AG for NT 11

As last time, everything is Noetherian or of finite type.

1 More on intersections

Relation with conormal sheaves and blowups. Let X be a fibered surface. Let $D \subseteq X$ be a Cartier divisor. ($D = V(I)$ where $I = \mathcal{O}_X(-D)$.) Let $i : D \hookrightarrow X$

$$1 \to I \to \mathcal{O}_X \to i_*(\mathcal{O}_D) \to 1$$

Then the conormal sheave in this case is

$$\begin{aligned} C_{D/X} &= i^*(I/I^2) \\ &= I \otimes_{\mathcal{O}_X} \mathcal{O}_X/I \\ &= \mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_D \\ &= \mathcal{O}_X(-D)|_D \end{aligned}$$

The cononical sheave is $\omega_{D/X} = C_{D/X}^\vee = \mathcal{O}_X(D)|_D$.

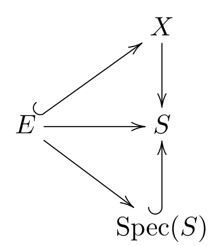
For a blowup: $Y = V(I) \hookrightarrow X$ both regular, we get $Y' \hookrightarrow X'$ where X' is the blowup of X in Y and Y' is the inverse image of Y (i.e., $V(I\mathcal{O}_{X'})$ and $Y' \cong \mathbb{P}_Y^{r-1}$. By construction $i : X' \hookrightarrow \mathbb{P}_X^{r-1}$ and $J = i^*(\mathcal{O}_{\mathbb{P}_X^{r-1}}(1))$. We have $J/J^2 = i^*(\mathcal{O}_{\mathbb{P}_Y^{r-1}}(1))$ and $\omega_{Y'/X} = (J/J^2)^\vee = \mathcal{O}_Y(-1)$. Specialising to Y being a point, we see that Y' has self-intersection -1, by taking the degree.

1.1 Adjunction

Let X be a fibered surface, suppose $E \subset X_S$ component of special fiber.

$$\omega_{E/k(s)} = \big(\mathcal{O}_X(E) \otimes \omega_{X/S}\big)_E$$

The proof of this formula is: One has



$$\underbrace{\omega_{E/S}}_{=\omega_{E/k(S)} \otimes \underbrace{\omega_{\mathrm{Spec}(s)|S}}_{\text{trivial}}} = \underbrace{\omega_{E/X}}_{\omega_{E/X}|_E = \mathcal{O}_X(E)|_E} \otimes \omega_{X/S}|_E$$

By taking degrees and using the relation between $\deg(\omega)$ and p_a, one gets

$$p_a(E) = 1 + \frac{1}{2}(E^2 + E \cdot K_{X/S})$$

where $\omega_{X/S} = \mathcal{O}_X(K_{X/S})$.

2 Cohomology

Let $\mathcal{F}$ be a sheaf on X be a scheme. We get an infinite sequence of groups $H^0(\mathcal{F}), H^1(\mathcal{F}), H^2(\mathcal{F}), \ldots$.

We have $H^0(\mathcal{F}) = \mathcal{F}(X)$

If

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

then we get the long exact sequence

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to H^1(\mathcal{G}) \to \ldots$$

Key Properties for Projective Space:

$X = \mathbb{P}^d_A = \mathbf{Proj}B$ where $B = A[T_0, \ldots, T_d]$, then

- $H^0(X, \mathcal{O}_X(n)) = B_n$ the nth graded parts
- $H^i(X, \mathcal{O}_X(n)) = 0$ if $i \neq 0, d$
- $H^d(X, \mathcal{O}_X(n)) = H^0(X, \mathcal{O}_X(-n-d-1))^\vee = 0$ for $n \gg 0$.
- Serre showed: Let $\mathcal{F}$ be a general sheaf on $X = \mathbb{P}^d_A$. Then $H^i(X, \mathcal{F}(u)) = 0$ for $i > 0$ if $n \gg 0$.
- Subschemes: if $i : Z \hookrightarrow X$ is a closed immersion then $H^i(Z, \mathcal{F}) \cong H^i(X, i_*\mathcal{F})$ (where $\mathcal{F}$ is a sheaf on Z)

Let X be an arithmetic surface $E \subset X_S$ irreducible such that

- $E \cong \mathbb{P}^1_{k'}$ for k'/k finite
- $E^2 < 0$

Let H be an effective divisor on X such that $H^1(X, \mathcal{O}_X(H)) = 0$. Let $r = \frac{-H \cdot E}{E^2} \in \mathbb{Q}$. Then

1. $H^1(X, \mathcal{O}_X(H + iE)) = 0$ for $i \leq r$
2. If $r \in \mathbb{Z}$, $\mathcal{O}_X(H)$ is generated by global sections: then $\begin{cases} \mathcal{O}_X(H + rE)|_E \cong \mathcal{O}_E \\ \mathcal{O}_X(H + rE) \text{ generated by global sections} \end{cases}$.

