Last time:

- Defined a sheaf of rings $\mathcal{O}$ on Spec A

$$U \longmapsto \mathcal{O}(U)$$

where $U \subseteq \operatorname{Spec} A$ is open and $\mathcal{O}(U)$ is the ring of functions

$$s : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

1. $s(\mathfrak{p}) \in A_{\mathfrak{p}}$

2. for all $\mathfrak{p} \in U$, there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in A$ such that $f \notin \mathfrak{q}$ for any $\mathfrak{q} \in V$ and $s(\mathfrak{q}) = \frac{a}{f}$, $\forall \mathfrak{q} \in V$.

- $(\operatorname{Spec} A, \mathcal{O})$ the spectrum of A
- $\mathcal{O}(D(f)) \cong A_f$
- $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$

Today

- Affine schemes and schemes
- Proj S
- Relation between varieties and schemes.

Defn Let A and B be two local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, resp. A ring homomorphism $\varphi: A \longrightarrow B$ is called a <u>local homo.</u> if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Defn

- A <u>ringed space</u> (RS) is a pair $(X, \mathcal{O}_X)$ such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.
- A <u>locally ringed space</u> (LRS) is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,P}$ is a local ring, $\forall P \in X$.

For example, $(\operatorname{Spec} A, \mathcal{O})$ is a LRS for any ring A.

Defn Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be RS's. A morphism of RS's from X to Y is a pair $(f, f^\#)$ where $f: X \longrightarrow Y$ is a continuous map and $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a sheaf of rings.

Remark: if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are LRS's and a morphism of RS's $(f, f^\#)$ between them, then $f^\#$ induces a ring homo. $f^\#_P: \mathcal{O}_{Y, f(P)} \longrightarrow \mathcal{O}_{X,P}$.

Indeed, if $P \in X$.

$$f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

So we have lots of homo, of the form

$$\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

We get

$$\mathcal{O}_{Y,f(P)} \longrightarrow \varinjlim_{f(P) \in U} \mathcal{O}_X(f^{-1}(U))$$
$$\downarrow$$
$$\varinjlim_{P \in V} \mathcal{O}_X(V) = \mathcal{O}_{X,P}$$

__Defn__ Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be LRS's. A __morphism of LRS's__ is a morphism of RS's $(f, f^\#)$ such that the induced map $f^\#_p$ is a local homomorphism, $\forall p \in X$.

__Prop__ (a) Let A and B be rings and $\varphi : A \longrightarrow B$ a ring homo. Then φ induces a morphism of LRS's

$$(f, f^\#) : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

(b) If we have a morphism of LRS's $(f, f^\#) : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, for some rings A, B, then $(f, f^\#)$ is induced by a ring homo. $\varphi : A \longrightarrow B$.

__proof__ (a) We have a ring homo. $\varphi : A \longrightarrow B$ and we want a continuous map $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$.

Just define $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

We know that every closed subset of $\operatorname{Spec} A$ is of the form $V(\mathfrak{a})$, for some ideal $\mathfrak{a} \triangleleft A$.

$$f^{-1}(V(\mathfrak{a})) = V((\varphi(\mathfrak{a})))$$

(check).

Now we want a sheaf of rings $f^\# : \mathcal{O}_A \longrightarrow f_* \mathcal{O}_B$.

So we will define ring homos.

$$\mathcal{O}_A(V) \longrightarrow \mathcal{O}_B(f^{-1}(V)).$$

The elements there are functions $s : V \longrightarrow \coprod_{\mathfrak{p} \in V} A_{\mathfrak{p}}$ ...

$$
\begin{tikzcd}
f(f^{-1}(V)) \arrow[r] & \coprod_{\mathfrak{p} \in f(f^{-1}(V))} A_{\mathfrak{p}} = \coprod_{\mathfrak{q} \in f^{-1}(V)} A_{f(\mathfrak{q})} \arrow[d] \\
f^{-1}(V) \arrow[u, "f"] \arrow[r] & \coprod_{\mathfrak{q} \in f^{-1}(V)} B_{\mathfrak{q}}
\end{tikzcd}
$$

We have a ring homo $\varphi : A \longrightarrow B$.

$\varphi_{\mathfrak{p}} : A_{\varphi^{-1}(\mathfrak{p})} \longrightarrow B_{\mathfrak{p}}$

Just check that this gives us what we wanted.

(b) Read in Hartshorne...

<u>Defn</u> (Affine Schemes)

- An <u>affine scheme</u> is a LRS which is isomorphic to the spectrum of some ring
- A <u>scheme</u> is a LRS $(X, \mathcal{O}_X)$ in which every point has a neighbourhood $U \subseteq X$ (open) such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

<u>Examples</u>...

1. k — field
   $\operatorname{Spec} k = \{(0)\}$.

   $\emptyset \longmapsto 0$
   $(0) \longmapsto \mathcal{O}((0))$

   $s : \{(0)\} \longrightarrow k = k_{(0)}$
   We actually get that any function $\{(0)\} \longrightarrow k$ is in $\mathcal{O}((0))$.

