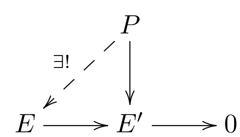
Modular representation theory

1 Definitions for the study group

Definition 1.1. Let A be a ring and let F_A be the category of all left A-modules. The *Grothendieck group* of F_A is the abelian group defined by generators and relations as follows:

1. Generators $[E]$ for $E \in F_A$.
2. Relations: for $E, E', E'' \in F_A$ if $0 \to E \to E' \to E'' \to 0$, then $[E'] = [E] + [E'']$.

Definition 1.2. A left A-module P is called *projective* if any of the following hold:

1. There exists a free A-module F such that $F = P \oplus Q$ for $Q \in F_A$
2. 

3. The functor $\mathrm{Hom}_A(P, -)$ is exact.

Definition 1.3. An A-module homomorphism $f : M \to M'$ is called *essential* if $f(M) = M'$ and $f(M'') \neq M'$ for any proper submodule $M'' \subseteq M$.

Definition 1.4. The *projective envelope* of a module M is a projective module P and an essential homomorphism $f : P \to M$.

Setup
Let G be a finite group, K be a complete field of characteristic 0 with a discrete valuation v, with residue field k of characteristic > 0. Let A be the ring of integers of K. We are going to look at $K[G], k[G]$and $A[G]$.

Definition 1.5. Let L be a field, denote by $R_L(G)$ the Grothendieck group of finitely generated $L[G]$-modules. We make it into a ring by setting $[E] \cdot [E'] = [E \otimes_L E']$.
Let S_L be the set of isomorphism classes of simple $L[G]$-modules.

Proposition 1.6. *The element of S_L form a basis for $R_L[G]$.*

Definition 1.7. Let $P_k[G]$ be the Grothendieck ring of the category of left $k[G]$-modules which are projective.
Let $P_A[G]$ be the Grothendieck ring of the category of the left $A[G]$-modules which are projective.

Fact. *Any $k[G]$-module has a projective envelope*
If E, P are $k[G]$-modules and P is projective, then $E \otimes_k P$ is also projective $\Rightarrow$ we have an action of $R_k[G]$ on $P_k[G]$.

Proposition 1.8. *Each projective $k[G]$-module can be written uniquely as a direct sum of indecomposable projective $k[G]$-modules*
Let P and P' be projective $k[G]$-modules, then they are isomorphic if and only if $[P] = [P']$.

Dualities: Let E, F be $K[G]$-modules. Set $\langle E, F\rangle = \dim \operatorname{Hom}_G(E, F)$ (i.e, $\langle\, ,\,\rangle : R_K[G] \times R_K[G] \to \mathbb{Z}$). This is bilinear with respect to short exact sequences.

Proposition 1.9. *If $E, E' \in S_K$ then* $\langle E, E'\rangle = \begin{cases} 0 & E \ncong E' \\ d_E = \dim \operatorname{End}_G(E) & E \cong E' \end{cases}$.

Definition 1.10. Say E is *absolutely simple* if $d_E = 1$.

We define $\langle\, ,\,\rangle : P_k[G] \times R_k[G] \to \mathbb{Z}$ by $(E, F) \mapsto \langle E, F\rangle = \dim \operatorname{Hom}_G(E, F)$. Since E is projective, $\langle\, ,\,\rangle$ is bilinear.

Proposition 1.11. *If $E, E' \in S_k$ then* $\operatorname{Hom}_G(P_E, E') = \operatorname{Hom}_G(E, E')$.

If $E, E' \in S_k$ then $\langle P_E, E'\rangle = \begin{cases} 0 & E \ncong E' \\ d_E & E \cong E' \end{cases}$

Proposition 1.12. *Let K be sufficiently large such that k contains all m-th roots of unity* ($m = \operatorname{lcm}(|g|)_{g\in G}$). *Then $d_E = 1$ for $E \in S_k$ and hence $\langle\, ,\,\rangle$ is non-degenerate and the base $[E], [P_E]$ are dual to each other.*

2 The cde Triangle

We will want to define three maps for the following triangle to commute

$$\begin{array}{ccc} P_k(G) & \xrightarrow{c} & R_k(G) \\ & \searrow^{e} & \uparrow d \\ & & R_K(G) \end{array}$$

Definition 2.1. *Cartan Homomorphism.* To each $k[G]$-module P, we associate the class of P, $[P]$ in $R_k(G)$. We get a homomorphism $c : P_k(G) \to R_k(G)$. We express c in terms of the canonical basis $[P_S], [S]$ for $S \in S_k$ (basis for $P_k(G)$ and $R_k(G)$ respectively). We obtain a square matrix C, of type $S_k \times S_k$, the *Cartan Matrix* of G with respect to k. The (S, T) coefficient, C_{ST} of C is the number of times that S appears in a composition series of the projective envelope P_T of T.

$$[P_T] = \sum_{S \in S_k} C_{ST}[S] \,\text{in}\, R_k(G)$$

Definition 2.2. *Decomposition Homomorphism.* Let E be a $K[G]$-module. Choose a lattice E_1 in E (finitely generated A-module which generates E as a K-module). We may assume that E_1 is G-stable (replace E_1 by the sum of its images under G). Denote the reduction $\overline{E}_1 = E_1/\mathfrak{m}E_1$, a $k[G]$-module.