Proof.

1. We use induction on i. For the given $i \leq r$, we have $(H + (i+1)E) \cdot E \geq 0$. So $\mathcal{O}_X(H + (i+1)E)|_E$ is of positive degree, hence isomorphic to $\mathcal{O}_E(a)$ for some $a \geq 0$. Use

   $$0 \to \mathcal{O}_X(H + iE) \to \mathcal{O}_X(H + (i+1)E) \to i_*\mathcal{O}_X(H + (i+1)E)|_E \to 0.$$

   Use cohomology, $H^1(X, \mathcal{O}_X(H + iE)) = 0$, by induction hypothesis.

   $$H^1(X, i_*\mathcal{O}_X(H + (i+1)E)|_E) = H^1(E, \mathcal{O}_E(a)) = 0$$

   by description for $\mathbb{P}^1$. Hence $H^1(X, \mathcal{O}_X(H + (i+1)E)) = 0$ because of the exact sequence

2. $\mathcal{O}_X(H + rE)|_E \cong \mathcal{O}_E$ because its degree is zero and $E \cong \mathbb{P}^1_{k'}$. Use the fact that it is generated by global sections, we only have to check this at points of E, since $H^0(X, H + rE) \supset H^0(X, H)$ and outsides E, these sheaves coincides.

   Now $H^0(X, \mathcal{O}_X(H + rE)) \twoheadrightarrow H^0(E, \mathcal{O}_X(H + rE)|_E) \to H^!(X, \mathcal{O}_X(H + \underbrace{(r-1)}_{=0}E)$

□

Theorem 2.1. *Let X be an arithmetic surface, $E \subseteq X_S$ irreducible component of special fiber such that*

1. $E \cong \mathbb{P}^1_{k'}$
2. $E^2 < 0$

Then a contraction $f : X \to Y$ of E exists.

Proof. Let $\mathcal{L}$ be an ample sheaf on X. If necessary replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$, we can assume $\mathcal{L}$ is very ample. Furthermore, by using Serre we can also assume $H^1(X, \mathcal{L}) = 0$. Let $\mathcal{L} = \mathcal{O}_X(H_0)$ where H_0 is an effective divisors.

Let Γ be a component of a special fiber, then $\mathcal{O}_X(H_0)|_\Gamma$ is still ample, so $H_0 \cdot \Gamma > 0$ (as ample if and only if $\deg > 0$ if $\mathbb{P}^1$). Let $m = -E^2 > 0$ and $r = H_0 \cdot E > 0$. Construct $D = mH_0 + rE$. By our previous result D is generated by global sections and so defines a morphism $f = f_D$:

1. E get contracted, because $D \cdot E = 0$ by construction, so $\deg(\mathcal{O}_X(D)|_E) = 0$, so $\mathcal{O}_X(D)|_E = \mathcal{O}_E$ (since $E \cong \mathbb{P}^1_{k'}$)
2. Other Γ do not get contracted: $\deg(\mathcal{O}_X(D)|_\Gamma) = D \cdot \Gamma = \underbrace{mH_0 \cdot \Gamma}_{>0\, H_0 \text{ ample}} + \underbrace{rE \cdot \Gamma}_{\geq 0, \text{ no common components}} > 0$. So $\mathcal{O}_X(D)|_\Gamma$ is not trivial.

□

Using the theorem on formal functions, one shows: if

$$\begin{aligned} d &= -E^2/[k' : k(s)] \\ &= \deg_{k'}(\mathcal{O}_X(-E)|_E) \\ &= \deg_{k'}(N_{E/X}) \end{aligned}$$

So $E \mapsto y$ is a contraction $\dim_{k(y)} T_{Y/y} = d + 1$.

