Further Representation Theory

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Contents

Aims:

- More character theory
- A bridge between representations and modules
- Representations of finite groups over fields (of characteristic 0) that are not algebraically closed.

Applications to Representation Theory

Theorem (Burnside). *If $|G| = p^\alpha q^\beta$ where p, q are primes, then G is soluble.*

Theorem (Frobenius). *If $H \leq G$ is such that $gHg^{-1} \cap H = \{1\}$ $\forall g \in G \setminus H$. Then $\exists N \lhd G$ such that $N \cap H = \{1\}$ and $NH = G$, i.e., $G = N \rtimes H$.*

Idea of Proof: Define $N = G \setminus \left(\cup_{g \in G} gHg^{-1} \cup \{1\}\right)$. We then use representation theory to prove that N is a normal subgroup of G.

Theorem. *If $G = S_n$, $1 \in G$ has the most square roots among all $g \in S_n$.*
More generally, can express the square root counting function through characters.

0 Revision

0.1 Representations

Let G be a finite group.

Definition 0.1. A *representation of G over a field K* is a K-vector space V together with a group homomorphism $\rho : G \to \mathrm{GL}(V) :=$\{invertible linear maps $V \to V$\}. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ s a basis of V, such a choice identifies $\mathrm{GL}(V)$ with $\mathrm{GL}_n(K) =$ \{invertible $n \times n$ matrices over K\}.

If $(V_1, \rho_1), (V_2, \rho_2)$ are two representations of G over K, a *homomorphism* $\phi : (V_1, \rho_1) \to (V_2, \rho_2)$ is a vector space homomorphism $\phi : V_1 \to V_2$ such that for all $v \in V_1, g \in G$ $\phi(\rho_1(g) \cdot v) = \rho_2(g) \cdot \phi(v)$.

Notation.

- Sometimes just say "V is a representation" when the map ρ is understood.
- Write ${}^g v$ or $g \cdot v$ instead of $\rho(g) \cdot v$.

Definition 0.2. If V is a representation, a *subrepresentation* is a a subvector space $W \subset V$ such that $G \cdot W = W$. We denote it $W \leq V$.

We have the obvious notion of V/W as a representation and the usual isomorphism theorems. (In particular kernels and images of homomorphism are subrepresentations.)

Example. Let $G = C_2 = \langle g \rangle$, let V be of dimension of 2, with basis v_1, v_2. We could have $\rho : g \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $W_1 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$ is a subrepresentation (one can easily see that it is invariant under ρ). The other subrepresentation is $W_2 = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$.

If $W_1, W_2 \leq V$ we say $V = W_1 \oplus W_2$ if this is true on the level of vector spaces.

Definition 0.3. A representation is *indecomposible* if it's not a direct sum of proper subrepresentation.

A representation is *irreducible* if it is non-zero and has no proper non-zero subrepresentation.

Example. Let $G = C_p = \langle g \rangle$ (where p is a prime). Let $K = \mathbb{F}_p$, V is 2 dimensional. Let the representation be $G \to \mathrm{GL}_2(\mathbb{F}_p)$, defined by $g \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This is not irreducible (i.e, reducible) since $W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is invariant under G (and $W \leq V$). But V is indecomposible, since there is no other proper subrepresentation.

Example.

- Given any group G and any field K, $G \to \mathrm{GL}_1(K) = K^*$ defined by $g \mapsto 1$. This is called the *trivial representation* denoted $\mathbb{I}$.
- • Given any group G, X a finite G-set (i.e., G acts on X by permutations) with $|X| = n$. Take an n-dimensional vectors space V over any field K, with a basis$\{v_x : x \in X\}$. Let G act on V by $g \cdot v_x = v_{g(x)}$. This representation is denoted by $K[X]$.
- Important special case: $X = G$, G acts by left multiplication. The resulting representation, $K[G]$, is called the *regular representation*.

Schur's Lemma. *Let G be a group, V_1, V_2 be two irreducible representations. Any homomorphism $V_1 \to V_2$ is either 0, or an isomorphism.*

Lemma 0.4. *Any irreducible representation V of G over K is isomorphic to a quotient of the regular representation $K[G]$.*

Proof. Take any $v \in V \setminus \{0\}$, define a map $K[G] \to V$ by $g \mapsto g \cdot v$. This is a homomorphism of representations, it is not the zero map, so it is onto. So we are done by the first isomorphism theorem. $\square$

Theorem 0.5 (Maschke)**.** *Suppose* $\mathrm{char} K \nmid |G|$. *Given any* $W_1 \leq V$, *representations of* G/K. *Then there exists a representation* $W_2 \leq V$ *such that* $V = W_1 \oplus W_2$.

Corollary 0.6. *Every irreducible representation* V *(in the case* $\mathrm{char} K \nmid |G|$*) is isomorphic to a subrepresentation of* $K[G]$.

0.2 Modules

Definition 0.7. An *algebra* A *over a field* K is a ring (with 1) that is also a K-vector space, such that $(x \cdot \alpha) \cdot (y \cdot \beta) = (xy) \cdot (\alpha \cdot \beta)$ for all $x, y \in K,\ \alpha, \beta \in A$.

Equivalently, A is a ring with $K \subset Z(A)$.

Example.

- $\mathbb{C}$ is a $\mathbb{C}$-algebra, but it is also an $\mathbb{R}$-algebra
- If A is any K-algebra, then the ring of $n \times n$ matrices over A, denoted $M_n(A)$, is also a K-algebra.
- $\mathbb{H} = \left\langle \mathbb{R} \cdot 1 + \mathbb{R} \cdot \underline{i} + \mathbb{R} \cdot \underline{j} + \mathbb{R} \cdot \underline{k} \mid ij = k, jk = i, ki = j, ij = -k, kj = -i, ik = -j, i^2 = j^2 = k^2 = -1 \right\rangle$ is an $\mathbb{R}$-algebra.
- If G is a group, K is a field, the group algebra $K[G]$ is a vector space spanned by vectors $v_g,\ g \in G$, with multiplication $v_g \cdot v_j = v_{gh}$.

Definition 0.8. If A is a K-algebra, a *left* A*-module* is an abelian group $(M, +)$ with a map $A \times M \to M$ such that

- $a \times (m_1 + m_2) = a \times m_1 + a \times m_2$
- $(a_1 + a_2) \times m = a_1 \times m + a_2 \times m$
- $0_A \times m = 0_M$
- $1_A \times m = m$
- $(a_1 \cdot a_2) \times m = a_1 \times (a_2 \times m)$

Equivalently, the map $A \to \mathrm{End}(M) = \mathrm{Hom}(M, M)$ defined by $a \mapsto (m \mapsto a \times m)$ is a ring homomorphism.

Moral: $K[G]$-modules are the same as representations of G over K.

We have the obvious notions of homomorphisms of modules, submodules, quotients, isomorphisms theorems, etc.

Example. Any algebra A can be thought as a module over itself: $M = A$ and $a \times m = a \cdot m$. This is called the *left regular module* of A.

The left regular module of $K[G]$ is the same as the regular representation of G over K.

Definition 0.9. A module M is *simple* if $M \neq 0$ and there exists no proper non-zero submodules.

A module M is *semi-simple* if it's a direct sum of simple modules.

Schur's Lemma. *If* M_1, M_2 *are simple* A*-modules, then any homomorphisms* $M_1 \to M_2$ *is either the* 0 *map, or an isomorphism.*

In particular, if M *is simple, then* $\mathrm{End}(M)$ *is a Division Ring (i.e., every non zero elements has a two-sided inverse)*

Note. A submodule of the left regular module of A is nothing but a left ideal.

Maschke's Theorem. *The left regular module of $K[G]$ is semi-simple, when $\mathrm{char}K \nmid |G|$.*

Theorem 0.10 (Artin - Wedderburn)**.** *Any algebra whose regular module is semi-simple is isomorphic to $\oplus_i M_{n_i}(D_i)$ where D_i are division rings.*

Hence, if $\mathrm{char}K \nmid |G|$ we have $K[G] \cong \oplus_i M_{n_i}(D_i)$, where D_i are division algebras over K.

Remark.

1. $M_n(D)$ is really semi-simple. $I_i = \begin{pmatrix} 0 & \cdots & 0 & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & 0 & \cdots & 0 \end{pmatrix}$ is clearly a left ideal, $M_n(D) = \oplus I_i$ as a module.

   Claim. I_i is simple.

   Proof. If $U \leq I_i$, $v \in U$ is non-zero, without loss of generality $v = \begin{pmatrix} \alpha \\ \vdots \end{pmatrix}$, $\alpha \in D^*$. Then

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \begin{pmatrix} & \alpha & \\ 0 & * & 0 \\ & \vdots & \\ & * & \end{pmatrix} = \begin{pmatrix} & \alpha & \\ 0 & 0 & 0 \\ & \vdots & \\ & 0 & \end{pmatrix} \in U$$

$$\begin{pmatrix} \alpha^{-1} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \begin{pmatrix} & \alpha & \\ 0 & * & 0 \\ & \vdots & \\ & * & \end{pmatrix} = \begin{pmatrix} & 1 & \\ 0 & 0 & 0 \\ & \vdots & \\ & 0 & \end{pmatrix} \in U$$

   So $U = I_i$. □

   Claim. The I_i are all pairwise isomorphic.

2.

   Corollary 0.11. *Then number of irreducible representations (up to isomorphism) of G over K (equivalently simple $K[G]$-modules) is equal to the number of conjugate classes of elements of G*

   Proof. Compute $\dim_K Z(K[G])$ on both sides. On the left had side $Z = \langle \sum_{h \in G} hgh^{-1} | g \in G \rangle$, so $\dim_K Z$ is precisely the number of conjugacy classes of elements of G. On the right hand side $Z = \langle (0, \ldots, 0, I_{n_i}, 0, \ldots 0) \rangle$ where $I_{n_i} \in M_{n_i}(D_i)$, so $\dim_K Z$ equals the number of distinct isomorphism classes of simple modules (one for each i). □

3. Suppose A is a semi-simple algebra. S is a simple A-module. Then Schur's Lemma says $\mathrm{End}_A(S) = D$ is a division algebra. Put $M = \underbrace{S \oplus \cdots \oplus S}_{n \text{ copies}}$. Then $\mathrm{End}_A(M) = M_n(D)$. (Each endomorphism of M is determined by the image of $(0, \ldots, 0, s, 0, \ldots, 0)$ (in the ith place), which is determined by projections to all components). The Wedderburn isomorphism comes by identifying A (actually A^{op}) with its endomorphism ring.

4. A decomposition $A = \oplus_i M_{n_i}(D_i)$ corresponds to writing $1 = \sum e_i$, where e_i are non-zero orthogonal primitive central idempotent.

   idempotent $e_i^2 = e_i$

   central $e_i a = a e_i$ for all $a \in A$.

orthogonal $e_ie_j = 0$ if $i \neq j$

primitive e_i is not a sum of non-zero orthogonal central idempotent elements.

If $A = \oplus M_{n_i}(D_i)$, then $1 = (1, \dots, 1) = (1, 0, \dots, 0) + \dots + (0, \dots, 0, 1)$. Conversely if $1 = \sum e_i$, then $U_i = e_i A$ gives $A = \oplus U_i$. Since they are orthogonal, we have $U_i \cap U_j = \{0\}$, since they are central idempotent, U_i are ideals, since they are primitive U_i are isotypical ($S \oplus \dots \oplus S$), and since their sum are 1, $A = \sum U_i$.

Example. If $G \cong C_3$, then $\mathbb{C}[G] \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. On the other hand $\mathbb{R}[G] \cong \mathbb{R} \oplus \mathbb{C}$.

To see the second statement, consider $e_1 = \frac{1}{3}(1+g+g^2)$. We have $e_1^2 = \frac{1}{9}(1+g+g^2)(1+g+g^2) = \frac{1}{9}(3+3g+3g^2) = e_1$. The ideal $e_1\mathbb{R}[G]$ had $\mathbb{R}$-dimension 1, so it is isomorphism to $\mathbb{R}$.

Consider $e_2 = 1 - e_1 = \frac{2}{3} - \frac{1}{3}g - \frac{1}{3}g^2$. The ideal $e_2\mathbb{R}[G]$ is generated (as $\mathbb{R}$-vector space) by $\alpha = e_2$ and $\beta = \frac{1}{\sqrt{3}}(g - g^2)$. We have$\beta^2 = \frac{1}{3}(g-g^2)^2 = \frac{1}{3}(g^2 - 2 + g) = -\alpha$. So $\alpha \mapsto 1$, $\beta \mapsto i$ is an isomorphism $e_2\mathbb{R}[G] \to \mathbb{C}$.

0.3 Characters

If $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a representation, the corresponding *character* $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$.

Theorem 0.12. *Let ρ_1, ρ_2 be two representations, then $\rho_1 \cong \rho_2$ if and only if $\chi_{\rho_1} = \chi_{\rho_2}$.*

Remark.

- $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.
- Recall: the number of distinct irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes of elements. The character table of G is a square table,

	1	g_2	g_3	$\dots$	$\dots$	g_k
$\mathbb{I}$	1	1	1	$\dots$	$\dots$	1
χ_1	$\dim\chi_1$	$\chi_2(g_2)$	$\chi_2(g_3)$	$\dots$	$\dots$	$\chi_2(g_k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
χ_k	$\dim\chi_k$	$\chi_k(g_2)$	$\chi_k(g_3)$	$\dots$	$\dots$	$\chi_k(g_k)$

Character are class functions, i.e., constant on conjugacy classes and the irreducible characters span the vector space of class functions.

Theorem 0.13 (Schur's Lemma in disguise, Row Orthogonality). *If χ_1, χ_2 are irreducible characters of G, then the inner product*

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}.$$

We also have column orthogonality. (Exercise: try to derive it using Row Orthogonality)

An arbitrary G representation ρ is a sum $\rho = \sum_i n_i\rho_i$ where ρ_i are distinct irreducible representations and $n_i \in \mathbb{Z}$. So $\chi_\rho = \sum n_i\chi_{\rho_i}$, and from the Row Orthogonality Theorem, $n_i = \langle \chi_\rho, \chi_{\rho_i} \rangle$, i.e.,$\chi_\rho = \sum \langle \chi_\rho, \chi_{\rho_i} \rangle \chi_{\rho_i}$.

Remark. The inner product can be defined for arbitrary class functions, so the theorem says that irreducible characters form an orthonormal basis of the space of class functions.

New characters / representations from old ones

If $N \lhd G$, then any group homomorphism from G/N induces a group homomorphism from G. So we can lift representations and characters from quotients.

Example. Let $G \cong S_3$ and $N \cong C_3$. Then $G/N \cong C_2$. Let $\epsilon : C_2 = \langle g \rangle \to \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$ be defined by $g \mapsto -1$

	1	(12)	(123)
$\mathbb{I}$	1	1	1
ϵ	1	-1	1
ρ	2	0	-1

Where the last row was worked out using the dimension (sum of the dimension squared need to equal $|G|$) and column orthogonality.

Another way to get new representations from old ones is using restriction. Any group homomorphism $G \to X$ (where X is anything) restricts to a group homomorphism $H \to X$ for any $H \leq G$. We will write this $\mathrm{Res}_{G/H}\rho$ or $\rho \downarrow_H^G$.

We also have induction: Let $H \leq G$ and $\rho : H \to \mathrm{GL}(V)$. Take a set of coset representatives $\{g_1 H, \ldots, g_n H\}$ for G/H. Define a new vector space $W = \oplus_{g_i} \underbrace{g_i \cdot V}_{\cong V \text{ as v.s.}}$. For any $g \in G$ and for each g_i, write (uniquely) $g \cdot g_i = g_j h$, with $h \in H$. Let g act on W by $g \cdot (g_i v) = g_j \rho(h)(v)$. This defines a representation of G on W, (note that $\dim W = \dim V \cdot |G/H|$). We write this as $\mathrm{Ind}_{G/H}\rho$, or $\rho \uparrow_H^G$.

