise we call it *linked*.

We then have the following criterion.

Theorem 7.9. *If $\{f, g\}$ and $\{f', g'\}$ are isotopic pairs of curves, we have*

$$\mathrm{link}(f, g) = \mathrm{link}(f', g').$$

Moreover, if $\mathrm{link}(f, g) \neq 0$, then the pair $\{f, g\}$ is linked.

Proof. Let $\{F, G\}$ be an isotopy between the isotopic pairs of curves $\{f, g\}$ and $\{f', g'\}$. For a fixed t, we associate to the pair $\{F_t, G_t\}$ the map $\mu_t\colon S^1 \times S^1 \to S^2$ given by

$$\mu_t(x, y) = \frac{F_t(x) - G_t(y)}{||F_t(x) - G_t(y)||}.$$

We can then define a smooth map

$$\mu\colon [0, 1] \times S^1 \times S^1 \to S^2$$

by $\mu(t, x, y) = \mu_t(x, y)$. We can apply Theorem 6.3 to conclude that $\deg(\mu_t)$ is constant in the family. Hence

$$\mathrm{link}(f, g) = \deg(\mu_0) = \deg(\mu_1) = \mathrm{link}(f', g').$$

We now show that if f and g are unlinked, then their linking number is zero. By definition, unlinkedness means that $\{f, g\}$ is isotopic to $\{f_0, g_0\}$, so by the first part we only have to show that the linking number of f_0 and g_0 is zero. But the images of f_0 and g_0 are contained in the (x, y)-plane, so the image of μ is also contained in the (x, y)-plane and cannot be all of S^2. Thus μ is not surjective, which means $\deg(\mu) = \mathrm{link}(f_0, g_0) = 0$ by Theorem 7.4. □

Example 7.10. We should show that $\mathrm{link}(f, g)$ is not by chance always identically zero. Consider the circles in Figure 1, top. The images of f and g lie in mutually perpendicular planes, the (x, y)- and (x, z)-planes, and their centres lie one the x-axis, the line of intersection of these two planes. We claim that $\mathrm{link}(f, g) = \pm 1$ (the sign depending on the chosen orientations). Consider the map

$$\mu\colon S^1 \times S^1 \to S^2$$

from Definition 7.7. Let $e_1 = (1,0,0) \in S^2$. We want to know what $\mu^{-1}(e_1)$ is. Now if $(a,b) \in \mu^{-1}(e_1)$, then $f(a) - g(b)$ must be parallel to the x-axis and point in the positive direction towards $+\infty$ on that axis: hence $f(a)$ and $g(b)$ must be the points shown in the figure. Therefore, $\mu^{-1}(e_1)$ is a single point.

