1. Blowing Up Varieties

2, 3, 4 : A 1-1 correspondence

Invertible sheaves on Varieties

$\longleftrightarrow$ Linear Systems of Divisors

(on Varieties).

We will use language of schemes

to study Varieties.

1. Blowing Up Varieties

We will construct the blowup of a Variety with respect to a non-singular closed subvariety.

This tool / technique is the main method to resolve singularities of an algebraic variety.

Defn 1.1

The blowup of $\mathbb{A}^n$ at O is constructed as follows:

Take the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

If $\{x_1, \ldots, x_n\}$ are affine coordinates for $\mathbb{A}^n$, $\{y_1 : \cdots : y_n\}$ are homogeneous coordinates of $\mathbb{P}^{n-1}$, the blowup of $\mathbb{A}^n$

$Bl_0 \mathbb{A}^n$ is the closed subset defined by

$$Bl_0 \mathbb{A}^n \text{ is } \left\{ x_i y_j = x_j y_i \;\middle|\; \begin{array}{c} 1 \leq i, j \leq n \\ i \neq j \end{array} \right\}$$

$$\begin{tikzcd} Bl_0 \mathbb{A}^n \arrow[r, hook] \arrow[dr, "\phi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "\text{Projection}"] \\ & \mathbb{A}^n \end{tikzcd}$$

Let ϕ be the morphism defined by this composition.

Lemma 1.2

1. If $P \in \mathbb{A}^n$, $P \neq 0$ then $\phi^{-1}(P)$ consists of one point.

   In fact ϕ gives an isomorphism of
   $$Bl_0 \mathbb{A}^n \setminus \phi^{-1}(0) \cong \mathbb{A}^n \setminus 0$$

2. $\phi^{-1}(0) \cong \mathbb{P}^{n-1}$

3. The points of $\phi^{-1}(0)$ are in 1:1 correspondence with lines of $\mathbb{A}^n$ through the origin.

4. $Bl_0 \mathbb{A}^n$ is irreducible.

Proof

1. let $P = (a_1, \dots, a_n) \in \mathbb{A}^n$

so assume $a_i \neq 0$

so if $P \times (y_1 : \dots : y_n) \in \phi^{-1}(P)$

then for each j, $y_j = \left(\frac{a_j}{a_i}\right) y_i$

so $(y_1 : \dots : y_n)$ is uniquely determined as a point in $\mathbb{P}^{n-1}$. By setting $y_i = a_i$ we have $(y_1 : \dots : y_n) = (a_1, \dots, a_n)$.

Hence $\phi^{-1}(P)$ consists of a single point.

Furthermore setting
$$\psi(P) = (a_1, \dots, a_n) \times (a_1 : \dots : a_n)$$
defines an inverse morphism to ϕ
$$\mathbb{A}^n \setminus 0 \longrightarrow Bl_0 \mathbb{A}^n \setminus \phi^{-1}(0).$$

2. $\phi^{-1}(0)$ consists of all points $0 \times Q$, $Q \in \mathbb{P}^{n-1}$ with no restrictions.

3. Immediate from 2.

4. $$Bl_0 \mathbb{A}^n = \left(Bl_0 \mathbb{A}^n \setminus \phi^{-1}(0)\right) \cup \phi^{-1}(0)$$
$$\underset{\mathbb{A}^n \setminus 0 \text{ irreducible}}{\cong}$$

each point in $\phi^{-1}(0)$ is contained in the closure of some line L' in $Bl_0 \mathbb{A}^n \setminus \phi^{-1}(0)$

Hence $Bl_0 \mathbb{A}^n \setminus \phi^{-1}(0)$ is dense in $Bl_0 \mathbb{A}^n$. So $Bl_0 \mathbb{A}^n$ is irreducible.

$\square$

Defn 1.3

If $Y \subset \mathbb{A}^n \setminus 0$ we define $Bl_0 Y$ to be $\tilde{Y} := \overline{\phi^{-1}(Y \setminus 0)}$

We see from Lemma 1.2 ϕ induces a birational morphism of $\tilde{Y}$ to Y.