Theorem 2.3. *The image $\overline{E}_1$ in $R_k(G)$ is independent of the choice of E_1.* (Warning! You can have that $\overline{E}_1 \ncong \overline{E}_2$ but with the same composition factors)

Proof. Let E_2 be another G-stable lattice, we want to show $\left[\overline{E}_1\right] = \left[\overline{E}_2\right]$ in $R_k(G)$. Replace E_2 by a scalar multiple, we may assume $E_2 \subset E_1$. There exists $n \geq 0$ such that $\mathfrak{m}^n E_1 \subset E_2 \subset E_1$.

If $n = 1$, we have $\mathfrak{m}E_1 \subset E_2 \subset E_1$. We have

$$0 \to E_2/\mathfrak{m}E_1 \to E_1/\mathfrak{m}E_1 \to E_1/E_2 \to 0$$

implies $[E_1/\mathfrak{m}E_1] = [E_2/\mathfrak{m}E_1] + [E_1/E_2]$. And

$$0 \to \mathfrak{m}E_1/\mathfrak{m}E_2 \to E_2/\mathfrak{m}E_2 \to E_1/\mathfrak{m}E_1 \to 0$$

implies $[E_2/\mathfrak{m}E_2] = [\mathfrak{m}E_1/\mathfrak{m}E_2] + [E_2/\mathfrak{m}E_1]$. But $[\mathfrak{m}E_1/\mathfrak{m}E_2] = [E_1/E_2]$ hence $[\overline{E}_1] = [\overline{E}_2]$. This covers our base case.

For general n, let $E_3 = \mathfrak{m}^{n-1}\mathfrak{m} + E_2$. We have $\mathfrak{m}^{n-1}E_1 \subset E_3 \subset E_1$, $\mathfrak{m}E_3 \subset E_2 \subset E_3 \Rightarrow [\overline{E}_2] = [\overline{E}_3] \Rightarrow \left[\overline{E}_1\right] = \left[\overline{E}_2\right] = \left[\overline{E}_3\right]$ ∎

The map $E \to [\overline{E}_1]$ extends to a ring homomorphism $d : R_K(G) \to R_k(G)$. The corresponding matrix is denoted D, type $S_k \times S_K$. For $F \in S_k$, $E \in S_K, D_{FE}$ is the number of times that F appears in the reduction mod $\mathfrak{m}$ of E_1

$$[\overline{E}_1] = \sum_F D_{FE}[F] \,\text{in}\, R_k(G).$$

Definition 2.4. *e.*

There exists a canonical isomorphism $P_A(G) \xrightarrow{\sim} P_k(G)$, i.e.,

- Let E be an $A[G]$-module, for E to be a projective $A[G]$-module then (only then) E is a to be free on A and $\overline{E}$ is a projective $k[G]$-module
- If F is a projective $k[G]$-module, then there exists a unique (up to isomorphism) projective $A[G]$-module whose reduction modulo $\mathfrak{m}$ is isomorphic to F.

The functor "tensor product with K" defines a homomorphism from $P_A(G)$ into $R_K(G)$. Combining both, we get a map $e : P_k(G) \to R_K(G)$ and an associated matrix E of type $S_K \times S_k$.

2.1 Basic properties of cde triangle

1. Commutativity: $c = d \circ e$, i.e., $C = DE$
2. d and e are adjoint of one another with respect to bilinear forms on $R_k(G), P_k(G)$. I.e., pick $x \in P_k(G)$ and $y \in R_K(G)$ then $\langle x, d(y)\rangle_k = \langle e(x), y\rangle_K$.
   Assume $x = [\overline{X}]$ where X is a projective $A[G]$-module, $y = [K \otimes_A Y]$ where Y is an $A[G]$-module which is free. Then the A-module $H_G(X, Y)$ is free. Let r be its rank: we have the canonical isomorphism $K \otimes \mathrm{Hom}_G(X, Y) = \mathrm{Hom}_G(K \otimes X, K \otimes Y)$ and $k \otimes \mathrm{Hom}_G(X, Y) = \mathrm{Hom}_G(k \otimes X, k \otimes Y)$. These both implies that $\langle e(x), y\rangle_K = t = \langle x, d(y)\rangle_k$.
3. Assume that K is sufficiently large. The canonical bases of $P_k(G)$ and $R_k(G)$ are dual to each other with respect to $\langle\, ,\, \rangle_k$. (Similarly $R_K(G), R_K(G)$ and $\langle\, ,\, \rangle_K$) Hence e can be identified with the transpose of d. $E = {}^tD$, $C = DE \Rightarrow C$ is symmetric.

3 Examples

Notation.

- K is a complete field
- A is the valuation ring of K, $\mathfrak{m}$ its maximal ideal
- k the residue field of K, with characteristic p
- K is sufficiently large, that is $\mu_n \subseteq K$ with $n = \mathrm{lcm}_{g \in G}(o(g))$

We have the cde triangle

$$\begin{array}{ccc} P_k(G) & \xrightarrow{c} & R_k(G) \\ & \searrow^{e} & \uparrow{\scriptstyle d} \\ & & R_K(G) \end{array}$$

- $c : P \to [P]$
- $d : E \to \overline{E}_1 = E_1/\mathfrak{m}E_1$ where E_1 is a stable lattice in E under G
- e: is the inverse of the canonical isomorphism $P_A(G) \to P_k(G)$ tensor with K

Definition 3.1. A *projective envelope* P of a L[G] module M is a projective module P with an essential homomorphism.

3.1 Example, p'-groups

Proposition 3.2. *Assume that the order of G is coprime to p. Then*

1. *Each $k[G]$-module (respectively each A-free $A[G]$-module) is projective*
2. *The operation of reduction $\mod \mathfrak{m}$ defines a bijection between S_K and S_k*
3. *If we identify S_K with S_k, then C, D, E are all identity matrices.*

Proof.