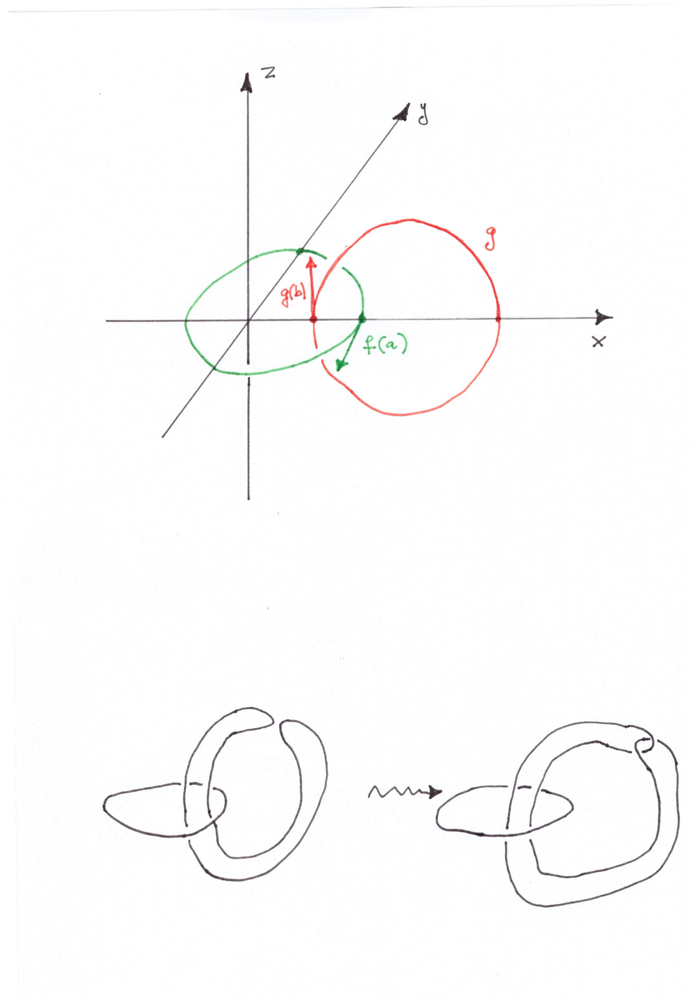

FIGURE 1. Linked curves with linking number ± 1 (top); the bottom shows curves with linking number zero, but the pair on the right hand side is linked.

We claim that e_1 is indeed a regular value for μ: indeed, $T_{(a,b)}(S^1 \times S^1) \simeq T_a(S^1) \times T_b(S^1)$ maps surjectively onto $T_{\mu(a,b)}(S^2)$ because the vectors $f'(a)$ and $g'(b)$ in Figure 1 are perpendicular. This implies that the two curves

$$\mu_a = \mu \,|_{\{a\} \times S^1} \colon \{a\} \times S^1 \to S^2, \quad \mu_b = \mu \,|_{S^1 \times \{b\}} \colon S^1 \times \{b\} \to S^2$$

passing through $\mu(a,b)$ have independent tangent vectors at that point. Thus by Theorem 7.4, $\deg(\mu) = \operatorname{link}(f,g) = \pm 1$.

Example 7.11. It should be noted that if $\operatorname{link}(f,g) = 0$, it is not necessarily true that the pair of curves $\{f,g\}$ is unlinked. Thus the converse to the second assertion in Theorem 7.9 is not true. Indeed, consider Figure 1, bottom. The pair of curves pictured on the left hand side is visibly unlinked, hence has linking number zero. We claim that the pair of curves on the right then also has to have linking number zero (though it is intuitively clear that this pair of curves is linked). Indeed, the proof of the first part of Theorem 7.9 shows a little more, namely that $\mathrm{link}(f, g)$ is not only invariant under isotopy, bot under what we may call *(smooth) homotopy*: here, in slight modification of Definition 7.6, we call pairs of curves $\{f, g\}$ and $\{f', g'\}$ *(smoothly) homotopic* if there are smooth maps

$$F\colon [0,1]\times S^1 \to \mathbb{R}^3, \quad G\colon [0,1]\times S^1 \to \mathbb{R}^3$$

such that $F_t(x) := F(t, x)$ and $G_t(x) := G(t, x)$ satisfy $F_0 = f, F_1 = f'$, $G_0 = g, G_1 = g'$ and

$$F_t(S^1) \cap G_t(S^1) = \emptyset$$

for every $t \in [0, 1]$. In other words, we relax the assumption that

$$\{F_t, G_t\}$$

be a pair of curves for every $t \in [0, 1]$, to allow more general maps F_t, G_t, $t \in (0, 1)$, that need not be smooth *embeddings* anymore: for example, the image of F_t (or G_t) could be a curve with self-intersections (so that F_t is no longer injective), or even wilder.

Then it is clear intuitively that the pairs of curves in Figure 1, bottom are smoothly homotopic.

Example 7.12. As an example, that the linking number $\mathrm{link}(f, g)$ in some sense really measures the number of times the curve f goes around the curve g, we en-