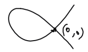
Fact 1.4

Blowing up is independent of your choice of embedding.

Example 1.5 (Node)

Let x, y be coordinates for $\mathbb{A}^2$.

$X : (y^2 = x^2(x+1))$



Let $(t:u)$ be hom. coord. for $\mathbb{P}^1$. Then

$$Bl_0 X = \{ y^2 = x^2(x+1),\ ty = ux \} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

On the affine piece $t \neq 0$ we have $y^2 = x^2(x+1)$, $y = ux$.

$\Rightarrow$ $u^2 x^2 = x^2(x+1)$. This factors

$\Rightarrow$ $\begin{cases} (x = 0) & = E \\ u^2 = x+1 & = \tilde{X} \end{cases}$

preimage of 0 under ϕ. "Exceptional Divisor"

"Proper transform of X"

Note that $\tilde{X} \cap E$ consists of two points, $u = \pm 1$.

Notice that these values for u are precisely the values of the slopes of X through the origin.

"Blowups separate points and tangent vectors".

Exercise (Tacnode)

$T: (y^2 = x^4(x+1))$. Blow this up at the origin and see what you get.

__Defn 1.7__ Blowing up with respect to a subvariety.

Let $X \subset \mathbb{A}^n$ be an affine variety

Let $Z \subset X$ be a closed non-singular subvariety, Z defined by the vanishing of polynomials $\{f_1, \dots, f_k\}$ in $\mathbb{A}^n$.

Let $\{y_1 : \dots : y_k\}$ be hom. coor. for $\mathbb{P}^{k-1}$. Define.

$$Bl_Z \mathbb{A}^n = \{ y_j f_i = y_i f_j \mid 1 \leq i, j \leq k,\ i \neq j \}.$$

as before we get a birational map

$$\begin{tikzcd} Bl_Z \mathbb{A}^n \arrow[r, hook] \arrow[dr, "\phi"'] & \mathbb{A}^n \times \mathbb{P}^{k-1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

with birational inverse

$$P = (a_1 \dots a_n) \to (a_1 \dots a_n) \times (f_1(P) \dots f_k(P))$$

also, define (as before):

$$Bl_Z X = \overline{\phi^{-1}(X \setminus 0)}$$

__Exercise 1.8__

compare Blowing up $y^2 = x^2(x+1)$ in $\mathbb{A}^3_{(x,y,z)}$ with respect to the z-axis.

[Note: $O \in \mathbb{A}^n$ is the subvariety defined by the vanishing of polynomials

$f_i = x_i$]

(For most purposes / "classifying all surfaces")

__Example 1.9__ Let X be the double cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3_{(x,y,z)}$.

Let Z be the line $\{y = z,\ x = 0\}$.

So $[t : u]$ are coordinates for $\mathbb{P}^1$.

$$Bl_Z X = \{ x^2 + y^2 = z^2,\ xt = (y-z)u \}.$$

On the piece $u \neq 0$ we get

$$xt = y - z \Rightarrow x^2 = xt(y+z)$$

this Coobonses:

$\hookrightarrow E : (x=0,\ y=z,\ t \text{ arbitrary}\}$

$\tilde{X}: \quad xt = y-z.$

↑ should be non-singular.

2. Invertible sheaves.

Let X be a variety.

Defn / Recall 2.1

- An invertible sheaf $\mathcal{F}$ on X is a locally free $\mathcal{O}_X$-module of rank 1,

[That is, $\exists$ open covering $\{U_i\}$ of X so that

$\mathcal{F}(U_i) \cong \mathcal{O}_X(U_i)$.]

- We will see soon that the Picard group is the group of iso. classes of invertible sheaves on X.

- On varieties, Weil divisors are "the same" as Cartier Divisors.

$D = \{(U_i, f_i)\}$ with $\{U_i\}$ an open covering of X, f_i on U_i is an element of

$\mathcal{O}_X(U_i)$. (Think of a rational function).