1. Let E be an A-free $A[G]$-module, then there is a free $A[G]$-module L such that E is L/R. Since E is A-free there exists a A-linear projection π of L onto R. If $g = \#G$, we know $\gcd(g, p) = 1$, hence we can replace π with $\frac{1}{g}\sum_{s\in G} s\pi s^{-1}$ and we get an $A[G]$-projection. By the fact $A[G]$-linear projection and the fact that $L/R \cong E$, we get that E is projective. Moreover $C = \mathrm{Id}$
2. and 3. Let $E \in S_k$, the projective envelope E_1 of E relative to $A[G]$ is a projective $A[G]$-module, whose reduction $\overline{E}_1 = E_1/\mathfrak{m}E_1$ is E. If $F = K \otimes E_1$, then $d([F]) = [E]$. Since E is simple, that implies that F is simple, thus $F \in S_K$. So, we obtain a map $E \to F$ of S_k to S_K which is the inverse of d.

□

3.2 Example, p-groups

Let G be a p-group with order p^n. Then

Proposition 3.3. *Let V be a vector space over k and $\rho : G \to \mathrm{GL}(V)$ a linear representation. There exists a non-zero element in V which is fixed by $\rho(s)$, for all $s \in G$.*

Corollary 3.4. *The only irreducible representation of a p-group in characteristic p is the trivial one.*

The Artinian ring $k[G]$ is a local ring with restriction field k.

The projective envelope of the simple $k[G]$-module k is $k[G]$.

The groups $R_k(G)$ and $P_k(G)$ can be identified with $\mathbb{Z}$ and C is multiplication by p^n. The map $d : R_K(G) \to \mathbb{Z}$ corresponds to the K-ranks, $e : \mathbb{Z} \to R_K(G)$ maps an integer n onto n times the class of the regular representation of G.

3.3 (p'-groups) $\times$ (p-groups)

Let $G = S \times P$ where S is a p'-group and P is a a p-groups. We have that $k[G] = k[S] \otimes k[P]$.

Proposition 3.5. *A $k[G]$-module E is if and only if P acts trivially on E.*

Proof. $\Leftarrow$) That follows from the fact every $k[S]$-module is semisimple.

$\Rightarrow$) Assume E to be simple. From the above the subspace E' of E consisting of elements fixed by P is not zero. Since P is normal, E' is stable under G, hence as E is simple, $E' = E$. □

Proposition 3.6. *A $k[G]$-module E is projective if and only if it is isomorphic to $F \otimes k[P]$ where F is a $k[S]$-module.*

Proof. Since F is a projective $k[S]$-module, then $F \otimes k[P]$ is projective $k[G]$-module. We can see that F is the largest quotient of $F \otimes k[P]$ on which P acts trivially. By the previous lectures $F \otimes k[P]$ is the projective envelope of F. However, every projective module is the projective envelope of its largest semisimple quotient. Thus we see that every projective module has the form $F \otimes k[P]$. □

Proposition 3.7. *An $A[G]$-module $\widetilde{E}$ is projective if and only if it is isomorphic to $\widetilde{F} \otimes A[P]$ where $\widetilde{F}$ is an A-free $A[G]$-module.*

4 More properties of the *cde* triangle

Notation.

- K is a complete field
- A is the valuation ring of K, $\mathfrak{m}$ its maximal ideal
- k the residue field of K, with characteristic p
- $$\begin{array}{ccc} P_k(G) & \xrightarrow{c} & R_k(G) \\ & \searrow^{e} & \uparrow d \\ & & R_K(G) \end{array}$$
- We say K is sufficiently large to mean that $\mu_n \subseteq K$ with $n = \mathrm{lcm}_{g\in G}(o(g))$

4.1 Brauer's Theorem (in the modular case)

Let $H \leq G$ be groups, we want to define homomorphism $R_K(G) \to R_K(H)$ and $R_K(H) \to R_K(G)$ (respectively R_k and P_k). Any $K[G]$-module is defines as a $K[H]$-module through restriction, and since it is projective if the module is projective, we have a *restriction map* $\mathrm{Res}_H^G : R_K(G) \to R_K(H)$ (respectively R_k and P_k)

Let E be a $K[H]$-module, then $K[G] \otimes_{k[H]} E$ is a $K[G]$-module, and is projective if E is projective. Hence we have an *induction map* $\mathrm{Ind}_H^G : R_K(H) \to R_K(G)$ (respectively R_k and P_k).

Fact. $\mathrm{Ind}_H^G(x \cdot \mathrm{Res}_H^G y) = \mathrm{Ind}_H^G(x) \cdot y$.

c, d and e commute with Res_H^G and Ind_H^G .

Theorem 4.1. *Let X be the set of all elementary subgroups of G. The homomorphisms*

$$\mathrm{Ind} : \bigoplus_{H\in X} R_K(H) \to R_K(G)$$

$$\mathrm{Ind} : \bigoplus_{H\in X} R_k(H) \to R_k(G)$$

$$\mathrm{Ind} : \bigoplus_{H\in X} P_k(H) \to P_k(G)$$

defined by the Ind_H^G are all surjective.

Note. A similar theorem is also true if we don't assume K to be sufficiently large, but we will not use this.