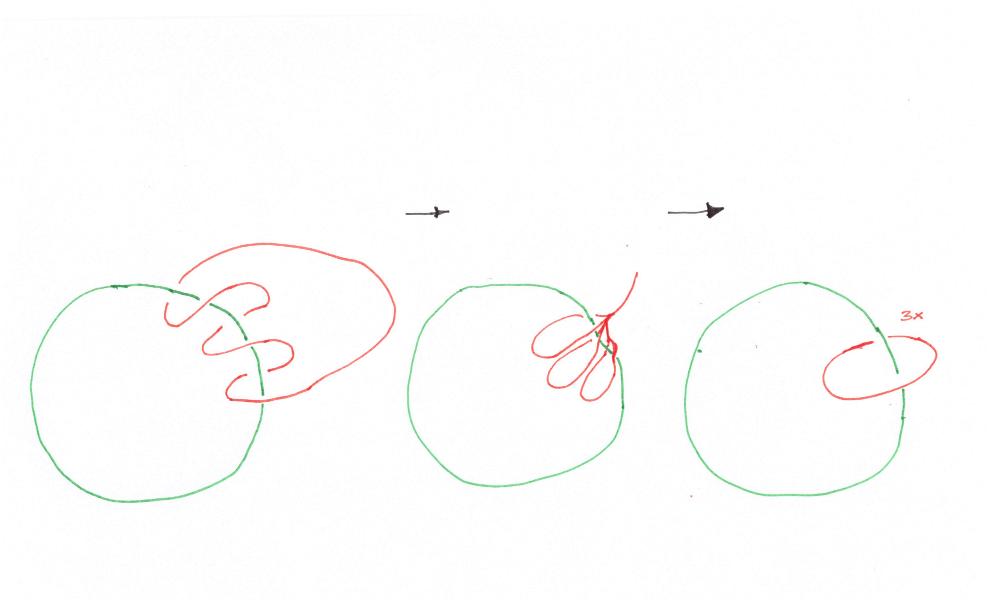

FIGURE 2. This is a geometric argument why the red coil winding around the green circle n-times gives a pair with linking number $\pm n$ (sign depending on orientations)

courage the reader to contemplate the homotopy depicted in Figure 2, where the red "coil" winds around the green curve n times ($n = 3$ is shown). Then that homotopy shows that this situation is homotopic to one with a red circle being linked to the green curve as shown, with the red circle being traversed n times (corresponding to the map $S^1 \to S^1$, $z \mapsto z^n$, $z \in \mathbb{C}, |z| = 1$), so $\operatorname{link}(f, g) = \pm n$.

We now leave linking numbers of curves and come to a final and very important concept that can be accessed using degree theory: the *index of an isolated singularity of a vector field*. As we will explain this has many wonderful applications and is also intimately related to the global topological properties of a manifold.

We start with the situation in $\mathbb{R}^n$, so let $U \subset \mathbb{R}^n$ be an open subset and let X be a vector field on U. We may view X simply as a smooth map $X\colon U \to \mathbb{R}^n$. We call a point $p \in U$ a *singularity* of X if $X(p) = 0$. In addition, we call it an *isolated singularity* if $X(q) \neq 0$ for every $q \neq p$ in a neighbourhood V of p in U.

Proposition 7.13. *Let p be an isolated singularity of the vector field X on the open subset U in $\mathbb{R}^n$, and denote by $S^{n-1}(p, \epsilon)$ the sphere of radius $\epsilon > 0$ and centre p in $\mathbb{R}^n$. Let $S^{n-1} = S^{n-1}(0, 1)$. If V is a neighbourhood of p that contains no further singularities of X than p and if ϵ is so small that $S^{n-1}(p, \epsilon) \subset V$, then the degree of the map*

$$\begin{aligned} f_\epsilon^X\colon S^{n-1} &\to S^{n-1}, \\ x &\mapsto \frac{X(p + \epsilon x)}{||X(p + \epsilon x)||} \end{aligned}$$

does not depend on ϵ.

We call this degree the *index of X at p* and denote it by $\operatorname{ind}_p X$.

Proof. Let ϵ and δ be small real numbers such that $S(p, \epsilon)$ and $S(p, \delta)$ are both contained in V. Define

$$F\colon [0, 1] \times S^{n-1} \to S^{n-1}$$

by

$$F(t, x) = f^X_{(1-t)\epsilon + t\delta}(x).$$

Since X does not vanish on V except in p, there exist an $\alpha > 0$ such that for every $t \in (-\alpha, 1 + \alpha)$ the denominator of f_t^X does not vanish for any x. Thus, since $F_0 = f_\epsilon^X$ and $F_1 = f_\delta^X$, we get the result from Theorem 6.3. $\square$

Example 7.14. You can easily calculate the following:

(1) If we put $X = \mathrm{id}_{\mathbb{R}^n}$, then the origin is an isolated singularity of index 1.
(2) For $X = -\mathrm{id}_{\mathbb{R}^n}$, the origin is an isolated singularity of index $(-1)^n$.
(3) If X on $\mathbb{R}^2$ is given by $X(x, y) = (-y, x)$, then the origin is an isolated singularity of index one.
(4) If $X(x, y) = (x^2 - y^2, 2xy)$ in $\mathbb{R}^2$, then the origin is an isolated singularity of index 2.