So the contraction is regular if and only if $\begin{cases} E \cong \mathbb{P}^1_{k'} \\ E^2 = -[k' : k(s)] \end{cases}$. In such a case E is called *exceptional*. The arithmetic surface obtained after successfully contracting all exceptional divisors is the *relatively minimal model* of X.

Criteria for a divisor to be exceptional:

1. E is exceptional if and only if $E^2 < 0$ and $K_{X/S} \cdot E < 0$.
2. $p_a(X_\eta) \geq 1$: E is exceptional if and only if $K_{X/S} \cdot E < 0$

Proof.

1. Use adjunctions,

$$\begin{aligned} K_{X/S} \cdot E + E^2 &= -2\chi_{k(s)}(\mathcal{O}_E) \\ &= -2 + 2\dim_{k'}(H^1(E, \mathcal{O}_E)) \end{aligned}$$

   This shows, $H^1 = 0$, which means E is a conic, and in fact $E \cong \mathbb{P}^1_{k'}$. In fact then $K_{X/S} \cdot E = E^2$

2. $H^0(X, \omega_{X/S}) \otimes K(S) \neq 0$ because of hypothesis. Therefore, $\omega_{X,S}$ is effective: $\omega_{X/S} = \mathcal{O}(K_{X/S})$ for $K_{X/S} > 0$. We have $K_{X/S} = aE + D$ where D has no common component with E. Because of the intersection number, we see$a \geq 1$. Then $aE^2 = \underset{<0}{K_{X/S} \cdot E} - \underset{>0}{D \cdot E} < 0$

□

Recall: If Y is an arithemtic surface, it is *minimal* if for all other X arithemtic surface, if we have that there is birational map $X \dashrightarrow Y$, it is in fact a morphism.

We want that relvative minimal implies minimal. This is true f $p_a(X_\eta) \geq 1$. This is beacuse for $p_a(X_\eta) = 0$, the statement is not true. (See diagram in notes)

Lemma 2.2. *Let X_1, X_2 be arithmetic birational surfaces without morphism between them. There exists a Z common birational cover and E_1, E_2 on Z such that either*

- *$p_2(E_1)$ is still exceptional or*
- *$(E_1 + \mu E_2)^2 \geq 0$ for some $\mu \geq 0$*

$$\begin{array}{ccc} & Z & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X_1 & \overset{\dashrightarrow}{\dashleftarrow} & X_2 \end{array}$$

Theorem 2.3. *Relative minimal implies minimal if $p_a(X_\eta) \geq 1$*

Proof.

$$\begin{aligned} 2p_a(X_\eta) - 2 &= 2p_a(Z_\eta) - 2 \\ &= K_{Z/S} \cdot Z_S \end{aligned}$$

Consider $D = E_1 + \mu E_2$, there are contained in the same fiber, so $D = rZ_s$ for some r because of negative-definiteness.

$$= \quad (K_{Z/S} E_1 + \mu E_2 \cdot K_{Z/S})/r < 0$$

Contraction. □

2.1 Other models

1. E/K elliptic curve where $K = K(S)$ for affine Dedekind scheme. Then there exists a normal model of E over S. So a minimal regular model $\mathcal{E}$ exists.

   Let $\mathcal{N}$ be the smooth locus of $\mathcal{E} \to S$. This is an open immersion in $\mathcal{E}$. We have $\mathrm{Hom}(S, \mathcal{E}) = \mathcal{E}(S) = E(K) = \mathrm{Hom}(\mathrm{Spec}\, K, E)$ (because $\mathcal{E}$ is proper). $\mathcal{E}(S) = \mathcal{N}(S)$ because rational points intersect special fibers once, so not in singular points.

   $\mathcal{N}$ is called the Neron model of E. It is the unique smooth model of E such that for X/S smooth, there is an bijection $\mathrm{Hom}_S(X, \mathcal{E}) = \mathrm{Hom}_K(X_K, E)$. This gives a filtration of $E(K)$: given by

   $$\underbrace{E^1(K)}_{\cong \mathfrak{m} \text{ for maximal ideal of DVR } R \text{ with FF } K} \hookrightarrow E^0(K) \hookrightarrow E(K)$$

   $k = R/\mathfrak{m}$, (first arrow) $N_J(k)$ is connected component containing 0 of the special fibers of $\mathcal{N}$ over S

   (second arrow) groups of connected components of special fiber.