The character of $\rho \uparrow_H^G$ is

$$\chi \uparrow_H^G (x) = \frac{1}{|H|} \sum_{g \in G} \chi^0(gxg^{-1}) \text{ where } \chi^0(y) = \begin{cases} \chi(y) & \text{if } y \in H \\ 0 & \text{if } y \notin H \end{cases}.$$

Frobenius reciprocity. *If $H \leq G$, χ is a character of G and τ is a character of H, then*

$$\left\langle \chi, \tau \uparrow_H^G \right\rangle_G = \left\langle \chi \downarrow_H^G, \tau \right\rangle_H.$$

More functorial statement of Frobenius reciprocity is the following:

If $H \leq G$, $\rho : H \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$, then there is a natural isomorphism $\mathrm{Hom}_G\left(\rho', \rho \uparrow_H^G\right) \cong \mathrm{Hom}_H\left(\rho' \downarrow_H^G, \rho\right)$.

This works over any field!

Some Properties of Characters

- Character values are sum of roots of unities, more specifically if $g \in G$ has order n, χ is d-dimensional, then $\chi(g)$ is the sum of d nth root of unity.

  In particular, $\chi(g)$ is an algebraic integer, i.e., roots of monic polynomial with integer coefficients.

  Also, it follows that $|\chi(g)| \leq |\chi(1)|$, with equality if and only if the matrix corresponding to g is in fact scalar (independent of basis on the vector space). Furthermore $\chi(g) = \chi(1)$ if and only if $g \mapsto I_n$ (the identity matrix). Hence define $\ker \chi = \{g \in G | \chi(g) = \chi(1)\}$. Define the centre, $Z(\chi) = \{g \in G | \, |\chi(g)| = \chi(1)\}$.

- There exists a bijection between irreducible characters χ of G with $\ker \chi \geq N \lhd G$ and irreducible characters of G/N lifted to G.

  All normal subgroups of G are obtained as intersections of $\ker \chi$ for suitable irreducible character χ. Also $Z(G) = \cap_{\chi \in \mathrm{Irr}(G)} Z(\chi)$

- Recall that $\mathbb{C}[G] = \oplus_{\rho_i \text{ irr}} \rho_i^{\oplus \dim \rho_i}$, so for any irreducible χ we have $\langle \mathbb{C}[G], \chi \rangle_G = \dim \chi$, hence $|G| = \sum_{\chi \in \mathrm{Irr}(G)} (\dim \chi)^2$

- Let $G' = \left\langle ghg^{-1}h^{-1} | g, h \in G \right\rangle \lhd G$. This is called the *derived subgroup* or *commutator subgroup*. It is the unique minimal normal subgroup with abelian quotient, i.e., if $N \lhd G$ is such that G/N is abelian then $N \geq G'$. It is easy to see that $G' = \cap_{\dim \chi = 1} \ker \chi$.

- If ϕ is any character and χ is a 1-dimensional character, then $\phi \otimes \chi(g) = \phi(g) \cdot \chi(g)$ is also a character (check!)

Example.

1. Cyclic groups, $C_n = \langle g \rangle$, of order n. All irreducible characters are 1-dimensional, $\chi_k : g \mapsto e^{\frac{2\pi i}{n}k}$ for $k = 0, \ldots, n-1$. Let $\zeta = e^{\frac{2\pi 1}{n}}$

	1	g	g^2	$\ldots$	g^{n-1}
$\mathbb{I}$	1	1	1	$\ldots$	1
χ_1	1	ζ	ζ^2	$\ldots$	ζ^{n-1}
χ_2	1	ζ^2	ζ^4	$\ldots$	$\zeta^{2(n-1)}$
$\vdots$					

2. Abelian groups, $A = C_{n_1} \times \cdots \times C_{n_r} = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle$. Then all irreducible characters are 1-dimensional, $\chi_{k_1 \ldots k_r} : g_j \mapsto e^{\frac{2\pi i}{n_j}k_j}$ for $0 \leq k_j \leq n_j - 1$.

3. Non-abelian group of order 8:

   - $G_1 = D_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = 1, \sigma\tau\sigma = \tau^{-1} \rangle$
   - $G_2 = Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik - -j \rangle$.

   First look at $G_1/G_1' \cong C_2 \times C_2$ where $G_1' = \langle \tau^2 \rangle$. So we can easily lift the characters of G_1/G_1' to G_1.

	1	σ	τ	τ^2	$\sigma\tau$
$\mathbb{I}$	1	1	1	1	1
ϵ_1	1	1	-1	1	-1
ϵ_2	1	-1	1	1	-1
ϵ_3	1	-1	-1	1	1
χ	2	0	0	-2	0

   G_2 is left as an exercise, but they do have the same character table (but $G_1 \ncong G_2$)

4. $G = S_4$. First recall that the character table for S_3 is

	1	(12)	(123)
$\mathbb{I}$	1	1	1
ϵ	1	-1	1
ρ	2	0	-1

   and recall that $S_4/V_4 \cong S_3$. Hence we can lift the three characters of S_3 into S_4. Then we use the dimensional formula to find that the last two characters must be 3 dimensional, so can complete using column orthogonality and the fact that $\epsilon \otimes \chi_1$ must be a character.

	1	(12)	(123)	(1234)	(12)(34)
$\mathbb{I}$	1	1	1	1	1
ϵ	1	-1	1	-1	1
ρ	2	0	-1	0	2
χ_1	3	1	0	-1	-1
χ_2	3	-1	0	1	-1

1 Mackey's Formula and Applications

Let $H \leq G$, and ρ is a representation of H, what is $\rho \uparrow^G \downarrow_H$?

Definition 1.1. Let $H, K \leq G$, a *double coset* is a set of the form $KgH = \{kgh | k \in K, h \in H\} = \cup_{k\in K} kgH = \cup_{h \in H} Kgh$.

$K\backslash G/H$ is the set of double cosets (or, by slight abuse of notation, the set of double coset representative).

Note. $Kg_1H = Kg_2H$ if and only if $g_2 \in Kg_1H$.

Warning: Different double cosets can have different size.

Example. Let $G = S_3$ and $H = K = \langle (12) \rangle$.

- $H \cdot 1 \cdot K = H$, size is 2.
- $H \cdot (123) \cdot K = \{(123), (12)(123)(12) = (132), (123)(12) = (23), (12)(123) = (13)\}$, size is 4.

Mackey's Formula. *Let $H, K \leq G$ and ρ a representation of H over any field L. Then*

$$\rho_H \uparrow^G \downarrow_K = \bigoplus_{g \in K\backslash G/H} {}^g\rho \downarrow_{K \cap gHg^{-1}} \uparrow^K$$

where ${}^g\rho(ghg^{-1}) = \rho(h)$ for all $h \in H$.

Proof. (Not Examinable) Let V be the vector space corresponding to ρ, then $\rho \uparrow^G$ is represented on $W = \oplus_{g \in G/H} gV$. Now G acts transitively on G/H, but K may not. Suppose

$$\underbrace{g_1H, \dots, g_{r_1}H}_{\cup = Kg_1H}, \underbrace{g_{r_1+1}H, \dots, g_{r_2}H}_{\cup = Kg_{r_1+1}H}, \dots$$

For a giving $k \in K$, there exists g_nH such that $kg_1H = g_nH$ if and only if $g_n \in kg_1H$. So $g_1V \oplus \cdots \oplus g_{r_1}V$ is a K-subrepresentation of W. By direct calculation, we see that this is isomorphism to ${}^{g_1}\rho \downarrow_{K \cap g_1Hg_1^{-1}} \uparrow^K$ □

Example. Take $\rho = \mathbb{I}_H$, then $\rho \uparrow^G = L[G/H]$ where L is the field as above. Now $\rho \uparrow^G \downarrow_K = \oplus_{g \in K\backslash G/H} \mathbb{I}_{K \cap gHg^{-1}} \uparrow^K = \oplus_{g \in K\backslash G/H} L[K/K \cap gHg^{-1}]$.

Check: The orbit of K acting on G/H are in bijection with $K\backslash G/H$.

1.1 Application I: principal series representation of $\mathrm{GL}_2(\mathbb{F}_p)$

Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ (where $p \in \mathbb{Z}$ is any prime). We have the subgroups $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \leq G$, $T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \leq B$ and $U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \leq B$. Note that $|U| = p$, $|T| = (p-1)^2$, $|B| = (p-1)^2 p$ and $|G| = (p^2-1)(p^2-p) = (p-1)^2 p(p+1)$. We have $T \cong \mathbb{F}_p^* \times \mathbb{F}_p^*$, $U \cong (\mathbb{F}_p, +) \lhd B$. We have $B/U \cong \mathbb{F}_p^* \times \mathbb{F}_p^*$, in fact $B = U \rtimes T$ (i.e, $U \lhd B$, $T \leq B$, $U \cap T = \{1\}$ and $UT = B$).

Let $\chi_1, \chi_2 : \mathbb{F}_p^* \to \mathbb{C}^*$ be two irreducible characters, then define $\tau = \chi_1 \otimes \chi_2 : B \to \mathbb{C}^*$ by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a) \cdot \chi_2(d)$. (Note $U \leq \ker \tau$).

Theorem 1.2. *$\tau \uparrow_B^G$ (which has dimension $p+1$) is either*

- *irreducible if $\chi_1 \neq \chi_2$ or*
- *(one dimensional)$\oplus$ irreducible if $\chi_1 = \chi_2$.*

Proof. Recall that a character is irreducible if and only if $\langle \tau, \tau \rangle_G = 1$. We have

$$\begin{aligned}
\langle \tau, \tau \rangle_B &\underset{\text{Frob rec}}{=} \left\langle \chi_1 \otimes \chi_2 \uparrow_B^G \downarrow_B, \chi_1 \otimes \chi_2 \right\rangle_B \\
&\underset{\text{Mackey}}{=} \sum_{g \in B \backslash G / B} \left\langle {}^g (\chi_1 \otimes \chi_2) \downarrow_{B \cap gBg^{-1}} \uparrow^B, \chi_1 \otimes \chi_2 \right\rangle_B \\
&\underset{\text{Frob rec}}{=} \sum_{g \in B \backslash G / B} \left\langle {}^g (\chi_1 \otimes \chi_2) \downarrow_{B \cap gBg^{-1}}, \chi_1 \otimes \chi_2 \downarrow_{B \cap gBg^{-1}} \right\rangle_{B \cap gBg^{-1}}
\end{aligned}$$

Claim. $B \backslash G / B = \{B \cdot 1 \cdot B, B \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot B\}$

Proof. It is enough to show that any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $c \neq 0$ is of the form $X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y$ with $X, Y \in B$. Let $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ and $Y = \begin{pmatrix} w & v \\ 0 & u \end{pmatrix}$, we compute that $X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & u \\ w & v \end{pmatrix} = \begin{pmatrix} yw & ux + vy \\ zw & vz \end{pmatrix}$. For a, b, c, d with $c \neq 0$ and $ad - bc \neq 0$, we can solve x, y, z, w, u, v. □

Going back to the equality above, we have

$$\begin{aligned}
\langle \tau, \tau \rangle &= \left\langle {}^1 (\chi_1 \otimes \chi_2) \downarrow_{B \cap 1 \cdot B \cdot 1} \uparrow^B, \chi_1 \otimes \chi_2 \right\rangle_B \\
&\quad + \left\langle {}^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (\chi_1 \otimes \chi_2) \downarrow_{B \cap \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \chi_1 \otimes \chi_2 \downarrow_{B \cap \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \right\rangle \\
&= 1 + \left\langle \underbrace{{}^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (\chi_1 \otimes \chi_2)}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_1(d) \chi_2(a)} \downarrow_T, \chi_1 \otimes \chi_2 \downarrow_T \right\rangle_T \\
&= 1 + \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2 \\ 1 & \text{if } \chi_1 = \chi_2 \end{cases} \\
&= \begin{cases} 1 & \text{if } \chi_1 \neq \chi_2 \\ 2 & \text{if } \chi_1 = \chi_2 \end{cases}.
\end{aligned}$$

We deduce that if $\chi_1 \neq \chi_2$, then τ is irreducible and otherwise it is the sum of two distinct irreducible.

Claim. If $\chi_1 = \chi_2$, then $\langle \tau, \chi_1 \circ \det \rangle_G = 1$.

Proof. We have

$$\langle \tau, \chi_1 \circ \det \rangle_G = \left\langle \underbrace{\chi_1 \otimes \chi_2}_{\chi_1(a)\chi_2(d)}, \underbrace{(\chi_1 \circ \det)}_{\chi_1(ad)} \downarrow_B \right\rangle_B = 1$$

□

So if $\chi_1 = \chi_2$, then $\tau = \chi_1 \circ \det \oplus$ (a p-dimensional irreducible character) □

Example. If $\chi_1 = \chi_2 = \mathbb{I}$, then $\tau \cong \mathbb{C}[G/B] = \mathbb{I} + (\underline{\text{Steinberg representation}})$

1.2 Application II: Semi-direct products by Abelian groups

Let $G \lhd N$, such that $N \cap H = \{1\}$, $NH = G$. Then $G = N \rtimes H$ (called semi-direct product). Note that this implies that for any $g \in G$ there exists unique $n \in N, h \in H$ such that $g = nh$. So as sets $G \leftrightarrow N \times H$. Under this bijection, $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \underbrace{(h_1 n_2 h_1^{-1})}_{\in N}, h_1 h_2)$. We have that H acts on N by conjugation, i.e., ${}^h n = hnh^{-1}$. This defines a map $H \to \mathrm{Aut}(N)$. So G is uniquely determined by N, H and the map $\phi : H \to \mathrm{Aut}(N)$. Conversely, given N, H and ϕ, we can construct the group G defined by as a set $N \times H$, with $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \phi(h_1) \cdot n_2, h_1 h_2)$

Example.

- $D_{2n} = C_n \rtimes C_2$ with $\phi : C_2 = \langle \sigma \rangle \to \mathrm{Aut}(C_n)$ is defined by $\sigma \mapsto (\tau \mapsto \tau^{-1})$.
- If $\phi : H \to \mathrm{Aut}(N)$ is defined by $h \mapsto \mathrm{id}$, then we get the direct product, $G = N \times H$.

Caution: We can get things that are isomorphic to $N \times H$ even if ϕ is non-trivial.

If $N \lhd G$, then G acts on the irreducible characters of N by ${}^g\chi(n) = \chi(g^{-1} n g)$. Note that N acts trivially on its own characters, so we get a well-defined action of G/N on $\mathrm{Irr}(N)$.

Remark. ${}^g\chi$ is certainly a class function (if χ is) but why is it a character, i.e., what is the corresponding representation? If $\rho : N \to \mathrm{GL}(V)$ is the representation attached to χ, then ${}^g\rho : N \to \mathrm{GL}(V)$ is defined by $n \mapsto \rho(g^{-1} n g)$. Note that the latter definition make sense over any fields.

Remark. If $N \lhd G$ and $H \cong G/N$, then in general $G \not\cong N \rtimes H$.

Example. Let $G = C_4$, $N \cong C_2$, $G/N \cong C_2$ but $G \not\cong C_2 \rtimes C_2$.

Now suppose $G = A \rtimes H$ where A is an Abelian group. We will completely describe $\mathrm{Irr}(G)$. Let $\chi \in \mathrm{Irr}(A)$ (hence χ is one dimensional), let $S_\chi := \mathrm{Stab}_H(\chi) = \{h \in H | \chi(h^{-1} a h) = \chi(a) \, \forall a \in A\}$. We extend χ to $A \rtimes S_\chi$ by $\chi(a \cdot s) = \chi(a)$. (Check that this is a 1-dimensional character of $A \rtimes S_\chi$). Take any irreducible character ρ of $S_\chi \cong (A \rtimes S_\chi)/A$, thought of as a character of $A \rtimes S_\chi$, and define $\tau_{\chi,\rho} = \chi \otimes \rho \uparrow^G$.

Theorem 1.3.