We just calculate (4) for fun (or punishment, whichever your view) and the rest is seen similarly, but easier. We recognise (4) as the map $z \mapsto z^2$ for $z = x + iy$, and thus the preimage of any point $p \in S^1$ consists of two points $\pm p$. Since the map preserves orientation, $\operatorname{ind}_0 X = 2$.

You should look at the primitive sketches of these vector field that I made in Figure 3, and use these to confirm that the index has the following geometric interpretation: traverse a little circle around the origin in counterclockwise direction; at each point of your way, the vector field furnishes you with an arrow that you can view as the position of the hand of a clock. Then the index is the number of times this hand turns on the clock as you traverse the circle and return to your starting

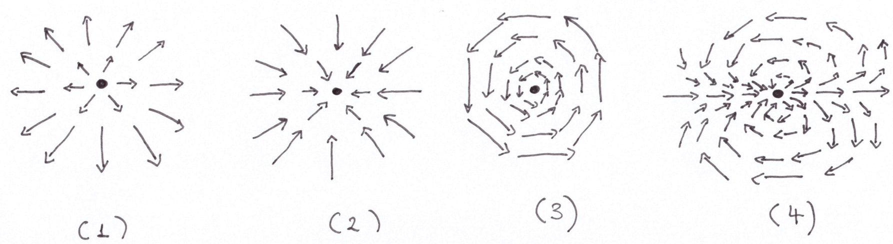

FIGURE 3. Sketches of the vector fields (1)-(4) in $\mathbb{R}^2$

point (where you count each full turn in counterclockwise direction as $+1$, and turns in clockwise direction as -1).

We now give a fun application of the concept of the index of a vector field, called the *argument principle* in complex analysis. Suppose we want to count the number of zeros of a complex polynomial

$$p(z) = z^m + a_1 z_{m-1} + \cdots + a_m$$

that lie in a smooth compact region Ω of the complex plane whose boundary we assume contains no zero of p; we will assume for simplicity and concreteness that Ω is just some some closed ball, i.e. a disc, in the plane, but other cases would be possible, too. We can view this as a manifold with boundary, oriented in a natural way using the standard orientation in the ambient $\mathbb{R}^2 = \mathbb{C}$.

Proposition 7.15. *The number of zeros of $p(z)$ in Ω, counted with multiplicity, equals the degree of the map*

$$\frac{p}{|p|} \colon \partial\Omega \to S^1.$$

This number is equal to the sum of the indices of singularities of the vector field $X(z) = p(z)$ contained in Ω.

Here the expression "counted with multiplicity" means the following: if

$$p(z) = \prod_{i=1}^{N} (z - \alpha_i)^{m_i}$$

and $\alpha_1, \dots, \alpha_t$ are the zeros of $p(z)$ lying inside Ω, the number in Proposition 7.15 is by definition

$$m_1 + \cdots + m_t.$$

Proof. We can find pairwise disjoint classed balls $D_i = \overline{B}(\alpha_i, \epsilon_i)$, $\epsilon_i > 0$ small real numbers, $i = 1, \ldots, t$, contained in the interior of Ω. We will now apply Stokes' Theorem to the oriented manifold with boundary

$$D = \Omega \backslash \bigcup_{i=1}^{t} B(\alpha_i, \epsilon_i)$$

(here $B(\alpha_i, \epsilon_i)$ denotes the open disk). Note that the map

$$F_p(z) := \frac{p(z)}{||p(z)||}$$

is defined and smooth in a neighbourhood of D. It takes values in $S^1 \subset \mathbb{C}$. Denoting by ω the canonical volume form giving the orientation on S^1 (normalised so that its integral over S^1 is 1), we thus have by Stokes' Theorem

$$0 = \int_D d(F_p^* \omega) = \int_{\partial\Omega} F_p^* \omega - \sum_{i=1}^{t} \int_{S^1(\alpha_i, \epsilon_i)} F_p^* \omega.$$