Also, on $U_i \cap U_j$, $\frac{f_i}{f_j}$ is invertible.

Notation 2.2

Let D be a divisor (Weil / Cartier)

Define $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$ module which is generated by f_i^{-1} on U_i.

This is well-defined, since f_i / f_j is invertible on $U_i \cap U_j$ so f_i^{-1} and f_j^{-1} differ by a unit.

This $\mathcal{L}(D)$ is the sheaf associated to $D = \{(U_i, f_i)\}$.

Prop 2.3

1. For any divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X, and the map

   $D \to \mathcal{L}(D)$ gives a 1:1 correspondence

   $$\frac{\text{Divisors on } X}{\text{linear equivalence}} \longleftrightarrow \text{Invertible sheaves on } X.$$

2. $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

3. $D_1 \sim D_2$ (linearly equivalent) iff $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$

Proof

1. The map $\mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}$ defined by $1 \mapsto f_i^{-1}$ is the isomorphism, so $\mathcal{L}(D)$ is an invertible sheaf.

Conversely, D can be recovered by $\mathcal{L}(D)$ by f_i on U_i to be the inverse of a generator for $\mathcal{L}(D)(U_i)$.

2. If $D_1 = \{(U_i, f_i)\}$, $D_2 = \{(V_j, g_j)\}$ then $\mathcal{L}(D_1 - D_2)$ on $U_i \cap V_j$ is generated by $f_i^{-1} g_j$

so $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

3. By 2. it is sufficient to show that $D = D_1 - D_2$ is principle $\Leftrightarrow \mathcal{L}(D) \cong \mathcal{O}_X$

If D is principal, defined by $f \in \Gamma(X, \mathcal{O}_X^*)$ then $\mathcal{L}(D)$ is globally generated by f^{-1}

so $1 \mapsto f^{-1}$ gives the iso. $\mathcal{O}_X \cong \mathcal{L}(D)$ $\square$

So we have 1:1 correspondence:

linear equivalence of Divisors $\longleftrightarrow$ iso. classes of invertible sheaves.

3. Morphisms to $\mathbb{P}^n$

On $\mathbb{P}^n$, the hom. coordinates $X_0, \dots, X_n$ give the standard affine cover $\{U_i = (X_i \neq 0)\}$ and on U_i, X_i^{-1} is a local generator for the invertible sheaf $\mathcal{O}(1)$.

For any variety X, let $\phi: X \to \mathbb{P}^n$.

Then $\mathcal{L} = \phi^*(\mathcal{O}(1))$ is an invertible sheaf on X. The global sections $s_0, \dots, s_n$ $(s_i := \phi^*(X_i))$, $s_i \in \Gamma(X, \mathcal{L})$. "generate" the sheaf $\mathcal{L}$.

Conversely, $\mathcal{L}$ and s_i determine ϕ.

(X_i)

<u>Thm 3-1</u>

1. If $\phi : X \to \mathbb{P}^n$ is a morphism, then $\phi^*(\mathcal{O}(1))$ is an invertible sheaf on X generated by global sections $s_i = \phi^*(x_i)$.

2. Any invertible sheaf $\mathcal{L}$ on X determines a unique morphism $\phi : X \to \mathbb{P}^n$.

<u>Proof</u>

1. From above

2. Lengthy argument Hartshorne pg 150.

<u>Prop 3-2</u>

Let k be an alg. closed field

Let X be a variety, $\phi : X \to \mathbb{P}^n$ be a morphism corresponding to $\mathcal{L}$. $s_0 \ldots s_n$ be as above.

Let $V \leq \Gamma(X, \mathcal{L})$ be a subspace spanned by $s_i = \phi^*(x_i)$. Then ϕ is a <u>closed immersion</u> iff

1. elements of V "separate points" i.e. for any $P \neq Q$ on X, $\exists s \in V$ with $s \in \mathfrak{m}_P \mathcal{L}_P$ but $s \notin \mathfrak{m}_Q \mathcal{L}_Q$

2. elements of V "separate tangent vectors": For each point $P \in X$, the set $\{ s \in V \mid s_P \in \mathfrak{m}_P \mathcal{L}_P \}$ spans the vector space $\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$.