- *$\tau_{\chi,\rho}$ are all irreducible,*
- *All irreducible characters of G are of this form,*
- *$\tau_{\chi,\rho} = \tau_{\chi',\rho'}$ if and only if χ, χ' lie in the same orbit under the H-action (i.e., there exists $h \in H$ such that $S_\chi = h S_{\chi'} h^{-1}$) and $\rho = {}^h\rho'$.*

Proof. Left as an exercise ☐

Example. We describe the characters of $D_{2n} \cong C_n \rtimes C_2$ in this way. If $C_n = \langle g \rangle$, and $C_2 = \langle h \rangle$, then we know the characters $\chi_k : C_n \to \mathbb{C}^*$ are defined by $g \mapsto e^{2\pi i k/n}$. Also ${}^h\chi_k(g) = \chi_k(h^{-1} g h) = \chi_k(g^{-1}) = e^{-2\pi i k/n}$, so

$$\mathrm{Stab}_{\chi_k}(C_2) = \begin{cases} C_2 & \text{if } e^{-2\pi i k/n} = e^{2\pi i k/n} \iff k = 0, n/2 \\ 1 & \text{otherwise} \end{cases}.$$

So if $k \neq 0, n/2$ then $\chi_k \uparrow^G$ is irreducible. If $k = 0, n/2$ then χ_k extends (in two ways) to a 1-dimensional character of G. Also $\chi_k \uparrow^G = \chi_{k'} \uparrow^G$ is and only if $k = \pm k'$.

Exercise.

- We describe the characters of $S_4 \cong V_4 \rtimes S_3$ (where $V_4 \cong C_2 \times C_2$ is the Klein group)
- Suppose $H \leq S_n$, A is any (Abelian) group. Consider $G = (\underbrace{A \times \cdots \times A}_{n-\text{times}}) \rtimes H =: A \wr H$, the *wreath product.*

  Describe $\mathrm{Irr}(G)$ using Theorem 1.3.

  Remark. $\mathrm{Syl}_p(S_{p^2}) \cong C_p \wr C_p$. (prove it!)

1.3 Application III: Clifford Theory (induction from and restriction to normal subgroups)

Theorem 1.4 (Clifford)**.** *Let G be any finite group, $N \lhd G$, F any fields, and ρ any irreducible representations of G over F, (equivalently a simple $F[G]$-module). Then $\rho\downarrow_N = \oplus \tau_i^{\oplus e}$, where τ_i are simple $F[N]$-modules, that form a single orbit under the G-action.*

Proof. Let τ_1 be a simple quotient submodule of $\rho\downarrow_N$. So then $\mathrm{Hom}_N(\rho\downarrow_N, \tau_1)$ is non-trivial. By Frobenius reciprocity, $\mathrm{Hom}_G(\rho, \tau_1\uparrow^G)$ is non-trivial. Equivalently, ρ is isomorphic to a submodule of $\tau_1\uparrow^G$. So $\rho\downarrow_N$ is a submodule of

$$\begin{aligned}\tau_1\uparrow^G\downarrow_N &= \bigoplus_{N\backslash G/N} {}^g\tau_1\downarrow_{N\cap gNg^{-1}}\uparrow^N \\ &= \bigoplus_{g\in G/N} {}^g\tau_1\end{aligned}$$

As τ_1 is simple, all ${}^g\tau_1$ are simple. Using the exercise below, we see that $\rho\downarrow_N = \oplus_{\text{some } g}\, {}^g\tau_1$. Note that $\mathrm{Hom}_N(\rho\downarrow_N, \tau_1) = \mathrm{Hom}_N({}^g\rho\downarrow_N, {}^g\tau_1)$. If $\rho\downarrow_N \cong \oplus\tau_i^{\oplus e_i}$ then $\mathrm{Hom}_N(\rho\downarrow_N, \tau_i) \cong D^{\oplus e_i}$, where $D = \mathrm{End}_N(\tau_i)$. So e_i are the same for all the distinct conjugates of τ_1. ☐

Exercise. Submodules of semisimple modules are semisimple.

Exercise. Prove the above theorem over $\mathbb{C}$, using characters.

Example.

- Let $G = S_3$ and $N = C_2$. Consider the 2-dimensional irreducible representation, ρ, over $\mathbb{C}$. Then $\rho\downarrow_N = \chi \oplus \overline{\chi}$ where χ is defined by $(123) \mapsto e^{2\pi i/3}$. Note ${}^{(12)}\chi = \overline{\chi}$.
- Let $G = S_n$ and $N = A_n$.

Now we want to translate Clifford's theorem into a statement about induction.

Example. $G = C_3 \rtimes C_4 = \langle x, y | x^3 = y^4 = 1, yxy^{-1} = x^{-1}\rangle$, where C_4 acts on C_3 through the quotient C_4/C_2. Let $N = C_3$, χ a non-trivial 1-dimensional character of C_3. We investigate $\chi\uparrow^G$ in two steps. Consider $\chi\uparrow^G = \chi\uparrow^{C_3\rtimes C_2 = C_6}\uparrow^G = (\tau\oplus\tau')\uparrow^G$ where τ and τ' are distinct irreducible $\mathrm{Irr}(C_6)$. Both $\tau\uparrow^G$ and $\tau'\uparrow^G$ are irreducible by the 2nd exercise sheet

Definition. Let $N \lhd G$ and χ an irreducible character of N. The *inertia* subgroup of χ in G is $I_G(\chi) = \mathrm{Stab}_G(\chi) = \{g \in G | {}^g\chi = \chi) = \{g \in G | \chi(g^{-1}ng) = \chi(n)\, \forall n \in N\}$.

Theorem 1.5. *Let $N \lhd G$, $\chi \in Irr(N)$, $T = I_G(\chi) \geq N$. Let τ be an irreducible summand of $\chi\uparrow^\tau$.*

1. *$\rho = \tau\uparrow^G$ is irreducible*
2. *$\tau \to \tau\uparrow^G$ is a bijection between the distinct irreducible summand of $\chi\uparrow^T$ and those of $\chi\uparrow^G$*
3. *$\rho\downarrow_T = \tau +$(stuff that is disjoint from $\chi\uparrow^T$). By disjoint we mean $\psi \in Irr(T)$ such that $\langle\psi, \chi\uparrow^T\rangle = 0$.*
4. *$\langle\rho\downarrow_N, \chi\rangle = \langle\tau\downarrow_N, \chi\rangle$.*

Proof. First note that $\tau\downarrow_N = e\cdot\chi$, and hence ${}^g\tau\downarrow_N = e\cdot{}^g\chi$. If $g \notin T$ then ${}^g\chi \neq \chi$, hence $\langle\tau\downarrow_N, {}^g\tau\downarrow_N\rangle = 0$ if $g \notin T$. Now compute

$$\begin{aligned}\langle\tau\uparrow^G, \tau\uparrow^G\rangle_G &= \langle\tau\uparrow^G\downarrow_T, \tau\rangle_T \\ &= \sum_{g\in T\backslash G/T} \langle{}^g\tau\downarrow_{T\cap gTg^{-1}}\uparrow^T, \tau\rangle_T \\ &= 1 + \sum_{\text{some} g\notin T} \langle{}^g\tau\downarrow_{T\cap gTg^{-1}}, \tau\downarrow_{T\cap gTg^{-1}}\rangle \qquad T\cap gTg^{-1} \geq N \\ &= 1 + 0\end{aligned}$$

This proves 1. We have that 2. and 4. follows from 3. .To prove 3. ,

$$\begin{aligned}
\rho\downarrow_T &= \tau\uparrow^G\downarrow_T \\
&= \sum_{T\backslash G/T} {}^g\tau\downarrow_{T\cap gTg^{-1}}\uparrow^T \\
&= \tau + \sum_{\text{some } g\notin T} {}^g\tau\downarrow_{T\cap gTg^{-1}}\uparrow^T
\end{aligned}$$

□

1.4 Application IV: Frobenius groups

Theorem 1.6 (Frobenius)**.** *Suppose $H \leq G$ is such that $H \cap gHg^{-1} = \{1\}$ for all $g \notin H$ (H is called a* Frobenius complement*). Then there exists $N \lhd G$ such that $G = N \rtimes H$.*

To prove this we will need several lemma. Define

$$N = G\backslash \left(\cup_g gHg^{-1}\right) \cup \{1\}.$$

Lemma 1.7. *Let N be defined as above, $|N| = \frac{|G|}{|H|}$, also is $M \lhd G$ intersect H trivially, then $M \subset N$*

Proof. The second part is by definition of N. For the first part

$$\begin{aligned}
|N| &= |G| - \frac{|G|}{|H|}(|H|-1) \\
&= |G| - |G| + \frac{|G|}{|H|}
\end{aligned}$$

□

Lemma 1.8. *Let G and H be as in Theorem 1.3. Let θ be a class function on H with $\theta(1) = 0$. Then $\theta\uparrow^G\downarrow_H = \theta$.*

Proof. By Machke we have

$$\begin{aligned}
\theta\uparrow^G\downarrow_H &= \sum_{g\in H\backslash G/H} {}^g\theta\downarrow_{H\cap gHg^{-1}}\uparrow^H \\
&= \theta + \sum_{g\notin H} {}^g\theta\downarrow_{H\cap gHg^{-1}}\uparrow^H \\
&= \theta + \sum 0_{\{1\}}\uparrow^H \\
&= \theta
\end{aligned}$$

□

Proof of Theorem 1.6. Motivation: if $\chi \in \mathrm{Irr}(G)$ is such that $\ker\chi \supseteq N$, then $\chi\downarrow_H$ is irreducible. We want to recover χ from $\chi\downarrow_H$.

Let $\mathbb{I}_H \neq \phi \in \mathrm{Irr}(H)$. Define $\theta_\phi = \phi - \phi(1)\mathbb{I}_H$, hence $\theta_\phi(1) = 0$. Note that

$$\begin{aligned}
\left\langle \theta_\phi\uparrow^G, \mathbb{I}_G\right\rangle_G &= \langle\theta_\phi, \mathbb{I}_H\rangle_H \\
&= -\phi(1).
\end{aligned}$$

Let us set $\chi_\phi = \phi_\psi \uparrow^G + \phi(1)\mathbb{I}_G$, hence $\langle \chi_\phi, \mathbb{I}_G \rangle_G = 0$. Furthermore

$$\begin{aligned}
\langle \chi_\phi, \chi_\phi \rangle + \phi(1)^2 &= \left\langle \theta_\phi \uparrow^G, \theta_\phi \uparrow^G \right\rangle_G \\
&= \left\langle \theta_\phi, \theta_\phi \uparrow^G \downarrow_H \right\rangle_H \\
&\underset{\text{Lemma 1.8}}{=} \langle \theta_\phi, \theta_\phi \rangle_H \\
&\underset{\text{def}\,\theta_\phi}{=} \underbrace{\langle \phi, \phi \rangle}_{=1} + \phi(1)^2.
\end{aligned}$$

Hence $\langle \chi_\phi, \chi_\phi \rangle_G = 1$ and is irreducible. Now θ_ϕ is the difference of two characters, therefore so is $\theta_\phi \uparrow^G$, and hence so is χ_ρ. But $\langle \chi_\phi, \chi_\phi \rangle = 1$, hence $\pm\chi_\phi \in \operatorname{Irr}(G)$. But also,

$$\begin{aligned}
\chi_\phi \downarrow_H &= \theta_\phi \uparrow^G \downarrow_H + \theta(1) \cdot \mathbb{I}_G \downarrow_H \\
&= \theta_\phi + \phi(1) \cdot \mathbb{I}_G \\
&= \theta.
\end{aligned}$$

So $\chi_\theta \in \operatorname{Irr}(G)$.

Define

$$M = \bigcap_{\mathbb{I}_H \neq \phi \in \operatorname{Irr}(H)} \ker(\chi_\phi) \lhd G.$$

Claim. $M \cap H = \{1\}$ (and hence $M \subseteq N$)

Indeed, if $h \in H$, then $\chi_\phi(h) = \phi(h)$. So $H \cap M = \cap_{\phi \in \operatorname{Irr}(H)} \ker \phi = \{1\}$

Claim. $N \subseteq M$.

If $n \notin gHg^{-1}$, then

$$\begin{aligned}
\chi_\phi(n) &= \theta_\phi \uparrow^G (n) + \phi(1) \cdot \mathbb{I}_G \\
&= \phi \uparrow^G (n) - \phi(1) \cdot \mathbb{I} \uparrow^G (n) + \phi(1) \cdot \mathbb{I}_G \\
&\underset{\text{def of irreducible char}}{=} 0 + 0 + \phi(1) \\
&= \chi_\phi(1)
\end{aligned}$$

for all $\mathbb{I}_H \neq \phi \in \operatorname{Irr}(H)$. Hence $n \in M$.

So $N = M \lhd G$ and we are done. □

2 Tensor Products, Frobenius - Schur indicators and much more

Let G be a finite group and K be any field.

Motivation: If χ, ϕ are characters of G, then so is $\chi + \phi$. But what about $\chi \cdot \phi$?

Definition 2.1. Let V and W be vector spaces (over K). The *tensor product* $V \otimes W$ is the vector space spanned by "symbols" $v \otimes w$ with $v \in V, w \in W$, with relations

- $(kv) \otimes w = v \otimes (kw) = k(v \otimes w)$
- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

Fact. *If $v_1, \dots, v_n$ is a basis for V and $w_1, \dots, w_m$ a basis of W, then $v_i \otimes w_j$ for $1 \leq i \leq n$,$1 \leq j \leq m$ is a basis for $V \otimes W$.*

Proposition 2.2. *Tensor products have the following properties:*

- $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$
- $(V \oplus U) \otimes W \cong V \otimes W \oplus U \otimes W$

Proof. Check that:

- $(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$
- $(u, v) \otimes w \mapsto (v \otimes w, u \otimes w)$

are isomorphisms. ∎

If V, W are G-representation, then G acts on $V \otimes W$ via $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$. Suppose that g is represented by $A = (a_{ij})_{1 \leq i,j \leq n}$ on V with respect to $v_1, \dots, v_n$, and it is represented by $B = (b_{ij})_{1 \leq i,j \leq m}$ on W with respect to $w_1, \dots, w_m$. Then $g \cdot (v_i \otimes w_k) = gv_i \otimes gw_k = (\sum a_{ij} v_j) \otimes (\sum b_{kl} w_l) = \sum_{j,l} a_{ij} b_{kl} (v_j \otimes w_l)$. So with respect to the basis $v_1 \otimes w_1, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_n \otimes w_n$ of $V \otimes W$, g is represented by

$$\left(\begin{array}{c|c|c|c} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \hline \vdots & & & \\ \hline a_{n1}B & & & a_{nn}B \end{array}\right) =: A \otimes B$$

Example. Let $G = S_3$ and take ρ to be the standard representation, that is ρ is defined by

$$\begin{aligned} (123) &\mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \\ (12) &\mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The basis of $\rho \otimes \rho$ is $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$. So $\rho \otimes \rho$ is defined by

$$\begin{aligned} (123) &\mapsto \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ \hline -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \\ (12) &\mapsto \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{aligned}$$

If V, W are complex representations with characters χ, ϕ respectively then the character τ of $V \otimes W$ is

$$\begin{aligned} \tau(g) &= \sum_{1 \leq i \leq n, 1 \leq k \leq m} a_{ii} b_{kk} \\ &= \left(\sum_i a_{ii}\right) \cdot \left(\sum_k b_{kk}\right) \\ &= \chi(g) \cdot \phi(g). \end{aligned}$$

Aside: Duals and homomorphism spaces

Definition 2.3. Let V be a representation of over K. The *dual* representation is $V^* = \{f : V \to K | f(v + \alpha w) = f(v) + \alpha f(w) \forall \alpha \in K, v, w \in V\}$ with G action on V^* by $(g \cdot f)(v) = f(g^{-1}v)$, i.e., $g \cdot f : v \mapsto f(g^{-1}v) \in K$

Lemma 2.4. *If V is a complex representation with character χ, then the character of V^* is $\overline{\chi}$.*

Proof. Let $v_1, \ldots, v_n$ be a basis of V. Take the dual basis V^* to be $f_1, \ldots, f_n$ such that $f_i(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Assume without loss of generality that $g \in G$ is represented by $\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$ with respect to $v_1, \ldots, v_n$. Then check that g in represented by $\begin{pmatrix} \alpha_1^{-1} & & \\ & \ddots & \\ & & \alpha_n^{-1} \end{pmatrix}$ with respect to $f_1, \ldots, f_n$. Since α_i are roots of unity, $\alpha_i^{-1} = \overline{\alpha_i}$. □

Corollary 2.5. *$V \cong V^*$ (as representations over $\mathbb{C}$) if and only if χ is $\mathbb{R}$-valued.*

Definition 2.6. If V, W are representations of G over K, then $\mathrm{Hom}_K(V, W)$ is a G-representation via $(g \cdot f)(v) = g \cdot f(g^{-1}v)$.