Thus we have that the degree of the map

$$\frac{p}{|p|} \colon \partial\Omega \to S^1$$

equals

$$\sum_{i=1}^{t} \operatorname{ind}_{\alpha_i} X.$$

Let us now check that

$$\operatorname{ind}_{\alpha_i} X = m_i.$$

Write

$$p(z) = (z - \alpha_i)^{m_i} q(z)$$

with $q(\alpha_i) \neq 0$. Then q is never zero in D_i. Recall that ϵ_i is the radius of D_i. Then

$$g \colon S^1 \to \partial D_i, \quad g(z) = \alpha_i + \epsilon_i z$$

is an orientation-preserving diffeomorphism of S^1 onto ∂D_i. Thus the degree of

$$\frac{p}{|p|} \colon \partial D_i \to S^1$$

is the same as the degree of

$$\frac{p \circ g}{|p \circ g|} \colon S^1 \to S^1.$$

Now define

$$h_t \colon S^1 \to S^1, \quad h_t(z) = \frac{z^{m_i} q(\alpha_i + t\epsilon_i z)}{|q(\alpha_i + t\epsilon_i z)|}$$

for $t \in [0, 1]$. Note that the division is permissible since q is never zero in D_i. Then

$$h_1 = \frac{p \circ g}{|p \circ g|}$$

and

$$h_0(z) = c z^{m_i}$$

where c is the nonzero constant

$$\frac{q(\alpha_i)}{|q(\alpha_i)|}.$$

Now by Theorem 6.3, h_1 and h_0 have the same degree, but the degree of h_0 is clearly equal to m_i. This concludes the proof. $\square$

Indeed, we can do somewhat better. The following result sometimes goes under the name *argument principle.*

Theorem 7.16. *The number of zeros (counted with multiplicity) of the complex polynomial $p(z)$ in the disk Ω (with zero-free boundary) is equal to*

$$\frac{1}{2\pi}\int_{\partial\Omega} d(\arg p(z)).$$

We first make some comments so that you can parse the formula in the statement of the Theorem. Each nonzero complex number w can be written uniquely as

$$w = re^{i\theta}$$

where r is the norm of w, and θ is, by definition, the *argument*. However, note that $\theta = \arg(w)$ is not really a well-defined function: all the values $\theta + 2\pi n$, n any integer, qualify equally to be the argument of w. Luckily, if we take the exteriro derivative d, this ambiguity disappears since any two choices of argument differ by addition of a constant. Indeed, in a suitable neighbourhood of any point in $\mathbb{C} - \{0\}$, we can always choose values of $\arg(w)$ so that this becomes a smooth function near the point; and any other local such choice will differ from the preceding one by some constant $2\pi n$. Thus we get a well-defined 1-form

$$d\arg$$

on all of $\mathbb{C} - \{0\}$, but this is not the differential of a well-defined function as the notation may erroneously suggest.

The integrand in Theorem 7.16 is the 1-form

$$z \mapsto d\arg p(z) = p^*(d\arg)$$

which is well-defined and smooth on the complex plane minus the zeros of p.

(Proof of Theorem 7.16). We already know by Proposition 7.15 that the number of zeros counted with multiplicity inside Ω is equal to the degree of the map

$$f := \frac{p}{|p|} \colon \partial\Omega \to S^1.$$

Now

$$\frac{p(z)}{|p(z)|} = e^{i\arg p(z)},$$

and all that remains to do is to identify the integral of $d(\arg p(z))$ over $\partial\Omega$ with $2\pi \deg(f)$.

We apply Theorem 7.4 to the restriction of the smooth 1-form $d\arg$ to S^1. If

$$w = f(z) = e^{i\arg p(z)}$$

then $\arg w = \arg p(z)$ and since $\arg p(z)$ is locally a well-defined smooth function, we get

$$d\arg p(z) = d(f^*\arg w) = f^*(d\arg w).$$

Thus

$$\int_{\partial\Omega} d(\arg p(z)) = \deg(f)\int_{S^1} d\arg w.$$

But calculating

$$\int_{S^1} d\arg w$$

is easy: indeed, parametrising S^1 by $\theta \mapsto e^{i\theta}$ gives

$$\int_{S^1} d\arg w = \int_0^{2\pi} d\theta = 2\pi.$$

$\square$

Suppose now that X is a vector field on a manifold M, having an isolated zero at $p \in M$. We would like to define the index of X at p, $\operatorname{ind}_p X$, to be the integer

$$\operatorname{ind}_{\varphi(p)} (\varphi_* X)$$

where (U, φ) is any chart around p. However, for this definition to be well-posed, we need the *invariance of the index under diffeomorphism*, namely:

Lemma 7.17. *Let U and U' be open subsets of $\mathbb{R}^n$ and let $f\colon U \to U'$ a diffeomorphism. If X is a vector field on U having an isolated zero at $x_0 \in U$, then $f(x_0)$ is an isolated zero of f_*X and*

$$\operatorname{ind}_{x_0} X = \operatorname{ind}_{f(x_0)} f_* X.$$

Before we can prove Lemma 7.17, we need one more auxiliary result.