Defn Let X be a topological space and Z an irreducible closed subset of X. A <u>generic point</u> for Z is a point $P \in Z$ s.t. $Z = \overline{\{P\}}$.

Prop If X is a scheme, every irreducible closed subset of X has a unique generic point.

2. If k is a field, we define the affine line as $\mathbb{A}^1_k = \operatorname{Spec} k[X]$.

The generic point is (0).

If k is an alg. closed field, each closed point in $\operatorname{Spec} k[X]$ corresponds to a point in the line k.

max ideals are $(X-a)$

k

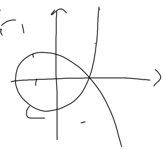
with the Zariski topology X^2+1

3. k - field, alg. closed

$$\mathbb{A}^2_k = \operatorname{Spec} k[X,Y]$$

- (0), the only generic point of $\mathbb{A}^2_k$
- if $f(X,Y) \in k[X,Y]$ is irreducible, then (f) is a prime ideals and it is a generic point for the closure of $\{(x-a, x-b) : f(a,b)=0\}$

$\mathbb{A}^2_k$,



In general, for any ring A,

we define $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \dots, x_n]$.

Proj S

Let S be a graded ring.

$$S = \bigoplus_{i \geq 0} S_i$$

$$S_i \cdot S_j \subseteq S_{i+j}$$

We will denote the ideal $\bigoplus_{i>0} S_i$ by S_+.

We define

Proj $S = \{ \mathfrak{p} \triangleleft S :$ homogeneous prime ideals which do not contain the whole of $S_+ \}$.

$V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a} \}$,

where $\mathfrak{a}$ is a homogeneous ideal of S.

Lemma 1. $\mathfrak{a}, \mathfrak{b}$ — homog. ideals, $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

2. $\{\mathfrak{a}_i\}$ family of homog. ideals, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$

Now we can define a topology on Proj S by setting the closed sets to be the sets of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is a homog. ideal of S.

We also define a sheaf of rings $\mathcal{O}$ in Proj S.

Notation: $\mathfrak{p}$ homog. prime ideal

$T_\mathfrak{p} = \{$ homog. elements of S not in $\mathfrak{p} \}$.

($\leftarrow$ this is a multiplicatively closed subset)

We localise S wrt $T_{\mathfrak{p}}$.
We define $S_{(\mathfrak{p})}$ to be the set of elements of deg 0 of $T_{\mathfrak{p}}^{-1}S$.
(The elements of $T_{\mathfrak{p}}^{-1}S$ look like $\frac{a}{b}$. The degree of $\frac{a}{b}$ is just $\deg a - \deg b$)

For any $U \subseteq \operatorname{Proj} S$ open we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$ s.t.

1. $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$.
2. for each $\mathfrak{p} \in U$, there is an open neighbourhood $V \subseteq U$ and homog. elements $f, g \in S$ of the same degree s.t. $g \notin \mathfrak{q}$, for any $\mathfrak{q} \in V$, and $s(\mathfrak{q}) = \frac{f}{g}$, $\forall \mathfrak{q} \in V$.

<u>Prop</u> Let S be a graded ring.

1. $\mathfrak{p} \in \operatorname{Proj} S \Rightarrow \mathcal{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$
2. $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$ is open and the sets of this form cover $\operatorname{Proj} S$. Also,
$$\left(D_+(f), \mathcal{O}|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$
3. $\operatorname{Proj} S$ is a scheme.

<u>Relation Varieties / Schemes</u>

X — topological space
$t(X)$ — set of irreducible closed subsets of X

<u>Some props</u>:

1. If Y is closed in X, $t(Y) \subseteq t(X)$.
2. $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, for Y_1 and Y_2 closed in X.
3. $t(\bigcap Y_i) = \bigcap t(Y_i)$, for $\{Y_i\}$ closed in X.

Define a top. on $t(X)$ by setting the closed subsets to be the sets $t(Y)$, where Y is closed in X

We define the map

$$\alpha : X \longrightarrow t(X)$$
$$p \longmapsto \overline{\{p\}}$$

α is easily seen to be continuous

Also, if $f : X_1 \longrightarrow X_2$ is continuous then we get an induced map $t(f) : t(X_1) \longrightarrow t(X_2)$.

__Prop__ Let V be an affine variety over an alg. closed field k. Then $(t(V), \alpha_*\mathcal{O}_V)$ is isomorphic to Spec A, where A is the affine coordinate ring of V.

__Defn__ A __scheme X over a scheme S__ is just a scheme X with a morphism $X \longrightarrow S$.

$\mathcal{V}ar(k)$ – category of varieties over k

$\mathcal{S}ch(k)$ – cat. of schemes over k (over Spec k).

__Prop__ Let k be an alg. closed field. The map $t : \mathcal{V}ar(k) \longrightarrow \mathcal{S}ch(k)$ is a functor. Also, any variety V is homeo. to the subset of closed point of $t(V)$ and its associated sheaf is given by restricting $\alpha_*\mathcal{O}_V$ wrt the homeomorphism.

__e.g.__ V – affine variety
$t(V) \cong$ Spec A

Let's just define $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$ for any ring A