Lemma 2.7. *If V and W are representations over $\mathbb{C}$ with characters χ, ϕ respectively then the character τ of $\mathrm{Hom}_K(V, W)$ is $\overline{\chi} \cdot \phi$.*

Proof. Use matrices with respect to basis $f_{ik} : v_j \mapsto \delta_{ij} w_k$. □

In particular, over $\mathbb{C}$, $V \otimes W \cong \mathrm{Hom}(V^*, W)$ (by comparing characters)

Lemma 2.8. *Over any field K, $V \otimes_K W \cong \mathrm{Hom}_K(V^*, W)$.*

Proof. Check that $V \otimes W \to \mathrm{Hom}(V^*, W)$ defined by $v \otimes w \mapsto (f \mapsto f(v) \cdot w)$ is an isomorphism (of G-representations). □

Remark. The fixed subspace of $\mathrm{Hom}(V, W)$ under the G-action is

$$\mathrm{Hom}_G(V, W) = \{f : V \to W \text{ linear} | g \cdot f(v) = f(g \cdot v) \forall v \in V, g \in G\}$$

Assume: For the rest of this chapter that the characteristic of K is 0

Notation. $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$

Example. Going back to the case that $G = S_3$, ρ the standard representation. The character table is

	1	(123)	(12)
$\rho^{\otimes 2}$	4	1	0

$$\begin{aligned}\langle \rho^{\otimes 2}, \mathbb{I} \rangle &= \frac{1}{6}(4+2) = 1 \\ \langle \rho^{\otimes 2}, \text{sign} \rangle &= \frac{1}{6}(4+2) = 1 \\ \langle \rho^{\otimes 2}, \rho \rangle &= 1\end{aligned}$$

Hence we have that $\rho^{\otimes 2} = \mathbb{I} + \epsilon + \rho$.

$V^{\otimes n}$ carries an action of S_n, $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for all $\sigma \in S_n, v_i \in V$. This action commutes with the G-action. (So we get an action of $G \times S_n$).

Claim. If $\rho_1, \ldots, \rho_k$ is a complete set of irreducible K-representations of S_n,then $V^{\otimes n} = \oplus_{i=1}^k V_{(\rho_i)}^{\otimes n}$ as G-representations.

Proof. If $g \in G$, then (for $t \in V^{\otimes n}$) $t \mapsto g \cdot t$ is a homomorphism of S_n-representation. So if $t \in V_{(\rho_i)}^{\otimes n}$, then the projection of $g \cdot t$ to any $V_{(\rho_j)}^{\otimes n}$ for $j \neq i$ is 0 by Schur's lemma. □

Example. Let $v_1, \ldots, v_n$ be a basis for V. We consider S_2 and hence $V^{\otimes 2}$, which has basis $v_i \otimes v_j$ for $1 \leq i, j \leq n$.

- $V_{(\mathbb{I})}^{\otimes 2}$ has basis $v_i \otimes v_j + v_j \otimes v_i$ for $1 \leq i \leq j \leq n$.
- $V_{(\text{sign})}^{\otimes 2}$ has basis $v_i \otimes v_j - v_j \otimes v_i$ for $1 \leq i < j \leq n$.

So in part, $V_{(\mathbb{I})}^{\otimes 2}$ has dimension $\frac{n(n+1)}{2}$, $V_{(\text{sign})}^{\otimes 2}$ has dimension $\frac{n(n-1)}{2}$.

Definition 2.9. $V_{(\mathbb{I})}^{\otimes 2}$ is called the *symmetric square of* V, written S^2V. $V_{(\text{sign})}^{\otimes 2}$ is called the *alternating square of* V, written $\wedge^2 V$.

Lemma 2.10. *The characters of* S^2V *and* $\wedge^2 V$ *are*

$$\begin{aligned}\chi_{S^2V}(g) &= \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \\ \chi_{\wedge^2 V}(g) &= \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)\end{aligned}$$

Proof. Let $g \in G$, take a basis of V to be $v_1, \ldots, v_n$ such that $g = \text{diag}(\alpha_1, \ldots, \alpha_n)$ with respect to that basis. Then $g \cdot (v_i \otimes v_j + v_j \otimes v_i) = \alpha_i \alpha_j (v_i \otimes v_j + v_j \otimes v_i)$. So

$$\begin{aligned}\chi_{S^2V}(g) &= \sum_{1 \leq i \leq j \leq n} \alpha_i \alpha_j \\ &= \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right).\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}\chi_{S^2V}(g) &= \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \\ &= \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right).\end{aligned}$$

□

Remark. $S^2\chi + \wedge^2\chi = \chi^2$.

Definition 2.11. Let χ be an irreducible character of G, the *Frobenius - Schur indicator* of χ is

$$s_2(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Theorem 2.12. *$s_2(\chi) \in \{0, 1, -1\}$ and $s_2(\chi) = 0$ if and only if $\chi \neq \overline{\chi}$.*

Proof. Note that $\chi(g^2) = S^2\chi(g) - \wedge^2\chi(g)$, so $s_2(\chi) = \left\langle S^2\chi, \mathbb{I} \right\rangle_G - \left\langle \wedge^2\chi, \mathbb{I} \right\rangle_G$.

Claim. $\left\langle \underbrace{S^2\chi + \wedge^2\chi}_{\chi^2}, \mathbb{I} \right\rangle_G = 0$ or 1

Let V be the vector space attached to χ. Then $\left\langle V^{\otimes 2}, \mathbb{I} \right\rangle = \dim\left((V \otimes V)^G\right) = \mathrm{Hom}_G(V^*, V)$ (where $(-)^G$ are elements fixed by G.) By Schur's lemma, these G-homomorphism are 1-dimensional if $V^* \cong V$ (i.e., if $\chi \cong \overline{\chi}$) and 0 otherwise.

$\square$

So what does this ± 1 mean for s_2 of real-valued characters?

Example. Let $G = S_3$ and χ the standard character, i.e., $\chi(1) = 2, \chi((123)) = -1$ and $\chi((12)) = 0$. Now $s_2(\chi) = \frac{1}{6}(2 + 3 \cdot 2 + 2 \cdot (-1)) = 1$.

Pairings on vector spaces

Definition 2.13. Let V be a vector space over a field K. A *pairing* on V is a bilinear map $\langle\, , \rangle : V \times V \to K$.

Given a paring, we get a linear map $V \to V^*$ defined by $v \mapsto (w \mapsto \langle v, w \rangle)$. Conversely, given a homomorphism $\phi : V \to V^*$, we can define a pairing by $\langle v, w \rangle = \phi(v)(w)$. These operations are inverses to each other.

Definition 2.14. A pairing is non-degenerate if, given $v \in V$, $\langle v, w \rangle = 0 \forall w \in V$ then $v = 0$. (This is equivalent to "right non-degenerate" for finite dimensional vector spaces.)

In the language of $\phi : V \to V^*$, this is equivalent to ϕ being an isomorphism.

Definition 2.15. $\langle\, , \rangle$ is *symmetric* if $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$

It is *alternating* if $\langle v, w \rangle = -\langle w, v \rangle$ for all $v, w \in V$.

Let G act on V. We say that $\langle\, , \rangle$ is *G-invariant* if $\langle gv, gw \rangle = \langle v, w \rangle\ \forall v, w \in V, \forall g \in G$.

This is equivalent to $\phi : V \to V^*$ being a G-homomorphism. So we have a bijection between $\mathrm{Hom}_G(V, V^*)$ and G-invariant pairings on V.

If U, V, W are vector spaces, then bilinear maps $U \times V \to W$ are "the same things as" linear maps $U \otimes V \to W$ in the following sense: there is a canonical map $U \times V \to U \otimes V$ defined by $(u, v) \mapsto u \otimes v$, and given any bilinear map

$$\begin{array}{ccc} U \times V & \longrightarrow & U \otimes V \\ & \searrow & \Big\downarrow \exists! \\ & & W \end{array}$$

so that the diagram commutes.

In particular, bilinear maps $V \times V \to K$ correspond canonically to maps $V \otimes V \to K$, and the set of G-invariant pairings on V is in bijection with $\mathrm{Hom}_G(V \otimes V, \mathbb{I})$.

- The pairing is symmetric if the map $V \otimes V \to K$ is 0 on $\wedge^2 V$, i.e., such pairings correspond to maps $V \otimes V / \wedge^2 V \cong S^2 V \to K$.
- The pairing is alternating if the map $V \otimes V \to K$ is 0 on $S^2 V$, i.e., such pairings correspond to maps $V \otimes V / S^2 V \cong \wedge^2 V \to K$.

- $V \cong V^*$ if and only if there exists a G-invariant non-degenerate pairing on V. Conversely, given $f : V \to V^*$, take $\langle u, v\rangle = f(u)v$ if and only if $\chi = \overline{\chi}$. If $V \cong V^*$, then $\mathrm{Hom}_G(V, V^*)$ is 1-dimensional
- If $\langle\, ,\, \rangle$ is a non-degenerate G-invariant pairings, we can write it

$$\langle u, v\rangle = \frac{1}{2}\underbrace{(\langle u, v\rangle + \langle v, u\rangle)}_{\langle\, ,\, \rangle_s} + \frac{1}{2}\underbrace{(\langle u, v\rangle - \langle v, u\rangle)}_{\langle\, ,\, \rangle_a}$$

  We cannot have both $\langle\, ,\, \rangle_s$ and $\langle\, ,\, \rangle_a$ non-degenerate, since they would have to be multiple of each other. Another way of saying this: since $V \otimes V = S^2V \oplus \wedge^2V$ we either have

  - $\dim(\mathrm{Hom}_G(S^2V, \mathbb{I})) = 1$ and $\mathrm{Hom}_G(\wedge^2V, \mathbb{I}) = 0$ or
  - $\dim(\mathrm{Hom}_G(S^2V, \mathbb{I})) = 0$ and $\mathrm{Hom}_G(\wedge^2V, \mathbb{I}) = 1$

- There exists a symmetric G-invariant, non-degenerate pairing on V if and only if $\mathrm{Hom}_G(S^2V, \mathbb{I}) \neq 0$, if and only if $V \cong V^*$ and $\mathrm{Hom}_G(\wedge^2V, \mathbb{I}) = 0$.

  Explicitly, if $f : S^2V \to \mathbb{I}$, construct the pairing by $\langle u, v\rangle = f(u \otimes v + v \otimes u)$

- There exists an alternating G-invariant, non-degenerate pairing on V if and only if $\mathrm{Hom}_G(\wedge^2V, \mathbb{I}) \neq 0$, if and only if $V \cong V^*$ and $\mathrm{Hom}_G(S^2V, \mathbb{I}) = 0$.

Theorem 2.16. *Let V be an irreducible complex representation of G.*

1. *There exists a non-degenerated G-invariant pairing on V if and only if $V \cong V^*$ if and only if $\chi \cong \overline{\chi}$*
2. *There exists a symmetric non-degenerate G-invariant pairing on V if and only if there exists a basis on V with respect to which G is represented by real matrices.*
3. *There exists an alternating non-degenerate G-invariant pairing on V if and only if χ is real-valued, but V cannot be defined over $\mathbb{R}$ (in the above sense)*

Example. Let $G = D_{10} = \langle \tau, \sigma | \tau^2 = \sigma^5 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$. Rotations and reflections of the 5-gon gives the following representation: $\tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma \mapsto \begin{pmatrix} e^{2\pi i/5} & 0 \\ 0 & e^{-2\pi i/5} \end{pmatrix}$. But with respect to "the right" basis, this can be written as $\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma \mapsto \begin{pmatrix} \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}$

Theorem (Part 2).

1. *holds if and only if $s_2(\chi) = \langle S^2\chi, \mathbb{I}\rangle - \langle \wedge^2\chi, \mathbb{I}\rangle \neq 0$*
2. *holds if and only if $s_2(\chi) = 1$*
3. *holds if and only if $s_2(\chi) = -1$*

Partial proof. Suffices to show 2. (as we already know 1.). We will only prove the implication $\Leftarrow$.

Suppose that V is definable over $\mathbb{R}$. This means that if $V$is regarded as an $\mathbb{R}$-vector space (of twice its dimension over $\mathbb{C}$), then $V = W + iW$, where W is invariant under G. Let $(\, ,\,)$ be any positive-definite pairing on W. Define $\langle\, ,\, \rangle_1$ on W by $\langle u, v\rangle_1 = \frac{1}{|G|}\sum_{g\in G}(g \cdot u, g \cdot v)$. This is clearly G-invariant and positive-definite.

Define $\langle u, v\rangle_2 = \langle u, v\rangle_1 + \langle v, u\rangle_1$. Then this is still G-invariant and positive-definite, furthermore it is symmetric. Define $\langle u, v\rangle_3$ on V by $\langle u + iu', v + iv'\rangle_3 = \langle u, v\rangle_2 - \langle u', v'\rangle_2 + i\left(\langle u, v'\rangle_2 + \langle u', v\rangle_2\right)$, this is the required pairing.

For the implication "$\Rightarrow$" see for example Serre, chapter 2, Theorem 31, or Curtis - Reiner, Vol II, Section 73.13. □

Application

Define $r_2(g) = \#\{h \in G | h^2 = g\}$. First observe: $h \mapsto xhx^{-1}$ gives a bijection between square roots of h^2 and those of xh^2x^{-1}. So r_2 is a class function. Thus $r_2 = \sum_{\chi \in \mathrm{Irr}(G)} \alpha_\chi \cdot \chi$ for $\alpha_\chi \in \mathbb{C}$. Now

$$\begin{aligned}
\alpha_\chi &= \langle r_2, \chi \rangle_G \\
&= \frac{1}{|G|} \sum_{g \in G} r_2(g)\chi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \#\{h \in G | h^2 = g\}\chi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{h \in G, h^2 = g} \chi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{h \in G, h^2 = g} \chi(h^2) \\
&= \frac{1}{|G|} \sum_{h \in G} \chi(h^2) \\
&= s_2(\chi) \in \{-1, 0, 1\}.
\end{aligned}$$

Hence $r_2 = \sum_{\rho \text{ irr reps realisable over } \mathbb{R}} \chi_\rho - \sum_{\text{self dual irr reps not realisable over } \mathbb{R}} \chi_\rho$.

Corollary 2.17. *Let G be an abelian group, then r_2 takes its maximum at the identity element.*

Proof. $r_2(g) = \left| \sum_{\text{real } \chi} \chi(g) \right| \leq \sum_{\text{real } \chi} |\chi(g)| \leq \sum_{\text{real } \chi} \chi(1) = r_2(1)$. □

Similarly for dihedral groups, symmetric groups, alternating groups, and for all groups that don't have χ with $s_2(\chi) = -1$.

Remark. One can define higher Frobenius - Schur indicators:

$$s_k(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^k),\ k \in \mathbb{N}.$$

For $k \geq 3$, these are unbounded as G varies (hint for proof: consider the Heisenberg group of order p^3, i.e., $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{GL}_3(\mathbb{F}_p)$)

To finish our discussion of $\mathbb{R}[G]$-modules, we should talk about Wedderburn components.

Recall: $R[G] \cong \oplus_i M_{n_i}(D_i)$ where D_i are division algebras. In fact $D_i = \mathrm{End}_{\mathbb{R}[G]}(\rho_i)$, where ρ_i are the distinct simple $\mathbb{R}[G]$-modules.