Lemma 7.18. *Let $U \subset \mathbb{R}^n$ be open and star-shaped at the origin (meaning the line segment connecting the origin to any point in U is entirely contained in U). Let*

$$f\colon U \to f(U)$$

be a diffeomorphism that takes the origin 0 to itself and with $\det(df_0) > 0$. Then f is isotopic to the identity, by which we mean there is a family

$$F\colon [0,1] \times U \to \mathbb{R}^n$$

of diffeomorphisms F_t such that $F_0 = f$ and $F_1 = \mathrm{id}$.

(Proof of Lemma 7.18). We define $G\colon [0,1] \times U \to \mathbb{R}^n$ by

$$G(t,x) = \begin{cases} \frac{f(tx)}{t} & \text{for } t \in (0,1], \\ (df_0)(x) & \text{for } t = 0. \end{cases}$$

This shows that f is isotopic to the linear map df_0. But the linear maps with positive determinant form a connected open subset of $\mathbb{R}^{n^2}$, hence we can connect df_0 to the identity by a path. $\square$

We now turn to

(Proof of Lemma 7.17). We first consider the case when $\det(df_0) > 0$. We can then assume that $x_0 = 0$ and that $f(0) = 0$, and that U is star-shaped (by possibly replacing U by some smaller open set). Then there is an isotopy F between f and id_U by Lemma 7.18. There is a small positive number $\epsilon > 0$ such that $Y_t = (F_t)_* X$ does not vanish on $B(0,\epsilon)\backslash\{0\}$ for all $t \in [0,1]$ (use the compactness of $[0,1]$). Using Theorem 6.3 and the definition of index (and degree), we get that $\operatorname{ind}_0 Y_t$ is constant, and in particular,

$$\operatorname{ind}_0 f_* X = \operatorname{ind}_0 Y_0 = \operatorname{ind}_0 Y_1 = \operatorname{ind}_0 X.$$

If $\det(df_0) < 0$, we reduce to the previous case by replacing X by $\varrho_* X$ and f by $f \circ \varrho^{-1}$ where ϱ is the reflection in some hyperplane in $\mathbb{R}^n$. Then

$$f_* X = (f \circ \varrho^{-1})_* (\varrho_* X)$$

and $\operatorname{ind}_0(\varrho_* X) = \operatorname{ind}_0 X$, which concludes the proof. $\square$

We can use the argument in Lemma 7.18 to calculate the index of a vector field in a much simpler way in certain special cases:

Proposition 7.19. *Suppose $X\colon U \to \mathbb{R}^n$ is a vector field on some open subset U of $\mathbb{R}^n$, and let $p \in U$ be an isolated singularity of X. If $\det(dX_p) \neq 0$, then we have*

$$\operatorname{ind}_p X = \begin{cases} 1 & \text{if } \det(dX_p) > 0, \\ -1 & \text{if } \det(dX_p) < 0. \end{cases}$$

Proof. Indeed, we can assume $p = 0$, and then X is a diffeomorphism on a neighbourhood of 0, which we can assume to be star-shaped. If $\det(dX_p) > 0$, then X is isotopic to the identity by Lemma 7.18, and each F_t in this isotopy can be considered as a vector field having an isolated singularity at p; by the invariance of degree (which again follows from Theorem 6.3), we get

$$\operatorname{ind}_p X = 1.$$

If X has $\det(dX_p) < 0$, we reduce to the previous case by considering $X \circ \sigma$ where σ is the reflection in a hyperplane. □

We conclude by mentioning a celebrated result, the *Poincaré-Hopf Index Theorem*, that relates the concept of indices of isolated singularities of vector fields on compact manifolds with a basic topological invariant of the manifold, the *Euler characteristic* (sometimes also called the *Euler number*).

Theorem 7.20. *If M is a compact manifold and X a vector field on M having only isolated singularities $p_1, \dots, p_n$, then*

$$\sum_{i=1}^{n} \operatorname{ind}_{p_i} X = \chi(M)$$

is the so-called Euler characteristic *of M, which can be computed in the following alternative way: all de Rham cohomology groups $H^i(M)$ are finite-dimensional $\mathbb{R}$-vector spaces if M is compact, and*

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim_{\mathbb{R}} H^i(M).$$

We do not have time in the course of this lecture to prove this result, but did not want to pass it by in silence because it is so beautiful and important; a proof can be found, for example, in the wonderful book by Raoul Bott and Loring W. Tu, Differential Forms in Algebraic Topology, Springer (1982), p. 126 ff.