Corollary 4.2. *Each element of $R_K(G)$ (respectively $R_k(G)$, $P_k(G)$) is a sum of elements of the form $\mathrm{Ind}_H(\gamma_H)$, where H is an elementary subgroup of G and $\gamma_H \in R_K(H)$ (respectively $R_k(H)$, $P_k(H)$).*

4.2 Surjectivity of d

Theorem 4.3. *The homomorphism $d : R_K(G) \to R_k(G)$ is surjective.*

Note. This is true for all K, but we will prove it only in the case K is sufficiently large.

Proof. $R_k(G)$ can be express as a sum of Ind_H^G with H elementary, and d commutes with Ind_H^G. Hence we just need to show $R_k(H) = d(R_K(H))$, i.e., we can assume that G is elementary.

Let $G = P \times H$ where P is a p-group and H has order coprime to p. As $S_k(G)$ forms a basis of $R_k(G)$, we just need to show that given E a simple $k[G]$-module it lies in the image of d. Let

$$E' = \{v \in E : gv = v \forall g \in P\}$$

which as P is a p-group using Theorem 3.3. As P is normal in G, we have that E' is stable under G, hence is a $k[G]$-submodule of E. But E is simple, hence $E = E'$, i.e. P acts trivially on E, i.e., the action of $k[G]$ on E factors through the projection map $k[G] \to k[H]$, i.e., E comes from $F \in R_k(H)$. But H as order prime to p, so by Proposition 3.2, we can find the lift of F and view it as a $K[G]$-module through the projection map $K[G] \to K[H]$. This is the lift of E. □

Theorem 4.4. *The homomorphism $e : P_k(G) \to R_K(G)$ is a split injection* (i.e., there exist $r : R_K(G) \to P_k(G)$ such that $r \circ e = 1$, or equivalently, e is injective and $e(P_k(G))$ is a direct factor of $R_K(G)$)

Proof. As we are assuming that K is large enough, we know $E = D^t$ hence e is injective follows from the fact that d is surjective. □

Corollary 4.5. *Let P and P' be projective $A[G]$-modules. If the $K[G]$-modules $K \otimes P$ and $K \otimes P'$ are isomorphic, then $P \cong P'$ as $A[G]$-modules.*

Proof. e is injective. □

4.3 Characterisation of the image of c

Theorem 4.6. *Let $\mathrm{ord}_p(|G|) = n$. Then every element of $R_k(G)$ divisible by p^n belongs to the image of $c : P_k(G) \to R_k(G)$.*

Note. This is true for all K, but we will prove it only in the case K is sufficiently large.

Proof. As in Theorem Theorem 4.3, as K is sufficiently large we can assume G is elementary, i.e., $G = P \times H$. The theorem is equivalent to showing that the cokernel $c : P_k(G) \to R_k(G)$ is killed by p^n. But as seen in the example 3.2 (first two theorems), the matrix C is the scalar matrix p^n. Hence we see that the cokernel c must be killed by p^n. □

Corollary 4.7. *The cokernel c is a finite p-group and c is injective.*

Proof. The fact that the cokernel is a finite p-group is clear. Then since the cokernel is finite and $P_k(G)$ and $R_k(G)$ are free $\mathbb{Z}$-modules of the same rank, as $\mathrm{im} c$ must have the same rank as $R_k(G)$ we get c is injective. □

Corollary 4.8. *If two projective $k(G)$-modules have the same composition factors (with multiplication), they are isomorphic*

Corollary 4.9. *Let K be sufficiently large. The matrix C is symmetric, and the corresponding quadratic form is positive definite. The determinant of C is a power of p.*

4.4 Characterisation of the image of e

Definition 4.10. An element $g \in G$ is said to be *p-singular* if $p | \mathrm{ord}(g)$.

Theorem 4.11. *Let K' be a finite extension of K. An element of $R_{K'}(G)$ is in the image e of $P_A(G) = P_k(G)$ if and only if its character take values in K, and is zero on the p-singular element of G.*

5 Characterisation of Projective $A[G]$-modules by their characters

Recall:

- Let K be a field, complete with a discrete valuation $\mathrm{char}(K) = 0$
- A a valuation ring, with unique maximal ideal $\mathfrak{m}$, $k = A/\mathfrak{m}$ its residue field with $\mathrm{char}(k) = p > 0$
- G a finite group, $g \in G$ is "p-singular" if $p|\mathrm{ord}(g)$
- $S_{F[G]} = \{s : s \text{ a simple } F[G]\text{-module}\}$, $F \in \{K, k, A\}$

Technical

- By the character of an $A[G]$-module E, we mean the character of $K \otimes E$
- A $K[G]$-module "comes from" the $A[G]$-module E if $V = K \otimes E$

Fact.

1. *Let F be a field, V an $F[G]$-module, S a simple $F[G]$-module, denoted by $V_{(S)}$ for the sum of all submodules of $V \cong S$. So if V is semisimple, then $V = \oplus_{s\in S_{F[G]}} V_{(S)}$*
2. *If $I \lhd G$, and an $F[I]$-module V, with rep $\rho : I \to \mathrm{Aut}(V)$, then for $g \in G$ then ${}^g\rho : I \to \mathrm{Aut}(V)$, $x \in I \mapsto \rho(g^{-1}xg)$ corresponding module ${}^gS$*
3. *If K is sufficiently large, then each simple $K[G]$-module is absolutely simple (i.e., simple as a $\mathbb{C}[G]$-module)*
4. *If S is a simple $F[G]$-module ($F \in \{K, A\}$), then $\mathrm{End}_G(S) = F^*$. (Schur's lemma)*
5. *E a semi-simple $F[G]$-module, $I \lhd G$, then $E\downarrow_I$ is semisimple (Clifford)*
6. *$p \nmid |G|$, then every representation of G over k can be lifted to a representation of G/A.*
7. *All simple $k[P]$-modules are trivial, where P is a p-group*
8. *Since $A/\mathfrak{m} = k$ has characteristic p, A^* can't contain any elements of order p*
9. *$p \nmid |G|$ implies that reduction mod $\mathfrak{m}$ contains a bijection $S_{k[G]} \to S_{A[G]}$.*

Aim: A $K[G]$-modules comes from a projective $A[G]$-modules if and only if its character has some properties.