Fact. *The only associative division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$.*

Theorem 2.18. *Let ρ be a complex irreducible representation of G.*

1. *If $\rho \neq \rho^*$, then $\rho \oplus \rho^*$ is realisable over $\mathbb{R}$, is simple as an $\mathbb{R}[G]$-module and the corresponding Wedderburn block is isomorphic to $M_{n_i}(\mathbb{C})$*

2. *If ρ is realisable over $\mathbb{R}$, then the Wedderburn component is isomorphic to $M_{n_i}(\mathbb{R})$*

3. *If $\rho \cong \rho^*$ but not realisable over $\mathbb{R}$ (i.e., ρ is* simpletic *or* quaternion*) then $\rho \oplus \rho$ is realisable over $\mathbb{R}$, it is simple and the corresponding Wedderburn block is isomorphic to $M_{n_i}(\mathbb{H})$.*

Proof (sketch). In cases 1. and 3. , to prove realisability over $\mathbb{R}$, we construct a symmetric non-degenerate, G-invariant pairing.

E.g., in case 3. let $[\,,\,]$ be a G-invariant non-degenerate alternating pairing on ρ. Define $\langle\,,\,\rangle$ on $\rho \oplus \rho$ by $\langle (u_1, v_1), (u_2, v_2) \rangle = [u_1, v_2] - [v_1, u_2]$. We can see that this is symmetric.

Case 1. is omitted

To find the corresponding Wedderburn component, notice that $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ have different dimensions over $\mathbb{R}$. So just need to know $\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}[G]}(-)$, which we use the following lemma for.

Lemma 2.19. *If τ is an $\mathbb{R}[G]$-module, $\tau \otimes_{\mathbb{R}} \mathbb{C}$ the corresponding $\mathbb{C}[G]$-module. Then* $\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}[G]}(\tau) = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}[G]}(\tau \otimes_{\mathbb{R}} \mathbb{C})$

We can calculate the following:

1. $\rho \neq \rho^*$, $\langle \rho \oplus \rho^*, \rho \oplus \rho^* \rangle = 2$ hence $\mathrm{End}_{\mathbb{R}[G]} = \mathbb{C}$
2. $\langle \rho, \rho \rangle_G = 1$ hence $\mathrm{End}_{\mathbb{R}[G]} = \mathbb{R}$
3. $\langle \rho \oplus \rho, \rho \oplus \rho \rangle_G = 4$ hence $\mathrm{End}_{\mathbb{R}[G]} = \mathbb{H}$.

□

Recall that $V^{\otimes n} \cong \oplus_{\chi \in \mathrm{Irr}(S_n)} V^{\otimes n}_{(\chi)}$. In general, if V is a $\mathbb{C}[G]$-module, $V = \oplus_{\chi \in \mathrm{Irr}(G)} V_{(\chi)}$, to find $V_{(\chi)} \subset V$, use idempotent:

- $e_\chi = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g \in \mathbb{C}[G]$ - a primitive central idempotent.

Hence $V_{(\chi)} = e_\chi \cdot V = \{e_\chi \cdot v | v \in V\}$.

Example. Let $G = S_3, V = \mathbb{C}[S_3]$, χ be the the standard character (2-dimensional).

$$e_\chi = \frac{1}{6}\,(2 \cdot \mathrm{id} - (123) - (132)) \in \mathbb{C}[G],$$

e.g, $e_\chi \cdot \mathrm{id} = e_\chi$

$$\begin{aligned} e_\chi(12) &= \frac{1}{6}\,(2 \cdot (12) - (23) - (13)) \\ e_\chi(13) &= \frac{1}{6}(2 \cdot (13) - (12) - (23)) \end{aligned}$$

etc, we get a 4 dimensional subrepresentation of $\mathbb{C}[S_3] \cong \rho^{\oplus 2}$ (where ρ is the standard representation)

3 Permutation representation, monomial representation, induction theorems

Recall: Let X be a finite G-set, i.e., $X = \{1, \dots, n\}$, and there is a group homomorphism $G \to S_n$. Then $\mathbb{C}[X]$ is the associated permutation representation.

Philosophy: these are easy, so we want to express pother representations in terms of these.

Recall: If $X = G/H$, then $\mathbb{C}[X] \cong \mathbb{I}_H \uparrow^G$. In particular, $\langle \mathbb{C}[X], \mathbb{I} \rangle_G = \left\langle \mathbb{I}_H \uparrow^G, \mathbb{I} \right\rangle_G = \langle \mathbb{I}_H, \mathbb{I} \rangle_H = 1$. So we can write $\mathbb{C}[X] \cong 1 \oplus \rho$. When is ρ irreducible?

Lemma 3.1. *If X is transitive, i.e., $\forall x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$, define $H = Stab_G(X)$ for a fixed $x \in X$. Then $X \cong G/H$, i.e., there is a bijection of sets that commutes with the G-action.*

Proof. Define $X \to G/H$ by $g \cdot x \mapsto g \cdot H$.

- This is well-defined and one to one:

$$\begin{aligned} g \cdot x = g' \cdot x, g, g' \in G &\iff g^{-1} g x = g^{-1} g' x \\ &\iff g^{-1} g' \in \mathrm{Stab}_G(x) = H \\ &\iff g \cdot H = g' \cdot H \end{aligned}$$

- Surjective by Orbit-Stabiliser:

$$|G/H| = \frac{|G|}{|H|} = |\mathrm{Orbit}(x)| = |X|$$

- An isomorphism of G-sets: $g(hx) = (gh)x \mapsto (gh)H = g(hH)$

□

Remark. $\mathrm{Stab}_G(g \cdot x) = g \mathrm{Stab}_G(x) g^{-1}$. In particular $G/H \cong G/(gHg^{-1})$

An arbitrary set X can be written as a union of orbits, $X = \coprod_{i=1}^r G/H_i$. Then $\mathbb{C}[X] = \oplus_{i=1}^r \mathbb{C}\left[G/H_i\right]$ and

$$\begin{aligned} \langle \mathbb{C}[X], \mathbb{I} \rangle_G &= \sum_{i=1}^r \langle \mathbb{C}[G/H_i], \mathbb{I} \rangle_G \\ &= r \\ &= \text{number of orbit of } X \text{ under } G. \end{aligned}$$

Lemma 3.2. *Let X be a transitive set, $X \cong G/H$ and χ be the permutation character. Then $\langle \chi, \chi \rangle_G =$ the number of orbits on X under the action of H*

Proof. Let number of orbits under H is

$$\begin{aligned} \langle \chi \downarrow_H, \mathbb{I}_H \rangle_H &= \left\langle \chi, \mathbb{I}_H \uparrow^G \right\rangle_G \\ &= \langle \chi, \chi \rangle_G \end{aligned}$$

□

Corollary 3.3. *$\mathbb{C}[G/H] \cong \mathbb{I} \oplus \rho$ with ρ irreducible if and only if H acts transitively on the non-trivial cosets. We say that $X = G/H$ is* doubly transitive.

Example.

- S_n, $n \geq 2$, acts doubly transitively on $\{1, \dots, n\}$, so we get an $(n-1)$-dimensional irreducible character χ. E.g., $n = 4$,

$$\begin{aligned} \chi'((123)) &= \chi((123)) - \mathbb{I}((123)) \\ &= \underset{\#\text{fixed pts}}{1} - 1 = 0 \end{aligned}$$

- $G = \mathrm{GL}_2(\mathbb{F}_p)$ acts doubly transitively on the $(p+1)$ lines through 0 in $(\mathbb{F}_p)^2$; e.g. the stabiliser of $\langle(1,0)\rangle$ is $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and that acts transitively on the remaining p lines $\langle(a,1)\rangle$, $a \in \mathbb{F}_p$.

  So we get a p-dimensional irreducible representation, $\mathbb{C}[G/H] - \mathbb{I}$.

Artin's Induction. *Let χ be a $\mathbb{Q}$-valued character (i.e., $\chi(g) \in \mathbb{Q}$ for all $g \in G$). Then $\chi = \sum_{H \le G} \frac{a_H}{[N_G(H):H]} \cdot \mathbb{I}_H \uparrow^G$, where the sum runs over representatives over conjugacy classes of cyclic subgroups and $a_H \in \mathbb{Z}$ and $N_G(H) = \{g \in G | gHg^{-1} = H\}$.*

Proof. The idea is to express χ as linear combinations of characteristic functions $\Phi_x(y) = \begin{cases} 1 & \text{if } \langle x\rangle \sim \langle y\rangle \\ 0 & \text{else} \end{cases}$, and then express Φ_x as linear combinations of $\mathbb{I}_H \uparrow^G$.

Lemma 3.4. *$\chi(x) = \chi(y)$ whenever $\langle x\rangle \sim \langle y\rangle$, i.e., χ is a $\mathbb{Q}$-linear combinations of characteristics functions of Φ_x.*

Proof. Suppose $\langle x\rangle \sim \langle y\rangle$, i.e., x is a conjugate to some y^m, where m is coprime to the $\mathrm{ord}(x) = \mathrm{ord}(y) = n$. Up to replacing x by a conjugate, let $x = y^m$. We can diagonalise x and y in the representation corresponding to χ, $x = \mathrm{diag}(\epsilon_1^m, \ldots, \epsilon_d^m)$ and $y = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_d)$, where ϵ_i are nth-roots of unity. There is an automorphism σ of $\mathbb{Q}(e^{2\pi i/n})$ such that for any nth-root of unity ϵ, $\sigma(\epsilon) = \epsilon^m$. So $\sigma(\chi(y)) = \sigma(\sum \epsilon_i) = \sum \epsilon_i^m = \chi(x) \in \mathbb{Q}$. But $\sigma|_{\mathbb{Q}} = \mathrm{id}$, hence $\chi(x) = \chi(y)$ ☐

Let $H_1, \ldots, H_s$ be representatives of conjugacy classes of cyclic subgroups, $\Phi_i(g) = \begin{cases} 1 & \text{if } \langle g\rangle \sim H_i \\ 0 & \text{else} \end{cases}$. By the above lemma, $\mathbb{I}_{H_i} \uparrow^G = \sum_{j=1}^s b_{ij}\Phi_j$. So we want to invert $B = (b_{ij})$. First note that the Φ_i are orthogonal with respect to $\langle\,,\,\rangle_G$:

$$\begin{aligned} \langle \Phi_i, \Phi j\rangle &= \frac{1}{|G|}\sum_{g \in G} \Phi_i(g)\Phi_j(g) \\ &= \begin{cases} 0 & i \ne j \\ \frac{1}{|G|}\phi(|H_i|) \cdot \frac{|G|}{|N_G(H_i)|} = \frac{\phi(|H_i|)}{|N_G(H_i)|} & i = j \end{cases} \end{aligned}$$

By definition, $\Phi_j \downarrow_{H_i}$ is identically 0 unless $H_j \le_G H_i$ in this case $\Phi_j \downarrow_{H_i} = 1$ on the $\phi(|H_j|)$ generators of H_j in H_i. Now

$$\begin{aligned} b_{ij} \cdot \frac{\phi(|H_i|)}{|N_G(H_j)|} &= \langle \mathbb{I}_{H_i} \uparrow^G, \Phi_j\rangle_G \\ &= \langle \mathbb{I}_{H_i}, \Phi_j \downarrow_{H_i}\rangle_{H_i} \\ &= \frac{1}{|H_i|} \cdot \phi(|H_j|) \text{ if } H_j \le_G H_i. \end{aligned}$$

Hence $b_{ij} = \frac{|N_G(H_j)|}{|H_i|}$ if $H_j \le_G H_i$ and otherwise $b_{ij} = 0$.

Now order the H_i by size, then we have established that B is triangular, integer entries, and in the ith row, all entries are divisible by $[N_G(H_i) : H_i]$,because if $H_j \le H_i$ then $|N_G(H_i)| \,\big|\, |N_G(H_j)|$. It follows that B is invertible, with denominators in the ith row of the inverse dividing $[N_G(H_i) : H_i]$. ☐

Remark. It is still an open question, how "bad" these denominators can get, e.g., we do not know for what groups G, any $\mathbb{Q}$-valued χ can be written as $\sum_{H \le G, \text{cyclic}} c_H \mathbb{I}_H \uparrow^G$ with $c_H \in \mathbb{Z}$. This is possible in S_n.

Example. Let $G = C_p \times C_p$. There are $p+1$ cyclic subgroups of order p, denote them $H_1, \ldots, H_{p+1}$. Any irreducible, non-trivial character χ factors through a unique G/H_i. In fact, $\mathbb{I}_{H_i} \uparrow^G = \sum_{\ker \chi_{ij} \ge H_i} \chi_{ij}$ with $\chi_{i1} = \mathbb{I}$. After solving the system of linear equations, we find that $\mathbb{I}_{\{1\}} \uparrow^G - \sum_i \mathbb{I}_{H_i} \uparrow^G = -p \cdot \mathbb{I}$.

Corollary 3.5. *The number of irreducible $\mathbb{Q}[G]$-modules is equal to conjugacy classes of cyclic subgroups.*

Proof. This corollary also depends on the theory of Schur indices, which we will cover later. □

Corollary 3.6. *Two $\mathbb{Q}[G]$-modules V_1, V_2 are isomorphic if and only if $\dim V_1^H = \dim V_2^H$ for all cyclic $H \leq G$.*

Proof. Exercise □

Example. Let $G = S_3$,

- $\mathbb{I}_{\{1\}} \uparrow^G = \mathbb{I} \oplus \epsilon \oplus \rho^{\oplus 2}$, where ϵ is the sign representation and ρ is the standard representation
- $\mathbb{I}_{C_2} \uparrow^G = \mathbb{I} \oplus \rho$
- $\mathbb{I}_{C_3} \uparrow^G = \mathbb{I} \oplus \epsilon$

Now $\phi = \mathbb{I}$, $-\mathbb{I}_{\{1\}} \uparrow^G + 2 \cdot \mathbb{I}_{C_2} \uparrow^G + \mathbb{I}_{C_3} \uparrow^G = 2 \cdot \mathbb{I}$.

Example. Let $G = C_p \rtimes C_{p-1}$, then $-\mathbb{I}_{\{1\}} \uparrow^G + (p-1)\mathbb{I}_{C_{p-1}} \uparrow^G + \mathbb{I}_{C_p} \uparrow^G = (p-1) \cdot \mathbb{I}_G$. Prove this as an exercise

Remark. Even if one was allowed to use $\mathbb{I}_H \uparrow^G$ for all $H \leq G$, one would still have denominators

Example. Let $G = Q_8 \times C_3$ and let ρ be the standard representations of Q_8, χ a 1-dimensional non-trivial character of C_3. Then $\rho \otimes (\chi \otimes \overline{\chi})$ is a representation that can be defined over $\mathbb{Q}$, but it is not a $\mathbb{Z}$-linear combination of $\mathbb{I}_H \uparrow^G$ for all $H \leq G$, but twice that is.

Definition 3.7. A group is called *p-quasi-elementary* if it's of the form $G = C \rtimes P$ where C is cyclic and P a p-group (i.e., order p^n for some n).

Without loss of generality, we can assume $p \nmid |G|$.

Solomon Induction. *There exists $a_H \in \mathbb{Z}$ for $H \leq G$ quasi-elementary subgroups such that $\mathbb{I} = \sum_{H \leq G} a_H \mathbb{I}_H \uparrow^G$.*

Brauer's Induction Theorem. *Let $\phi \in \mathrm{Irr}(G)$. Then there exists $a_{H,\lambda} \in \mathbb{Z}$ for H of the form $H = C \times P$ (with C cyclic and P a p-group, these are called elementary groups), such that $\phi = \sum a_{H,\lambda} \lambda \uparrow^G$, where λ are 1-dimensional characters of elementary subgroups.*

We will first deduce Brauer from Solomon, to do so we will use for the first time the ring structure of the ring of class functions.

Definition 3.8. We define $R(G) = \langle \mathrm{Irr}(G) \rangle_{\mathbb{Z}} = \left\{ \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \cdot \chi \middle| a_{\chi in} \mathbb{Z} \right\}$.

For any family of subgroups of G, $\mathcal{H}$, we define $I_{\mathcal{H}}(G) = \left\{ \sum_{H \in \mathcal{H}, \lambda \in \mathrm{Irr}(H)} a_{H,\lambda} \lambda_H \uparrow^G \middle| a_{H,\lambda} \in \mathbb{Z} \right\}$.