We just note that from our earlier computation of the de Rham cohomology of spheres and tori, it follows that for $M = S^n$, we get

$$\chi(S^n) = 1 + (-1)^n$$

and for $M = \mathbb{T}^n$,

$$\chi(\mathbb{T}^n) = 0.$$

Moreover, for $M = \mathbb{P}^n_{\mathbb{R}}$ we get

$$\chi(\mathbb{P}^n_{\mathbb{R}}) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

It is easy in low dimensions to picture vector fields with isolated singularities on these manifolds that allow us to confirm the equality stated in the Poincaré-Hopf

Index Theorem explicitly for these examples. The reader should compare Figure 4 and check that in each case the sum of the indices of the vector fields depicted is equal to the Euler characteristic.

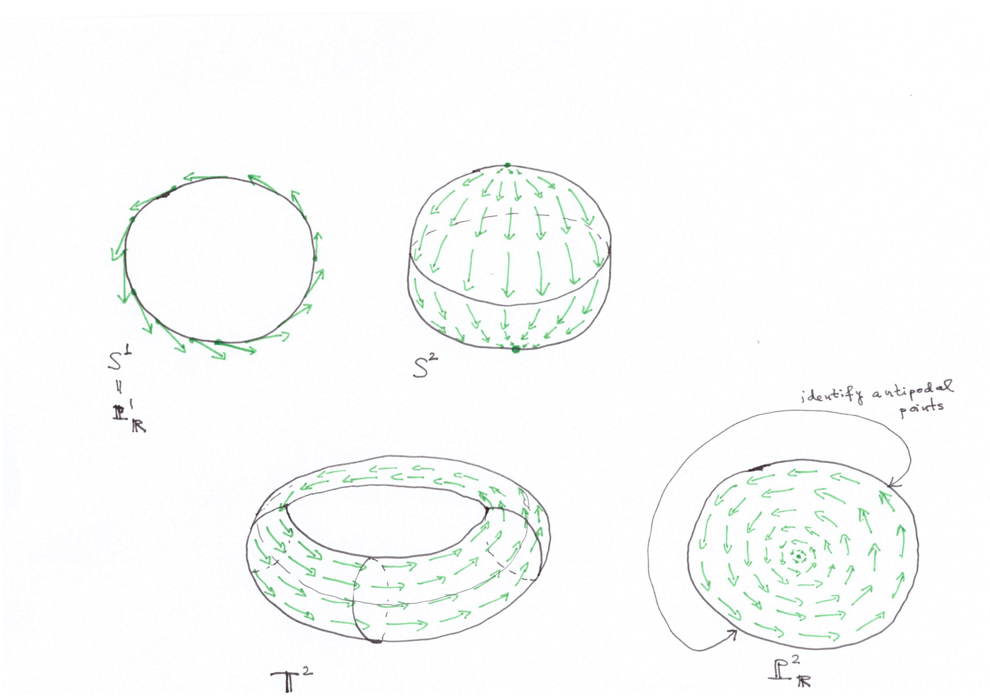

Figure 4. Vector fields with isolated singularities on some compact manifolds.

References

[BeGo88] Berger, M., Gostiaux, B., *Differential Geometry: Manifolds, Curves and Surfaces*, GTM **115**, Springer (1988)

[BoTu82] Bott, R., Tu, L.W., *Differential Forms in Algebraic Topology*, GTM **82**, Springer (1982)

[GuPo74] Guillemin, V., Pollack, A., *Differential Topology*, reprint of the 1974 edition, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island (2014)

[Hitchin] Hitchin, N., *Differentiable Manifolds*, online notes available at http://people.maths.ox.ac.uk/ joyce/Nairobi2019/Hitchin-DifferentiableManifolds.pdf

[KoNo63] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry, Volume I*, John Wiley & Sons Inc. (1963)

[MoKo71] Morrow, J., Kodaira, K., *Complex Manifolds*, reprint of the 1971 edition, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island (2006)

[Tu11] Tu, L.W., *An Introduction to Manifolds*, 2nd Edition, Springer Universitext, Springer (2011)

[Warner83] Warner, F.W., *Foundations of Differentiable Manifolds and Lie Groups*, GTM **94**, Springer (1983)