Definition 5.1. If G has a series $1 = L_0 \lhd L_1 \lhd \cdots \lhd L_n = G$ where each $L_i \lhd G$, and L_i/L_{i-1} is either a p-group or a p'-group. Then G is p-soluble of height n.

Example. All soluble groups are p-soluble
All subgroups and quotient groups of a p-soluble group are p-soluble.

Theorem (Fang-Swan)**.** *Let G be p-soluble and K sufficiently large. Then every simple $k[G]$-module is the reduction mod m of an $A[G]$-module (necessarily simple)*

Corollary 5.2. *Let G be a p-soluble, K sufficiently large, then a representation V of G over K comes from a projective $A[G]$-module if and only if its character χ_V is 0 on the p-singular elements of G*

Proof. Follow from Fran-Swan and Theorem 4.11 □

Proof. (Of Fang-Swan) We do it by induction on the p-soluble height of G, and on $|G|$ for p-soluble groups of height h

The case $h = 0$ is trivial, so assume $h \leq 1$, so G has a series $1 = L_0 \lhd L_1 \lhd \cdots \lhd L_n = G$. Let $I = L_1$, so I is a p-group or a p'-group and height of $G/I \leq h - 1$. Let E be a simple $k[G]$-module

Step 1 We may assume I is a p'-group

By 5), $E\downarrow_I$ is semisimple and hence by 7), I acts trivially on E so we can view E as a $k[G/I]$-module and then the result follows from induction

Step 2 We may assume that $E\downarrow_I$ is isotopic

Write $E\downarrow_I = \oplus_{\overline{s}\in S_{K[I]}} E\downarrow_{I_{(\overline{S})}}$ and let $G_{(\overline{s})} = \mathrm{Stab}_G(E\downarrow_{I_{(\overline{s})}})$. Since E is simple, G acts transitively on $\{E\downarrow_{I_{(\overline{s})}} : \overline{s}\in S_{k[I]}\}$. Then $E\downarrow_{I_{(s)}}$ is a $k[G_{(s)}]$-module, and $E = (E\downarrow_{I(\overline{s})})\uparrow^G_{G_{(\overline{s})}}$. If $E_{(s)} \subsetneq E\downarrow_I$, then $G_{(\overline{s})}$ is a proper subgroup of G, so by induction, $E_{(\overline{s})}$ can be lifted. But since E is the induced module and the operations of induction and reduction mod$\mathfrak{m}$ commute, E can be lifted.

We assume for the rest of the proof that I is a p'-group and $E\downarrow_I = E_{(\overline{s})}$ for a simple $k[I]$-module $\overline{s}$. By 6) $\overline{s}$ can be lifted to an $A[I]$-module s. Let $\rho : I \to \mathrm{Aut}(s)$ be the associated representation. By 3), $K\otimes s$ is absolutely simple. So $\dim(s) = \dim(K\otimes s)$ divides $|I|$. So $p \nmid \dim(s)$ (†).

Step 3 There exists a finite group G_2 such that

- there exists $N \lhd G_2$ such that N is cyclic of p'-order and $G/N \cong G$.
- I can be embedded in a normal subgroup of G_2 with $I\cap N = 1$
- There exists a representation $\rho_2 : G_2 \to \mathrm{Aut}(S)$ extending

Since $E = E_{(\overline{s})}$, ${}^g\overline{s} = \overline{s}$ for all $g\in G$ so ${}^g s = s$ for all $g \in G$. Thus for all $g\in G$ there exists $t\in \mathrm{Aut}(s)$ such that $t\rho(x)t^{-1} = \rho(g^{-1}xg)$ for all $x\in I$. Let $U_g = \{t\in \mathrm{Aut}(s),\ t\rho(x)t^{-1} = \rho(g^{-1}xg)\forall x\in I\}$. So $U_g \neq \emptyset$ for all $g\in G$. Set $G_1 = \{(g,t) : g\in G, t\in U_g\}$, it is easy so see that G_2 is a group. Also $\ker(G\to G) = A^*$. By (†) if $d = \dim(s)$, then $\{\det(t) : t\in U_g\}$ is a coset of A^{*d} in A^* for any $g\in G$. By enlarging K (which is ok, since it doesn't change $R_K(G)$) we may assume that all of these cosets are trivial. So for all $g\in G$, there exists $t\in U_g$ such that $\det(t) = 1$. Let $C = \{\det\rho(x) : x\in I\} \le A^*$, $G_2 = \{(g,t)\in G_1 : \det(t)\in C\} \le G_1$. By the above $G_2 \twoheadrightarrow G$ and $\ker(G_2 \twoheadrightarrow G) := N \cong \{\alpha\in A^* : \alpha^d \in C\}$. Also $I \hookrightarrow G_2$ by $x\mapsto (x,\rho(x))$. Finally, the last point holds by defining $\rho_2 : G_2 \to \mathrm{Aut}(s)$, $(s,t)\mapsto t$.