Lemma 3.9. *Let $H \leq G$, ϕ a class function of H, ψ a class function of G. Then $\phi_H \uparrow^G \cdot \psi = (\phi \cdot \psi \downarrow_H) \uparrow^G$.*

Proof. Just do it. (exercise) □

Corollary 3.10. *$I_{\mathcal{H}}(G)$ is an ideal in $R(G)$*

Proof. If $\phi = \sum a_{H,\lambda} \lambda_H \uparrow^G \in I_{\mathcal{H}}(G)$, $\psi \in \mathrm{Irr}(G)$, then $\psi \cdot \phi = \sum a_{H,\lambda} \psi \lambda_H \uparrow^G = \sum a_{H,\lambda} (\psi \downarrow_H \cdot \lambda) \uparrow^G \in I_{\mathcal{H}}(G)$. □

Now let $\mathcal{H} = \left\{ C \times P \leq G \middle| C \, \text{cyclic}, P \, \text{a} \, p \, \text{group} \right\}$. We will prove Brauer if we can show

- $\mathbb{I} \in I_{\mathcal{H}}(G) =: I(G)$
- All elementary groups are M-groups, i.e., every irreducible characters is monomial, i.e., induced from a 1-dimensional character. (This is left as an exercise)

Theorem 3.11. $\mathbb{I} \in I(G)$.

Proof (assuming Solomon). We do this by induction on $|G|$. We can use elementary groups G as our base case.

Assume that the theorem holds for all proper subgroups of G, i.e., for all $H \leq G$, $\mathbb{I}_H = \sum_{U \leq H \text{ elem}, \lambda \in \mathrm{Irr}(U)} a_{U,\lambda} \lambda_U \uparrow^H$. Then it is enough to show that

$$\mathbb{I} = \sum_{H \lneq G, \lambda \in \mathrm{Irr}(H)} b_{H,\lambda} \lambda_H \uparrow^G, \quad (\dagger)$$

because then, each $\lambda = \sum_{U \leq H \text{ elem}} a_{u,\mu}^{(\lambda)} \mu \uparrow^H$ and $\mathbb{I}_G = \sum \sum b_{H,\lambda} a_{U,\mu}^{(\lambda)} \mu \uparrow^H \uparrow^G$. If G is not quasi-elementary, then Solomon shows ($\dagger$).

So we are left with proving the statement

$$\mathbb{I} = \sum_{H \lneq G, \lambda \in \mathrm{Irr}(H)} b_{H,\lambda} \lambda_H \uparrow^G, \ b_{H,\lambda} \in \mathbb{Z}$$

for $G = C \rtimes P$, where P acts non-trivially on C by conjugation, so that $G \neq C \times P$. Let $Z = Z_C(P) = \{x \in C | xpx^{-1} = p \forall p \in P\}$. Since $G \neq C \times P$, $Z \neq C$. Set $E = ZP \neq G$.

We have $\mathbb{I}_E \uparrow^G = \mathbb{I}_G + \Xi$. It is enough to show that any irreducible summand of Ξ is induced from a proper subgroup. Let ξ be an irreducible summand of Ξ. Let χ be an irreducible summand of $\xi \downarrow_C$. Let $S = \mathrm{Stab}_G(\chi)$. Recall that if ι is an irreducible summand of $\chi \uparrow^S$, then $\iota \uparrow^G$ is irreducible and all summands of $\chi \uparrow^G$ are of this form (in particular this is true for ξ). So we now just need to know that $S \neq G$, i.e., that χ is not invariant under the G-action. To do so, we use the following lemma.

Lemma 3.12. *Let $G = C \rtimes P$, $p \nmid |C|$, $\chi \in \mathrm{Irr}(C)$, $Z = Z_C(P)$ and assume $Z \in \ker \chi =: K$. If χ is invariant in G, then $\chi = \mathbb{I}_C$.*

Proof. χ is a faithful character on C/K, so for χ to be invariant, G has to preserve each coset cK. But if P acts on cK, then the number of points moved is divisible by p (by Orbit - Stabiliser). But $p \nmid |C|$ so $p \nmid |K|$, hence $p \nmid |cK|$. So there is at least one point in cK that is normalised by P, i.e., $cK \cap Z \neq \emptyset$. But $Z \subseteq K$, so $cK = K$ for all c, i.e., $\ker \chi = K = C$. □

In our situation, $Z \lhd G$, so by Clifford, $Z \subset \ker(\mathbb{I}_E \uparrow^G)$, so in particular $Z \subset \ker \chi$. We claim that $\chi \neq \mathbb{I}$: note

$$\begin{aligned} \langle \mathbb{I}_G \downarrow_C + \Xi \downarrow_C, \mathbb{I} \rangle &= \langle \mathbb{I}_E \uparrow^G \downarrow_C, \mathbb{I} \rangle \\ &= \langle \oplus_{E \backslash G / C} \mathbb{I} \downarrow \uparrow, \mathbb{I} \rangle \ \text{ but } E\backslash G/C = E \cdot 1 \cdot C \\ &= 1 \end{aligned}$$

So $\langle \xi \downarrow_C, \mathbb{I} \rangle = 0$, so $\chi \neq \mathbb{I}$. Hence χ is not invariant. □

Lemma 3.13 (Banashewski)**.** *Let S be a finite set, and R be a rng (i.e., a ring which does not necessarily contain 1) of functions $f : S \to \mathbb{Z}$. Then either $R \ni \mathbb{I}_S$ or there exists $s \in S$ and a prime p such that $p | f(x) \forall x \in R$.*

Proof. Suppose there is no such x, p. Then for any $x \in S$, $\gcd\{f(x) | f \in R\} = 1$. So there exists $x \in R$ such that $f_x(x) = 1$. Consider $\prod_{x \in S}(f_x - \mathbb{I}_S) \equiv 0$ on S. So expanding the product gives an expression for $\mathbb{I}_S$ as a linear combination of products of $f_x \in R$. □

Definition 3.14. We define $P_{\mathcal{H}}(G) = \left\{ \sum_{H \in \mathcal{H}} a_H \mathbb{I}_H \uparrow^G | a_H \in \mathbb{Z} \right\}$.

Lemma 3.15. *Suppose that $\mathcal{H}$ is closed under taking subgroups, i.e., $H \in \mathcal{H}$ implies $U \in \mathcal{H}$ for all $U \leq H$. Then $P_{\mathcal{H}}(G)$ is a rng.*

Proof. Either use

$$\begin{aligned} \mathbb{I}_H \uparrow^G \cdot \mathbb{I}_H \uparrow^G &= \left(\mathbb{I}_H \cdot \mathbb{I}_H \uparrow^G \downarrow_H \right) \uparrow^G \\ &= \sum_{H \backslash G / H'} \mathbb{I} \downarrow_{H \cap gH'g^{-1}} \uparrow^G \in P_{\mathcal{H}}(G) \end{aligned}$$

or note that if $v_1, \ldots, v_n$ is a permutation basis of V, $w_1, \ldots, w_m$ is a permutation basis of W, then $v_i \otimes w_j$ is a permutation basis. and the point stabiliser of $v_i \otimes w_j = \mathrm{Stab}(v_i) \cap \mathrm{Stab}(w_j) \in \mathcal{H}$ if one of the other stabilisers was in $\mathcal{H}$. □

We want to use Banaschewski's lemma to conclude that if $\mathcal{H} = \{\text{quasi-elementary subgroups}\}$, then $\mathbb{I}_G \in P_{\mathcal{H}}(G)$.

Lemma 3.16. *For any prime p, any $x \in G$, there exists a quasi-elementary $H \leq G$ such that $p \nmid \mathbb{I}_H \uparrow^G (x)$.*

Proof. For $x \in G$, write $\langle x \rangle = C_p \times C_{p'}$, where C_p is a p-group and $C = C_{p'}$ has order not divisible by p. Let $N = N_G(C)$, let P be a p-Sylow group in N containing C_p. Set $H = C \rtimes P$.
Claim. $p \nmid \mathbb{I}_H \uparrow^G (x)$

Indeed,

$$\begin{aligned} \mathbb{I}_H \uparrow^G (x) &= \# \left\{ gH \in G/H \middle| xgH = gH \right\} \\ &= \# \left\{ gH \in G/H \middle| g^{-1}xg \in H \right\}. \end{aligned}$$

If $g^{-1}xg \in H$, then $g^{-1}Cg = C$. So

$$\begin{aligned} \mathbb{I}_H \uparrow^G (x) &= \# \left\{ gH \in G/H \middle| g^{-1}Cg = C \,\text{and}\, g^{-1}xg \in H \right\} \\ &= \# \left\{ gH \in N/H \middle| g^{-1}xg \in H \right\}. \end{aligned}$$

The action of $\langle x \rangle$ on N/H factors through $\langle x \rangle / C$, i.e., C acts trivially on N/H: indeed $C \lhd N$ and $C \leq H$, so $c \cdot nH = n \cdot c' \cdot H = nH$. But $\langle x \rangle / C$ is a p-group, so the number of elements of N/H that are <u>not</u> fixed by $\langle x \rangle / C$ is a multiple of p. So

$$\begin{aligned} \mathbb{I}_H \uparrow^G (x) &\equiv |N/H| \mod p \\ &\not\equiv 0 \mod p \end{aligned}$$

because H/C is a p-Sylow of N/C.

□

Remark. There is a counterpart to Brauer's theorem, which is called "Brauer's characterisation of Characters":

Theorem 3.17. *A class function ϕ of G is a $\mathbb{Z}$-linear combination of characters ($\phi \in R(G)$) if and only if $\phi \downarrow_H \in R(G)$ for all $H \leq G$ elementary subgroups.*

Idea of Proof. Define $R_{\mathcal{H}}(G) = \{\text{class functions}\, \phi \,\text{of}\, G | \phi \downarrow_H \in R(H) \,\forall H \in \mathcal{H}\}$. Note that $I_{\mathcal{H}}(G) \subset R_{\mathcal{H}}(G)$ is an ideal (exercise). But $\mathbb{I} \in I_{\mathcal{H}}(G)$, so $I_{\mathcal{H}}(G) = R_{\mathcal{H}(G)}$. □

For consequences, see Isaacs, chapter on Brauer's Theorem

4 Rationality questions, Schur indices.

Definition 4.1. Let M be a $K[G]$-module, $F \subset K$ a subfield. We say that M is *realisable* over F if there exists an $F[G]$-module M_F such that $M_F \otimes_F K \cong M$.

In the language of representations: In the language of representations, this means that we can find a K-basis on M such that all $g \in G$ are represented by matrices with entries in F with respect to this basis.

Example. Consider $G = S_3$. We have ρ: $(123) \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{pmatrix}, (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. But we can change this basis in such a way that ρ becomes $(123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So the $K[G]$-representation ρ ($K = \mathbb{Q}(e^{2\pi i/3}) = \mathbb{Q}(\sqrt{-3})$) is realisable over $\mathbb{Q}$.

- In fact, ρ is induced from $\chi : (123) \mapsto e^{2\pi i/3}$, $\langle (123) \rangle \cong C_3 \lhd S_3$. This χ is definitely <u>not</u> realisable over $\mathbb{Q}$, since $\mathrm{GL}_1(K)$ is commutative, so change of basis doesn't do anything to $\chi((123))$.

  More generally, the character of a representation is independent of basis, so if ρ realisable over F, then we need $F \supseteq \mathbb{Q}(\chi_\rho)$, where $\mathbb{Q}(\chi_\rho)$ is the field generated over $\mathbb{Q}$ by $\chi_\rho(g)$ for all $g \in G$.

Let $G = Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1}\rangle$. Let $\rho : x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this is a $K[G]$-representation where $K = \mathbb{Q}(i)$. We already know that ρ can not be defined over $\mathbb{R}$, so it is certainly not realisable over $\mathbb{Q}$. But we have exactly two copies of ρ inside $\mathbb{C}[G]$, which is realisable over $\mathbb{Q}$. The other summands in $\mathbb{C}[G]$ are 1-dimensional, all realisable over $\mathbb{Q}$. So $\mathbb{Q}[G]/(\text{all 1-dimensional subrepresentations}) \cong \rho^{\oplus 2}$. So $\rho^{\oplus 2}$ is realisable over $\mathbb{Q}$.

Definition 4.2. Let $K \subset \mathbb{C}$, ρ an irreducible (complex) representation of G, the *Schur index*, $m_K(\rho)$, of ρ over K is the the smallest integer m such that there exists an irreducible $K[G]$-representation τ with $\langle \tau, \rho\rangle = m$. Equivalently, it's the unique integer m such that $m | \langle \rho, \tau\rangle$ for all $K[G]$-module τ.

Example. We have:

- $m_{\mathbb{Q}}(\text{standard representation of } Q_8) = 2$
- $m_{\mathbb{Q}}(\text{standard representation of } S_3) = 1$.
- $m_{\mathbb{Q}}(\chi : (123) \mapsto e^{2\pi i/3}) = 1$, although χ is not realisable over $\mathbb{Q}$. (Note that $\chi + \overline{\chi}$ is definable over $\mathbb{Q}$)

Definition 4.3. A representation over K is said to be *absolutely irreducible* if it is irreducible over $\mathbb{C}$.

A field $K \subset \mathbb{C}$ is called a *splitting field* of G if every irreducible $K[G]$-representation is absolutely irreducible, equivalently if every complex G-representation is realisable over K.

Lemma 4.4. *Let χ be an irreducible character of G, $F \subset \mathbb{C}$ such that $F(\chi) = F$, i.e., χ takes values in F. Let τ be an irreducible $F[G]$-representations such that $\langle \tau, \chi\rangle \neq 0$. Then $\tau \otimes \mathbb{C} = m_F(\chi) \cdot \chi$.*

Proof. The element $e_\chi = \frac{1}{|G|}\sum_{g\in G} \chi(g)g^{-1} \in F[G]$. So $e_\chi \cdot \tau = \tau$. But for any complex representation V, $e_\chi \cdot V \cong \chi^{\oplus n}$, for some n. But this n has to be $m_F(\chi)$. □

If χ_ρ is a character, why is χ_ρ^σ a character for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi_\rho)/\mathbb{Q}) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\chi_\rho))$? We have that if $\rho : G \to \mathrm{GL}_n(\overline{\mathbb{Q}})$ is defined by $g \mapsto (a_{ij})$, then $\rho^\sigma : G \to \mathrm{GL}_n(\overline{\mathbb{Q}})$ is defined by $g \mapsto (a_{ij}^\sigma)$. Now if σ fixes $\mathbb{Q}(\chi_\rho)$, then by definition $\chi_{\rho^\sigma} = \chi_\rho$. Hence $\rho^\sigma \cong \rho$.

Theorem 4.5. *Let $K \subset \mathbb{C}$ be arbitrary, $\rho \in \mathrm{Irr}(G)$, τ a simple $K[G]$-module such that $\langle \tau, \rho\rangle \neq 0$. Then*

$$\tau \otimes \mathbb{C} = m_K(\rho) \sum_{\sigma \in \mathrm{Gal}(K(\chi_\rho)/K)} \rho^\sigma.$$

Proof. Let $F = K(\chi_\rho)$, let ψ be a simple $F[G]$-module such that $\langle \tau, \psi\rangle \neq 0$. So by the lemma, $\psi \otimes \mathbb{C} = m_F(\rho) \cdot \rho$. Let $\sigma \in \mathrm{Gal}(F/K)$. Since $\tau^\sigma = \tau$ (as τ is a $K[G]$-module),

$$\begin{aligned} \langle \tau, \psi^\sigma\rangle &= \langle \tau^\sigma, \psi^\sigma\rangle \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_\tau^\sigma \overline{\chi_\psi^\sigma} \\ &= \left(\frac{1}{|G|}\sum_{g\in G} \chi_\tau \overline{\chi_\psi}\right)^\sigma \\ &= \langle \tau, \psi\rangle^\sigma \\ &= \langle \tau, \psi\rangle. \end{aligned}$$

So each Galois conjugate occurs with equal multiplicity inside τ, i.e., $\tau \otimes F = \alpha \sum \psi^\sigma$+stuff that is not Galois conjugate to ψ. So $\tau \otimes \mathbb{C} = \alpha m_F(\rho) \sum_{\sigma \in \mathrm{Gal}(F/K)} \rho^\sigma$+stuff that has nothing to do with ρ. In particular $m_F(\rho) | m_K(\rho)$. We just need to prove now that $m_F(\rho)\sum \rho^\sigma$ is realisable over K.