<u>Pf</u> (only do $\Rightarrow$).

If ϕ is a closed immersion, think of X as a closed subvariety of $\mathbb{P}^n$.

So $\mathcal{L} = \mathcal{O}_X(1)$. and the v.space $V \subseteq \Gamma(X, \mathcal{O}_X(1))$ is spanned by the images of $x_0 \ldots x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Given $P \neq Q \in X$, $\exists$ a hyperplane H containing P and not Q.

$H = (\sum a_i x_i = 0)$. $a_i \in k$ then

$s = \sum a_i x_i \big|_X$ satisfies Job property.

For 2nd, each hyperplane passing through P gives rise to sections which generate $\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$.

E.g. if $P = (1:0:0\ldots 0)$.

then U_0 has local coordinates $y_i = \frac{x_i}{x_0}$

so $P \doteq (0,\ldots,0)$ and $\mathfrak{m}_P / \mathfrak{m}_P^2$ is the vectorspace spanned by y_i. $\square$

So we have 1:1 correspondences

linear equivalence classes of divisors on X $\longleftrightarrow$ iso classes of invertible sheaves

morphisms to $\mathbb{P}^1$

4. Linear systems of Divisors

Defn 4.1 A complete linear system $|D_0|$ on a non-singular projective variety is the set of all effective divisors linearly equivalent to D_0.

That is, $|D_0|$ is in 1:1 correspondence to this set

$$\left(\Gamma(X, \mathcal{L}(D_0)) \setminus \{0\}\right) / k^*$$

i.e. $|D_0|$ "is" a projective space.

Def. 4.2

A linear system δ on X is a subset of a complete linear system $|D_0|$ which is linear subspace for $|D_0|$.

I.e. δ is a sub-vectorspace of $\Gamma(X, \mathcal{L}(D_0))$

Defn 4.3

A point $P \in X$ is a base point for a linear system δ is $P \in \operatorname{supp} D$ for every $D \in \delta$

set of all prime divisors whose coeff. is non-zero.

Lemma 4.4. Let δ be a linear system, on X corresponding to the subspace

$$V \subset \Gamma(X, \mathcal{L}(D_0)).$$

Then a point $P \in X$ is a base point of δ

$\iff$ $s_P \in \mathfrak{m}_P \mathcal{L}_P$ for all $s \in V$.

In particular, δ is base point-free iff $\mathcal{L}(D_0)$ is generated by global sections in V.

Pf This follows from the fact that

$\forall s \in \Gamma(X, \mathcal{L}(D_0))$, $s \mapsto D = (U_i, \varphi_i(s))$

[where ~~$\phi_i : \mathcal{L}(D_0)(U_i)$~~

$\phi_i : \mathcal{L}(D_0)(U_i) \xrightarrow{\cong} \mathcal{O}_X(U_i)$]

D is an effective divisor on X so the supp D is the complement of the open set $X_s := \{x \in X \mid s_x \notin m_x \mathcal{L}(D_0)\}$.

$\square$

Remarks

We can rephrase Prop 3.2 in terms of linear systems (without base points)

$\phi : X \to \mathbb{P}^n$ is a closed immersion iff:

1. δ "separates points" i.e. $\forall P \neq Q$ on X $\exists D \in \delta$ with $P \in \operatorname{supp} D$ and $Q \notin \operatorname{supp} D$.

2. δ "separates tangent vectors"

If $P \in X$, and $t \in m_P / m_P^2$ (is a tangent vector) then $\exists D \in \delta$ s.t. $P \in \operatorname{supp} D$ but $t \notin (m_{P,D} / m_{P,D}^2)$ considering $D \subset X$ as a closed subvariety.)