Step 4 Let $F = \mathrm{Hom}_I(\overline{s}, E)$. Then $\overline{s}\otimes F$ is a $k[G_2]$-module, and $\overline{s}\otimes F \cong_{G_2} E$

G_2 acts on $\overline{s}$ by reduction of ρ_2 and on E since $G_2/N \cong G$. Hence G_2 acts $\overline{s}\otimes F$. $u : \overline{s}\otimes F \to E$ defined by $a\otimes b \mapsto b(a)$. This is easy to see is an isomorphism.

Step 5 F can be lifted to an $A[G_2]$-module $\widetilde{F}$

Skip

Now, we've already shown that $\overline{s}$ can be lifted (to s). By step 5, $\widetilde{E} = s\otimes\widetilde{F}$ is an $A[G_2]$-module which reduces to E. But since N acts trivially on E, and E is a simple $k[G_2]$-module, by 9) N acts trivially on $\widetilde{E}$. Hence $\widetilde{E}$ is a $A[G_2/N]$-module.

□

6 Modular/Brauer Character

Fix K sufficiently large, i.e., contains μ_n where $n = \mathrm{lcm}_{g\in G}(\mathrm{ord}(g))$

Call this μ_L reduction mod p gives an isomorphism onto μ_R

Definition 6.1. Brauer Character of a $R[G]$-module. Let E be an n-dimensional $R[G]$-module, let $s\in G_{\mathrm{reg}}$, let s_E be the associated automorphism of E. We may diagonalise S_E with eigenvalues in μ_R call these λ_i and their lifts to μ_K, $\widetilde{\lambda}_i$. Let $\phi_E(s) = \sum_{i=1}^n \widetilde{\lambda}_i$. Then $\phi_E : G_{\mathrm{reg}} \to A$ is the Brauer character of E.

Properties

1. $\phi_E(e) = n$

2. $\phi_E(tst^{-1}) = \phi_E(s)$ for all $t \in G$, $s \in G_{\text{reg}}$
3. $0 \to E \to E' \to E'' \to 0$, $\phi_{E'} = \phi_E + \phi_{E'}$
4. $\phi_{E_1 \otimes E_2} = \phi_{E_1} \times \phi_{E_2}$.

"New Properties"

5. Let $t \in G$ with p-regular component $s \in G_{\text{reg}}$, t_E its associated endomorphism. $\text{Tr}(t_E) = \overline{\phi_E(s)}$.
6. F a $K[G]$-module with K-character χ, E its associated $k[G]$-module with Brauer character ϕ_E. Then $\phi_E = \chi$ on G_{reg}.
7. F a projective $k[G]$-module, $\widetilde{F}$ a lift of F to a projective $A[G]$-module. Let Φ_F be the K character of $k \otimes \widetilde{F}$, let E be any $k[G]$-module. $E \otimes F$ is projective $\Phi_{E \otimes F}(s) = \begin{cases} \phi_E(s)\Phi_F(s) & s \in G_{\text{reg}} \\ 0 & \text{else} \end{cases}$
8. $\dim \text{Hom}(F, E) = \langle F, E \rangle_k = \frac{1}{|G|} \sum_{g \in G_{\text{reg}}} \Phi_F(s^{-1})\phi(s) = \langle \phi_E, \Phi \rangle$

Note. $\dim F = \langle \mathbb{I}, \Phi_F \rangle := \frac{1}{|G|} \sum_{g \in G_{\text{reg}}} \Phi_F(g)$.

Theorem. *The irreducible modular character ϕ_E ($E \in \delta_k$) forms a basis of the K-vector space of class functions on G_{reg} with values in K.*

Corollary. *If F and F' are two $k[G]$-module and $\phi_F = \phi_{F'}$ then $[F] = [F']$ in $R_k(G)$.*

Corollary. $\ker d : R_K(G) \to R_k(G)$ *consists of the elements whose characters are* 0 *on* G_{reg}.

Corollary. *The number of isomorphism class of simple $R[G]$-module = the number of p-regular conjugacy class of G*

Example. S_4

	e	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	-1

$p = 2$: p-regular conjugates are e and (123).

	e	(123)
ϕ_1	1	1
ϕ_2	2	-1

. Then $D = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$. $\Phi_1 = \chi_1 + \chi_2 + \chi_4 + \chi_5$ and $\Phi_2 = \chi_3 + \chi_4 + \chi_5$. Finally $C = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, $\Phi_1 = 4\phi_1 + 2\phi_2$ on $G_{\text{reg}}$and $\Phi_2 = 2\phi_1 + 3\phi_2$ on G_{reg}.

We see that C, D and E gives us relations between χ_i, Φ_i and ϕ_i. To see this, we note that after tensoring with K, the cde triangle becomes

$$\begin{array}{ccc} K \otimes P_k(G) & \xrightarrow{K \otimes c} & K \otimes R_k(G) \\ & \searrow^{K \otimes e} & \uparrow{\scriptstyle K \otimes d} \\ & & K \otimes R_K(G) \end{array}$$

this gives:

- $\chi_F = \sum_{e \in S_k} D_{EF}\phi_E$ on G_{reg}
- $\Phi_E = \sum_{F \in S_K} D_{EF}\chi_F$ on G
- $\Phi_E = \sum_{E' \in S_K} C_{E'E}\phi_{E'}$ on G_{reg}

we now have the following orthogonality $\langle \Phi_E, \phi_{E'} \rangle = \delta_{EE'}$

7 Brauer Character II

Let K, A, k as previously.