Let V be the F-vector space on which ψ is represented. Regarding F as a $|\mathrm{Gal}(F/K)|$-dimensional K-vector space, we can think of V as a $|\mathrm{Gal(F/K)}| \cdot \dim_F V$-dimensional vector space over K. Inspecting the action of G on V with respect to this K-basis, one can see that $V \otimes_K F = \sum_{\sigma \in \mathrm{Gal}(F/K)} \psi^\sigma$, which is realisable over K. □

Example. Let $G = C_7 \rtimes C_9$, where C_9 acts on C_7 through $C_9/C_3 \cong C_3$. Let $\chi \in \mathrm{Irr}(C_7)$ be faithful, extended trivially to $S_\chi \cong C_3$. Let τ be a faithful character of $C_3 \cong C_7 \times C_3/C_7$, $\rho = (\chi \otimes \tau)\uparrow^G \in \mathrm{Irr}(G)$. The field $\mathbb{Q}(\chi_\rho)$ is the degree 4 subfield of $\mathbb{Q}(\zeta_{21})$, call it F. In particular, $\sum_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \rho^\sigma$ takes values in $\mathbb{Q}$. But $m_\mathbb{Q}(\rho) = 3$. So $\sum_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \rho^\sigma$ is not a character of a $\mathbb{Q}[G]$-module, but 3 times it is (and it's a simple $\mathbb{Q}[G]$-module).

Corollary 4.6. *We have $m_K(\rho) = m_{K(\chi_\rho)}(\rho)$.*

Theorem 4.7. *Let $e = \mathrm{exponent}(G) = \min\{n \in \mathbb{N} | g^n = \mathrm{id}\, \forall g \in G\}$. Let $F = \mathbb{Q}(\zeta_e)$. Then F is a splitting field for G.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. Write

$$\chi = \sum_{H \leq G \text{ elem}, \lambda \in \mathrm{Irr}(H)\ 1-\dim} a_{H,\lambda} \lambda_H \uparrow^G$$

with $a_{H,\lambda} \in \mathbb{Z}$. Clearly λ can be realised over F, therefore so can $\lambda\uparrow^G$. But $m_F(\chi) | \langle \chi, \text{every } F[G]-\text{module} \rangle$, so $m_F(\chi) | \langle \mathrm{RHS}, \chi \rangle = \langle \chi, \chi \rangle = 1$. So $m_F(\chi) = 1$, so χ is realisable over F. □

Remark.

- Schur indices can be arbitrarily large: let p, q be primes, $q \equiv 1 \mod p$ and $q \not\equiv 1 \mod p^2$. Let $G = C_q \rtimes C_{p^2}$ where C_{p^2} acts on C_q through $C_{p^2}/C_p \cong C_p$. Then using the same notation as in the above example $\chi \otimes \psi$ is a faithful irreducible character of $C_q \times C_p$. Now let $p = (\chi \otimes \psi)\uparrow^G$, then $m_\mathbb{Q}(\rho) = p$ (this claim can not be proven with techniques learn in this course)

- Schur indices don't behave very well under induction, restriction and tensor products (in part, if $G = H \times K$, $\chi \in \mathrm{Irr}(H)$, $\psi \in \mathrm{Irr}(K)$, $m_\mathbb{Q}(\chi \otimes \psi) \neq m_\mathbb{Q}(\chi) \cdot m_\mathbb{Q}(\psi)$).

- Schur indices are hard to calculate. There is an algorithm, based on another induction theorem (Witt - Berman). But it's difficult to understand Schur indices in natural families.

- There is no unique minimal field of definition for $\rho \in \mathrm{Irr}(G)$.

  Example. Let $G = Q_8$ and ρ be the standard representation. We know that ρ is not realisable over $\mathbb{Q}$. But it is realisable over $\mathbb{Q}(i)$. In fact ρ is realisable over K if and only if $-1 = x^2 + y^2$ for some $x, y \in K$. In particular, ρ is realisable in infinitely many quadratic fields.

- Note: $m_\mathbb{Q}(\rho) = 1$ if and only if $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi_\rho)/\mathbb{Q})} \rho^\sigma$ is realisable over $\mathbb{Q}$, if and only if ρ is realisable over $\mathbb{Q}(\chi_\rho)$.

4.1 Schur indices and Artin - Wedderburn

Recall: $K[G] \cong \oplus_i M_{n_i}(D_i)$ with $n_i \in \mathbb{N}$, and D_i are division algebras over K. These D_i are isomorphic to $\mathrm{End}_G(A_i)$ as A_i ranges over simple non-isomorphic $K[G]$-modules. Each D_i contains K in its centre, $Z = Z(D_i)$. Also Z is a field, but D_i can also contain bigger fields.

Example. Let $D = \mathbb{H} = \langle 1, i, j, k \rangle_\mathbb{R}$. Note that $\mathbb{Z} \cong \mathbb{R} = \langle 1 \rangle$, but $\langle 1, i \rangle_\mathbb{R} \cong \mathbb{C}$.

We will show the following theorem:

Theorem 4.8. *If $\chi \in \mathrm{Irr}(G)$, $K = \mathbb{Q}(\chi)$, M a simple $K[G]$-module with $\langle M, \chi \rangle \neq 0$, i.e., $M \otimes \mathbb{C} = m_K(\chi) \cdot \chi$, and let $D = \mathrm{End}_{K[G]}(M)$. If $F \subset D$ is a maximal subfield of D, then χ can be realised by a representation over F. Moreover $[F : K] = m_K(\chi)$, and χ cannot be realised over any smaller degree extension.*

Lemma 4.9. *Let $\rho : G \to \mathrm{GL}_n(F)$ be an irreducible representation over F. Then the following are equivalent:*

1. *ρ is absolutely irreducible*
2. *The centraliser of $\rho(G)$ in $M_n(F)$ is just scalar matrices.*

Proof. Note that the centraliser of $\rho(G)$ in $M_n(F)$ is $\mathrm{End}_G(\rho) \cong F$.

1. $\Rightarrow$2. Suppose that ρ is absolutely irreducible, let $M \in M_n(F)$ commute with $\rho(g)$ for all $g \in G$. Let L/F be an extension in which M has an eigenvalue, λ. Then $M - \lambda I$ commutes with $\rho(g)$ for all $g \in G$. But $\rho \otimes L$ is still simple, so Schur's lemma implies $M - \lambda I$ is either 0 or invertible. But, its singular, so $M - \lambda I = 0$, so $M = \lambda I$.

2. $\Rightarrow$1. Suppose that ρ is not absolutely irreducible. Let L/F be a finite Galois extension of F such that $\rho \otimes L \cong \rho_1 \oplus \rho_2$, so with respect to a suitable L-basis, $\rho \to \left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right)$. Let X be the change of basis matrix form the original basis to this one. Any matrix of the form $A_{\lambda_1 \lambda_2} = \left(\begin{array}{c|c} \lambda_1 I_{d_1} & \\ \hline & \lambda_2 I_{d_2} \end{array}\right)$ commutes with $\rho(G)$, note $\lambda_1, \lambda_2 \in L$. So $X A_{\lambda_1, \lambda_2} X^{-1}$ commutes with $\mathrm{im}(\rho \to M_n(F))$. Also $\overline{A} = \sum_{\sigma \in \mathrm{Gal}(L/F)} \left(X A_{\lambda_1 \lambda_2} X^{-1}\right)^{\sigma} \in M_n(F)$ commutes with $\mathrm{im}(\rho \to M_n(F))$. Exercise: by varying λ_1, λ_2, we can arrange $\overline{A}$ to not be scalar.

□

Proof of Theorem 4.8. Since $F \subset \mathrm{End}_{G[K]}(M)$, M can be thought as a vector space over F, and this F-commutes with the G-action, so M becomes an $F[G]$-module. But $\mathrm{End}_{F[G]}(M)$ is the centraliser of $F[G]$ in D. But by the maximality of F, $\mathrm{End}_{F[G]}(M) \cong F$, so M is an absolutely simple $F[G]$-module. So $M \otimes_F \mathbb{C}$ has character χ.

It remains to compute $[F : K]$. Regarding F as a K-vector space, M becomes a $K[G]$-module of K-dimension equal to $\dim_F(M) \cdot [F : K]$. But also, this dimension is $m_K(\chi) \cdot \chi(1)$; on the other hand, M is an absolutely simple $F[G]$-module, so $\dim_F(M) = \chi(1)$. So $m_K(\chi) = [F : K]$. Also, if L/K is a degree d-extension such that χ is realised by a simple $L[G]$-module M' then regarding L as a K-vector space, M' becomes a $K[G]$-module with $\dim_K(M') = \dim_L(M') \cdot [L : K] = [L : K] \cdot \chi(1)$. So by definition, $m_K(\chi) | [L : K]$. □

Remark.

- There is a local to global theory of Schur indices: If D is a division algebra over a number field K, one defines $m_{\mathfrak{p}}(D) = m(D \otimes K_{\mathfrak{p}})$ for all place (finite and infinite) $\mathfrak{p}$ of K, and $m(D) = \mathrm{lcm}(m_{\mathfrak{p}}(D))$.
- If p is an odd prime, and G is a p-group, then $m_{\mathbb{Q}}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$.

  If $p = 2$ and G is a 2-group, then $m_{\mathbb{Q}}(\chi) = 1$ or 2, and it's 1 if and only if χ is realisable over $\mathbb{R}$.
- $m(\chi)^2 \big| |G|$

For more details see Curtis - Reiner, Vol II.

5 Examples

1. Let $G = C_2 \wr C_3 = (C_2 \times C_2 \times C_2) \rtimes C_3$. The conjugacy classes:

   - $1 = ((0,0,0), \mathrm{id})$
   - $((1,0,0), \mathrm{id})$, this has 3 elements
   - $((1,1,0), \mathrm{id})$, this has 3 elements
   - $((1,1,1), \mathrm{id})$, this has 1 element
   - $((0,0,0), (123))$, this has 4 elements
   - $((0,0,0), (132))$, this has 4 elements
   - $((1,0,0), (123))$, this has 4 elements
   - $((1,0,0), (132))$, this has 4 elements

   All irreducible characters are $(\chi \otimes \psi) \uparrow^G$, with $\chi \in \mathrm{Irr}(C_2^3)$ and $\psi \in \mathrm{Irr}(\mathrm{Stab}_{C_3}(\chi))$. Let ϵ denote the sign representation.

   - $\chi = \mathbb{I}$. Then $\mathrm{Stab}(\chi) = C_3$, so we get 3 irreducible characters corresponding to the irreducible characters of C_3.
   - $\chi = (\epsilon, \mathbb{I}, \mathbb{I})$, then $\mathrm{Stab}(\chi) = \{1\}$. Hence $\chi \uparrow^G$ is irreducible.
   - $\chi = (\epsilon, \epsilon, \mathbb{I})$, then $\mathrm{Stab}(\chi) = \{1\}$. Hence $\chi \uparrow^G$ is irreducible.
   - $\chi = (\epsilon, \epsilon, \epsilon)$, then $\mathrm{Stab}(\chi) = C_3$. So $\chi \otimes \psi$ is irreducible for $\psi \in \mathrm{Irr}(C_3)$.

	1	$((1,1,1),\mathrm{id})$	$((1,0,0),\mathrm{id})$	$((1,1,0),\mathrm{id})$	$((0,0,0),(123))$	$((0,0,0),(132))$	$((1,0,0),(123))$	$((1,0,0),(132))$
$\chi_1 = \mathbb{I}$	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$
χ_3	1	1	1	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{2\pi i/3}$
χ_4	1	-1	-1	1	1	1	-1	-1
χ_5	1	-1	-1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$-e^{2\pi i/3}$	$-e^{4\pi i/3}$
χ_6	1	-1	-1	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	$-e^{4\pi i/3}$	$-e^{2\pi i/3}$
χ_7	3	-3	1	-1	0	0	0	0
χ_8	3	3	-1	-1	0	0	0	0

   We see that $s_2(\chi_{2,3,5,6}) = 0$, $s_2(\mathbb{I}) = 1$, $s_2(\chi_4) = 1$, and $s_2(\chi_{7,8}) = 1$ because the corresponding representations are inductions of rational representations.

   So for example, the number of square roots of

$$\begin{aligned} ((1,1,1), \mathrm{id}) &= \sum_{\chi \in \mathrm{Irr}} s_2(\chi) \cdot \chi((1,1,1), \mathrm{id}) \\ &= 1 - 1 - 3 + 3 = 0. \end{aligned}$$

   Indeed, there are no elements of order 4.

   We could explicate Brauer's induction theorem, e.g., $\chi_7 = \mathrm{Ind}_{c_2^3}$. But for $\chi_{1\ldots6}$, need to induce various characters of elementary subgroups and look for linear relationships.

2. Let $G = \mathrm{PSL}_2(\mathbb{F}_7) = \mathrm{SL}_2(\mathbb{F}_7)/\{\pm \mathrm{id}\}$. This is a simple group of order 168. This has conjugacy classes are id, $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 1 & 4 \end{pmatrix}$. Let $B = \left\langle \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \cong C_7 \rtimes C_3$. Using Machke, you can check that if χ is a one-dimensional non-trivial character of B, then $\chi\uparrow^G$ is irreducible (8-dimensional). (To compute $\langle \chi\uparrow^G, \chi\uparrow^G \rangle = \langle \chi, \chi\uparrow^G\downarrow_B \rangle = \langle \chi, \oplus_{B\backslash G/B}\, {}^g\chi\downarrow_{B\cap {}^gB} \rangle$).

   G acts doubly-transitively on lines in $(\mathbb{F}_7)^2$. (SL_2 acts doubly transitively, because, the line $\langle(1,0)\rangle$ is stabilised by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which permutes the lines $\langle(a,1)\rangle$; and $\pm\mathrm{id}$ acts trivially). Hence the (permutation character $-\mathbb{I}$) is irreducible. (7-dimensional).

   Also, $\mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{GL}_3(\mathbb{F}_2) \cong \mathrm{PSL}_3(\mathbb{F}_2)$, which acts on non-zero vectors in $(\mathbb{F}_2)^3$. So we get an irreducible $7-1=6$-dimensional character. By $|G| = \sum(\dim\chi)^2$, there are 2 3-dimensional irreducible characters.

   These can be obtained by column orthogonality. It then turns out that s_2(3-dimensionals) $= 0$. The 7-dimensional character is the character of a $\mathbb{Q}[G]$-module/$\mathbb{Q}[G]$-submodule, so the representation is defined over $\mathbb{Q}$. Similarly, the 6-dimensional one. The 8-dimensional is, a priory, defined over $\mathbb{Q}\left(\sqrt{-3}\right) = \mathbb{Q}\left(e^{2\pi i/3}\right)$. But on the other hand, it is real-valued, and $s_2(\chi\uparrow^G) = 1$ (check, but only if you feel adventurous). From general theory of Schur indices (beyond the scope of this course), $m_{\mathbb{Q}}(\chi\uparrow^G) = 1$.

3. Let $G = S_7$. The conjugacy classes are: id, $(12), (12)(34), (12)(34)(56), \ldots$ (there are 15 of them).

   Obvious Characters: $\mathbb{I}$, sign, $\chi =$ (standard permutation character $-\mathbb{I}$) which is 6 dimensional, and $\chi\otimes$ sign. Consider $\chi^{\otimes 2} = S^2\chi \oplus \wedge^2\chi$, note that $S^2\chi = \mathbb{I} + \ldots$, hence it is still reducible. We have $\wedge^2\chi(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$, so we can explicitly work out its values, and hence can calculate that $\langle \wedge^2\chi, \wedge^2\chi \rangle = 1$, so $\wedge^2\chi$ is irreducible. This also gives us $\wedge^2\chi\otimes$ sign for free.

   It turns out that $\wedge^k\chi(g)$ is $(-1)^k$ times the coefficient of x^k in $\det(\rho(g) - x\cdot\mathrm{id})$. So again, we can explicatively calculate $\wedge^3\chi(g)$ for all g, and find that $\wedge^3\chi$ is irreducible.