Example. A_5 has character table (in char 0)

	e	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$p = 2$ We have 4 2-regular conjugacy classes

	e	(123)	(12345)	(13524)
ϕ_1	1	1	1	1
ϕ_2	2	-1	$\frac{1+\sqrt{5}}{2}-1$	$\frac{1-\sqrt{5}}{2}-1$
ϕ_3	2	-1	$\frac{1-\sqrt{5}}{2}-1$	$\frac{1+\sqrt{5}}{2}-1$
ϕ_4	4	1	-1	-1

$\chi_3 + \chi_2 = \chi_1 + \chi_5$ on G_{reg}, hence $\chi_3|_{G_{\text{reg}}}$and $\chi_2|_{G_{\text{reg}}}$ are not irreducible. We have $\chi_3 - \chi_2$ is a Brauer character of an simple $R[G]$-module and so is $\chi_3 - \chi_1$.

Hence $D = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7.1 Cool stuff

Theorem 7.1. *Orthogonality Relations for Brauer Characters*

Let ϕ_i denote the Brauer character of a simple $k[G]$-module and let η_j denote the Brauer character of an indecomposable projective $k[G]$-module. $C = (c_{ij})$, $C^{-1}(\gamma_{ij})$

1. $\sum_{g \in G_{\text{reg}}} \phi_i(g)\eta_j(g^{-1}) = |G|\,\delta_{ij}$
2. $\sum_{g \in G_{\text{reg}}} \phi_i(g)\phi_j(g^{-1}) = |G|\,\gamma_{ij}$
3. $\sum_{g \in G_{\text{reg}}} \eta_i(g)\eta_j(g^{-1}) = |G|\,c_{ij}$

Proof. C is invertible (it is injective and its cokernel is a finite p-group). Let Z be a matrix given by the characteristic 0 character table, let W be the matrix given by the characteristic p character table, H the matrix given by the projective Invertible module character table mod p. Using the CDE triangle, $Z = DW$ and $H = CW$. ${}^tZZ = |G|\,h_K^{-1}\delta_{K\ell^*}$ where h_i is the size of the ith conjugacy class and $\delta_{ij^*} = \begin{cases} 1 & \text{if } g \in C_i \Rightarrow g^{-1} \in C_j \\ 0 \end{cases}$. Rewire i) as a sum over conjugacy classes

$$\begin{aligned} \sum_{k=1}^{r} h_k\phi_i(c_k)\eta_j^*(c_k) &= \sum\sum h_k\delta_{k\ell^*}\phi(C_k)\eta_j^*(C_\ell) \\ &= \frac{1}{|G|}W({}^tZZ)^{-1}\,{}^tH = \text{Id} \end{aligned}$$

Similarly:
ii) $W({}^tZZ)^{-1}W = C^{-1}$
iii) $H({}^tZZ)^{-1}\,{}^tH = C$ □

Definition 7.2. Let $\chi \in \mathrm{Irr}(G)$, its p-defect is $\mathrm{ord}_p(|G|/\chi(1))$

Proposition 7.3. *Let $\chi \in \mathrm{Irr}(G)$ with p-defect 0. Then χ is in fact a character of $K \otimes P$ where P is a* PIM *over $A[G]$. Moreover $\overline{P}$ is a simple and projective as a $R[G]$-module*

Proof. Let M be the simple $K[G]$-module with character χ, its corresponding idempotent $e = \frac{\chi(1)}{|G|}\sum_{x \in G}\chi(x^{-1})x \in K[G]$ is in fact defined over A. Let P_i be a PIM over $A[G]$ then the map e maps this to a $k[G]$-module. $\langle k \otimes P_i, \chi\rangle_K = \langle \eta_i, d(\chi)\rangle_R = \langle \eta_i, d_{ij}\phi_i\rangle = d_{ki}$. So in particular χ is a summand of one such, say $K \otimes P$. So $eP \neq 0$, either $eP = P$ or we have a decomposition $P = eP \oplus (1-P)P$. This is not possible as P is indecomposable, so $eP = P$. Hence $e(K \otimes P) = K \otimes P$, so $K \otimes P = \alpha\chi$. So χ must vanish on all P irregular/singular classes. It follows ($K \otimes P_i$ form a basis of such class function). So $\chi = \sum n_i(K \otimes P_i)$ but $\chi = \alpha^{-1}(K \otimes P)$ so $\alpha = 1$ and $\chi = K \otimes P$. Why is $\overline{P}$ simple? Because exactly one $d_{ij} \neq 0$ and is in fact to 1 but $C = {}^tDD$, hence $\overline{P}$ is simple. ☐

Steinburg character of $\mathrm{SL}_n(q)$. B is the subgroup of upper triangular matrices, $G = B \cup BxB$ for some $x \notin B$.

$$\begin{aligned} \left\langle \mathbb{I}_B \uparrow^G, \mathbb{I}_B \uparrow^G \right\rangle &= \left\langle \mathbb{I} \uparrow_B^G \downarrow_B, \mathbb{I} \right\rangle \\ &= \#_B \backslash G /_B = 2 \end{aligned}$$

So $\mathbb{I}_B \uparrow^G - \mathbb{I} = \chi$. χ is irreducible with degree $|G/B| - 1 = q$. But $|G| = q(q^1 - 1)$ so χ is q-defect 0.

8 Introduction to Block Theory

Let p be a fixed rational primes, K, A, k as before. $\pi : \mathcal{O}_K \to k = \mathcal{O}_K/P$ quotient map where P is a prime of $\mathcal{O}_K$ above p. $K_1 = \sum g$ class sum for any class C_i. $\mathrm{Irr}(G)$ = character 0 irreducible class, $\mathrm{Br}(G)$=irreducible Brauer class for prime p.

Definition 8.1. Let $\chi \in \mathrm{Irr}(G)$ afforded by ρ. Then for all $z \in Z(K[G])$, $\rho(z) = \epsilon_z I$. Define $\omega_x : Z(K[G]) \to K$ defined by $z \mapsto \epsilon_z$

Let $\chi, \psi \in \mathrm{Irr}(G)$, say $\chi \sim \psi$ if $\pi(\omega_\chi(K_i)) = \pi(\omega_\chi(K_i))$ for all i.

Definition 8.2. A subset $B \subset \mathrm{Irr}(G) \cup \mathrm{Br}(G)$ is a p-block if

1. $B \cap \mathrm{Irr}(G)$ is an equivalence class under $\sim$
2. $B \cap \mathrm{Br}(G) = \{\phi \in \mathrm{Br} | d_{\chi\phi} \neq 0$ for some $\chi \in B \cap \mathrm{Irr}(G)\}$

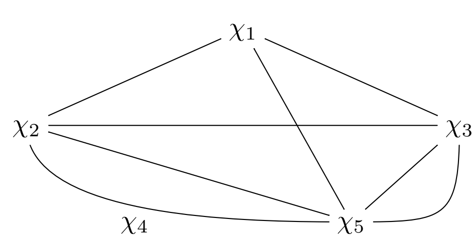
Notation. We set $Bl(G) = \{\text{set of } p \text{ blocks}\}$ We call the block containing $\mathbb{I}_G$ to be the principal block.

Theorem 8.3. *$\chi, \psi \in \mathrm{Irr}(G)$ are in the same p-block if and only if $\omega_\chi(K_i) - \omega_\phi(K_i) \in \mathcal{P}$ for all i and all primes $\mathcal{P} \in \mathcal{O}_K$ above p.*

Theorem 8.4. *Let $\phi \in \mathrm{Br}(G)$, then ϕ lies in a unique p-block.*

Define a graph $G = (V, E)$ by $V = \mathrm{Irr}(G)$ and $(\chi, \phi) \in E$ if there exists $\psi \in \mathrm{Br}(G)$ such that $d_{\chi\psi} \neq 0 \neq d_{\phi\psi}$. We call this the *Brauer Graph.*

Example. A_5 from last week, $p = 2$.



Fact. *$B \cap \mathrm{Irr}(G)$ is a single connected component, so $B_1 = \chi_1, \chi_2, \chi_3, \chi_5$ and ϕ_1, ϕ_2, ϕ_3 and $B_2 = \chi_4$ and ϕ_4*

Theorem 8.5. *Let B be a p-block. Then $|B \cap \mathrm{Irr}(G)| \geq |B \cap \mathrm{Br}(G)|$. Let $\chi \in \mathrm{Irr}(G)$ then the following are equivalent*

1. $|B \cap \mathrm{Irr}(G)| = |B \cap \mathrm{Br}(G)|$
2. $p \nmid \frac{|G|}{\chi(1)}$
3. $B \cap \mathrm{Irr}(G) = \{\chi\}$ *(in this case $B \cap \mathrm{Br}(G) = \{\widehat{\chi}\}$)*

Corollary 8.6. *If $p \nmid |G|$, then $|B \cap \mathrm{Br}(G)| = |B \cap \mathrm{Irr}(G)| = 1$*

Definition 8.7. Let $\chi \in \mathrm{Irr}(G)$. Then $e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g \in Z(K[G])$ is the idempotent for χ.

Note that if $\chi \neq \psi \in \mathrm{Irr}(G)$ then $e_\chi e_\psi = 0$, hence $e_\chi + e_\psi$ is an idempotent.

Definition 8.8. Let $B \in Bl(G)$. Then $f_B := \sum_{\chi \in B \cap \mathrm{Irr}(G)} e_\chi \in Z(\mathcal{O}_p[G])$. This is called the Osima idempotent. Let $e_B := \pi(f_B)$. Define $\lambda_B = \pi(\omega_\chi)$ for some $\chi \in B \cap \mathrm{Irr}(G)$.

Note that if ϕ is afforded by the k-representation η, then $\eta(z) = \lambda_B(z) I$ for all $z \in Z(k[G])$.

Theorem 8.9.

1. $\lambda_{B_i}(e_{B_j}) = \delta_{ij}$
2. *e_B are orthogonal idempotents*
3. *e_B is a k-linear combination of class sum of p-regular classes*
4. $\sum e_B = 1$
5. *If $\lambda_B(z) = 0$ for all $B \in Bl(G)$. Then z is nilpotent.*
6. $\{\lambda_B\} = \mathrm{Hom}(Z(k[G], k)$
7. *Every idempotent of $Z(k[G])$ is a sum of the e_B.*

Proof.

1. Let $\chi \in \mathrm{Irr}(G)$, then $\omega_\chi(f_B) = 1$ if $\chi \in B$, 0 else. If $\chi \in B$, then $\pi(\omega_\chi) = \lambda_B \Rightarrow \lambda_B(e_B) = 1$.
2. First note $f_B f_{B'} = \delta_{BB'} f_B$. So $e_B e_{B'} = \delta_{BB'} e_B$. Since $\lambda_B(e_B) = 1 \Rightarrow e_B \neq 0$.
3. Exercise
4. $\sum f_B = \sum e_\chi = 1$, hence $\sum e_B = \sum \pi(f_B) = \pi\left(\sum f_B\right) = \pi(1) = 1$

□