4. Generalised quaternions: $Q_{2^{n+2}} = \langle c, x | c^{2^n} = x^2, xcx^{-1} = c^{-1} \rangle$ (so $\mathrm{ord}(c) = 2^{n+1}, \mathrm{ord}(x) = 4$). Hence $|Q_{2^{n+2}}| = 2^{n+2}$.

   Let $n = 2$, so consider $G = Q_{2^4}$: We have the conjugacy classes:

Representative	Size	Order
id	1	1
x^2	1	2
x	4	4
c^2	2	4
cx	4	4
c	2	8
c^3	2	8

   Let $G' = \langle c^2 \rangle$, $G/G' \cong C_2 \times C_2$, so we get 4 1-dimensional characters.

	id	x^2	x	c^2	cx	c	c^3
$\mathbb{I}$	1	1	1	1	1	1	1
χ_1	1	1	-1	1	1	-1	1
χ_2	1	1	-1	1	-1	1	1
χ_3	1	1	1	1	-1	-1	-1
τ_1	2	2	0	-2	0	0	0
τ_2	2	-2	0	0	0	$\sqrt{2}$	$-\sqrt{2}$
τ_3	2	-2	0	0	0	$-\sqrt{2}$	$\sqrt{2}$

   We then calculate an induced character from a non-abelian quotient of order 8 (hence either Q_8 or D_8, but they have the same character table)

We now compute the Frobenius - Schur indicator: $s_2(\chi_i) = 1$, $s_2(\tau_1) = \frac{1}{16}\sum_{g\in G}\tau_1(g^2) = \frac{1}{16}(2+2+2\cdot 2+4\cdot 2+\dots) > 0$, hence $s_2(\tau_1) = 1$. This actually shows that $G/\left\langle c^4\right\rangle \cong D_8$ (and not Q_8), and τ_1 is realisable over $\mathbb{Q}$. Next we compute $s_2(\tau_2) = s_2(\tau_3) = -1$. So τ_2, τ_3 are nor realisable over $\mathbb{R}$ (in particular not over $\mathbb{Q}\left(\sqrt{2}\right)$). In particular, $m_{\mathbb{Q}}(\tau_{2,3}) > 1$. Let $H = \langle c\rangle \cong C_8$,

$$\begin{aligned}\left\langle \tau_2 \downarrow_H, \left(c \mapsto e^{2\pi i/8}\right)\right\rangle_H &= \frac{1}{|H|}\sum_{k=0}^{7}\tau_2(c^k)\cdot e^{2\pi i k/8} \\ &= 1 \\ &= \left\langle \tau_2, \left(c\mapsto e^{2\pi i/8}\right)\uparrow^G\right\rangle_G.\end{aligned}$$

Hence $\tau_2 = \left(c\mapsto e^{2\pi i/8}\right)\uparrow^G$. So we can define τ_2 over $\mathbb{Q}\left(e^{2\pi i/8}\right)$. Note that $[\mathbb{Q}\left(e^{2\pi i/8}\right) : \mathbb{Q}\left(\sqrt{2}\right)] = 2$, so $m_{\mathbb{Q}}(\tau_2) = 2$. Similarly for τ_3. This last calculation also verifies Brauer's induction theorem for τ_2 and τ_3.

Let us verify Artin's Induction:

- The cyclic subgroups are: $1, Z = \left\langle x^2\right\rangle, C_1 = \left\langle c^2\right\rangle, C_2 = \langle x\rangle, C_3 = \langle cx\rangle, C_4 = \langle c\rangle$.
- So we get $\mathbb{I}_{\{1\}}\uparrow^G = 1+\chi_1+\chi_2+\chi_3+\chi_4+2\cdot(\tau_1+\tau_2+\tau_3)$
- $\mathbb{I}_Z\uparrow^G = 1+\chi_1+\chi_2+\chi_3+2\tau_1$
- etc

6 Symmetric Groups

Motivation:

1. Artin - Wedderburn: $\mathbb{C}[G] \cong \oplus_{i=1}^{t} M_{n_i}(\mathbb{C})$. It's clear what the simple modules of the RHS look like (they are, as submodules of the regular module, columns). If $e_i = (0, \ldots, 0, I_{n_i}, 0, \ldots, 0) \in \oplus M_{n_i}(\mathbb{C})$, then $e_i \cdot \oplus M_{n_i}(\mathbb{C})$ is a direct sum of n_i mutually isomorphic simple modules. By finding that $e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \in \mathbb{C}[G]$, one finds the "$\chi_i$"-isotypical block $M_{n_i}(\mathbb{C})$ as a submodule of $\mathbb{C}[G]$. For example:

   - $\mathbb{I} = \left\langle \sum_{g \in G} g \right\rangle \leq \mathbb{C}[G]$.
   - $G = S_3$, χ the standard representation, so $e_\chi = \frac{2}{6}(2\,\mathrm{id} - (123) - (132)) \in \mathbb{C}[S_3]$, and $e_\chi \mathbb{C}[G]$ is 4-dimensional.

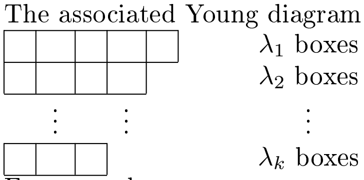
   But if we knew what $f_i = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \in M_{n_i}(\mathbb{C})$ looks like as an element of $\mathbb{C}[G]$, then $\mathbb{C}[G] f_i$ would give us a simple summand.

2. We know that the number of conjugacy classes of elements of G is the same as the number of isomorphism classes of simple $\mathbb{C}[G]$-modules. But there is, in general, no canonical bijection between these two sets.

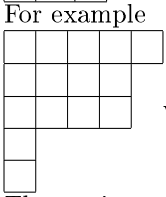
6.1 Young Diagrams

Conjugacy classes of S_n $\leftrightarrow$cycle types $\leftrightarrow$ partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n such that $\lambda_i \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum \lambda_i = n$.

The associated Young diagram is



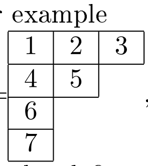
For example



which corresponds to $5 + 4 + 4 + 1 + 1 = 15$.

The conjugate partition is obtained by reflecting the Young diagram in the $\backslash$ diagonal. A Young tableau is a numbering of the boxes of a Young Diagram by numbers from 1 to n. S_n acts on the Young tableau of any given Young diagram. Fix any tableau of the Young diagram corresponding to λ (e.g., the obvious one), define $P_\lambda = \{g \in S_n | g \text{ fixes each row of the tableau}\}$.

For example

$\lambda =$ 

, then $P_\lambda = \langle (123), (45), (12) \rangle \cong S_3 \times S_2 \times S_1 \times S_1$.

We also define $Q_\lambda = \{g \in S_n | g \text{ fixes each columns of the tableau}\}$. Define:

- $a_\lambda = \sum_{g \in P_\lambda} g \in \mathbb{C}[S_n]$
- $b_\lambda = \sum_{g \in Q_\lambda} \mathrm{sgn}(g) \cdot g \in \mathbb{C}[S_n]$,
- $c_\lambda = a_\lambda \cdot b_\lambda$. This is the Young symmetriser corresponding to λ.

Example. Let $G = S_3$ and $\lambda = (2, 1)$, i.e., $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. Then $P_\lambda = \langle(12)\rangle$, $Q_\lambda = \langle(13)\rangle$, $a_\lambda = \mathrm{id} + (12)$, $b_\lambda = \mathrm{id} - (13)$ and hence $c_\lambda = \mathrm{id} + (12) - (13) - (123)$.

Theorem 6.1. *$V_\lambda := \mathbb{C}[G] \cdot c_\lambda$ is a simple S_n-module, only depending on λ up to isomorphism. $V_\lambda = V_\mu$ if and only $\lambda = \mu$, and all simple modules are isomorphic to some V_λ.*

Reference for the proof: Fulton, Harris: Representation Theory: a first course.

Example. Continuing from above, $G = S_3$ and $\lambda = (2, 1)$, we get $\mathbb{C}[G]c_\lambda = \langle c_\lambda, (13) \cdot c_\lambda\rangle := \langle v_1, v_2\rangle$. Then see how S_3 acts on v_1, v_2:

- $(13)v_1 = v_2$, $(13)v_2 = v_1$
- $(123)v_1 = (123) + (23) - (12) - (132) = -v_1 - v_2$, $(123)v_2 = v_1$.

Example. Let $G = S_n$:

- Consider $\lambda = (n)$. Then we have $P_\lambda = S_n$, $Q_\lambda = \{1\}$, $a_\lambda = \sum_{g \in S_n} g = c_\lambda$. So $V_\lambda \cong \mathbb{I}$.
- Consider $\lambda = (1, \dots, 1)$. Then we have $P_\lambda = \{1\}, Q_\lambda = S_n$, $b_\lambda = \sum_{g \in S_n} \mathrm{sgn}(g) \cdot g = c_\lambda$. So $V_\lambda \cong \mathrm{sign}$.
- Consider $\lambda = (n-1, 1)$. Then we have $P_\lambda = S_{n-1} = \mathrm{stab}_{S_n}(n) \leq S_n$, $Q_\lambda = \langle(1n)\rangle$. So $a_\lambda = \sum_{g \in \mathrm{Stab}(n)} g$, $b_\lambda = \mathrm{id} - (1n)$, hence $c_\lambda = \sum_{g \in \mathrm{Stab}(n)} g - \sum_{g \in \mathrm{Stab}(n) \cdot (1n)} g$. After some work, you should find that V_λ is the standard representation of S_n ($(n-1)$-dimensional)
- More generally: $\lambda = (n - i, 1, \dots, 1)$, then $V_\lambda = \wedge^i V_{(n-1,1)}$.

6.2 Frobenius's Formula

Let χ_λ be the character of V_λ. Let $g \in S_n$, i_1 = number of 1-cycles in g, i_2 = number of 2-cycles of g, ... , i_n = number of n-cycles of g. As usual $\lambda = (\lambda_1, \dots, \lambda_k)$. Consider the following symmetric function in $x_1, \dots, x_k$:

- $P_j(\underline{x}) = x_1^j + \cdots + x_k^j$, $1 \leq j \leq n$
- $\Delta(\underline{x}) = \prod_{i<j}(x_i - x_j)$

Set $l_1 = \lambda_1 + (k-1)$, $l_2 = \lambda_2 + (k-2)$, ... , $l_k = \lambda_k$.

Theorem 6.2 (Frobenius). *$\chi_\lambda(g)$ is the coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in*

$$\Delta(\underline{x}) \prod_{j=1}^{n} P_j(\underline{x})^{i_j}.$$

Example. Let $G = S_3$, $\lambda = (2, 1)$.

- $g = (12)$, then we have $i_1 = 1$, $i_2 = 1$, $i_3 = 0$, $l_1 = 2 + 2 - 1 = 3$, $l_2 = 1 + 2 - 2 = 1$. We have $\Delta(\underline{x}) = x_1 - x_2$, so $\chi_\lambda(g)$ is the coefficient of $x_1^3 x_2$ in $(x_1 - x_2)(x_1 + x_2)^1(x_1^2 + x_2^2)^1 = (x_1^4 - x_2^4)$. So $\chi_\lambda(g) = 0$.
- $g = (123)$, then we have $i_1 = 0$, $i_2 = 0$, $i_3 = 1$, $l_1 = 3$, $l_2 = 1$. So $\chi_\lambda(g)$ is the coefficient of $x_1^3 x_2$ in $(x_1 - x_2)(x_1^3 + x_2^3) = x_1^4 + x_1 x_2^3 - x_2 x_1^3 - x_2^4$. So $\chi_\lambda(g) = -1$.

Note that with manipulation, we can find that

$$\dim \chi_\lambda = \frac{n!}{l_1! \dots l_k!} \cdot \prod_{i<j}(l_1 - l_j).$$

Another dimension formula: Define the look length of a box in a Young Diagram is the number of boxes to the right and underneath it (counting the box itself once). E.g.,

7	4	3	1
5	2	1	
2			
1			

Theorem 6.3.

$$\dim \chi_\lambda = \frac{n!}{\prod \text{hook length in } \lambda}$$

E.g., with the example above, we find that $\chi_\lambda = \frac{9!}{2 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 7} = \frac{6 \cdot 8 \cdot 9}{2} = 216$.

Remark. Since $c_\lambda \in \mathbb{Q}[G]$, all simple $\mathbb{C}[S_n]$-modules are realisable over $\mathbb{Q}$.

7 Revision Quiz Session

True or False

- If $H \leq G$, $\chi \in \mathrm{Irr}(G)$, then all irreducible summands of $\chi\downarrow_H$ have the same dimension.

  No: Let $H = S_3$, we are looking for $G \geq S_3$ and $\chi \in \mathrm{Irr}(G)$, such that $\langle \chi, \mathrm{stand}\uparrow^{\mathrm{G}} \rangle = \langle \chi\downarrow_{S_3}, \mathrm{stand} \rangle \neq 0 \neq \langle \chi\downarrow_{S_3}, \mathrm{a\,1{-}dimensional\,char} \rangle = \langle \chi, 1{-}\mathrm{dim}\uparrow^G \rangle$. Let ρ be the standard representation and τ the 1-dimensional representation, then we want $\langle \rho\uparrow^G, \tau\uparrow^G \rangle \neq 0$. We have

$$\begin{aligned} \langle \rho\uparrow^G, \tau\uparrow^G \rangle &= \langle \rho\uparrow^G\downarrow_{S_3}, \tau \rangle \\ &= \sum_{S_3\backslash G/S_3} \left\langle {}^g\rho\downarrow_{S_3\cap {}^gS_3}\uparrow^{S_3}, \tau \right\rangle \end{aligned}$$

  If $G = S_6$, $S_3 = \mathrm{Stab}(4,5,6)$, there exists a $g \in G$ such that ${}^gS_3 = \mathrm{Stab}(1,2,3)$, so $S_3 \cap {}^gS_3 = \{1\}$, so $\rho\downarrow_{S_3\cap {}^gS_3} = \mathbb{I} + \mathbb{I}$.

- If $H = N \lhd G$, $\chi \in \mathrm{Irr}(G)$, then all irreducible summand of $\chi\downarrow_H$ have the same dimension.

  Yes: (Clifford).

- If $H \leq G$, $\chi \in \mathrm{Irr}(H)$, then all irreducible summand of $\chi\uparrow^G$ have the same dimension.

  No: Consider the regular representation with any groups.

- If $H = N \lhd G$, $\chi \in \mathrm{Irr}(H)$, then all irreducible summand of $\chi\uparrow^G$ have the same dimension.

  No: by the above reasoning.

- If χ is realisable over K, then so are all irreducible summands of $\chi\downarrow_H$.

  No: Let $G = S_3$, $H = C_3$, χ be the standard representation.

- If χ is realisable over K, then so are all irreducible summands of $\chi\uparrow^G$.

  No: Take the regular representation.

- If $\chi_H\uparrow^G$ is realisable over K, then so is χ.

  No: Let $\chi \in \mathrm{Irr}(C_3)$ be non-trivial and $G = S_3$.

- If F/K is an extension of fields, then $m_F(\chi) \leq m_K(\chi)$.

  Yes: By definition of $m(\chi)$. (Also using Tower laws we get $m_F(\chi) \big| m_K(\chi)$)

Recall: If F/K is a finite field extension, M an $F[G]$-module, by thinking of F as a K-vector space, we can think of M as a $K[G]$-module of K-dimension equal to $\dim_F(M)\cdot[F:K]$.

A completely different instance of the same "philosophy": Let C be a curve over $\mathbb{Q}$, e.g., $C : y^2 = x^3 + x$. We can think of this as a curve over $\mathbb{Q}[i]$. So now, write $x = u + iv$ and $y = w + iz$, with $u, v, w, z \in \mathbb{Q}$. So $C : (w + iz)^2 = (u + iv)^3 + (u + iv)$. By equating real and imaginary parts, we get two equations in four unknowns over $\mathbb{Q}$. So we now have a surface over $\mathbb{Q}$.